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the amalgamated product  $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$  is isomorphic to  $\mathbf{Z}^2$ . The circuit series of  $\mathcal{D}$ ,  $\mathcal{E}$  and  $\mathcal{F}$  have been calculated explicitly and are algebraic. The circuit series of  $\mathcal{X}$  was shown in Section 10 to be transcendental; so there can exist no algebraic definition of  $G_{\mathcal{X}}$  in terms of  $G_{\mathcal{D}}$ ,  $G_{\mathcal{E}}$  and  $G_{\mathcal{F}}$ . However, there exists some relations between these series, as given by [Voi90, Theorem 5.5].

Given a graph  $\mathcal{X}$ , one can construct a graph  $\mathcal{X}^{(k)}$  on the same vertex set, and with edge set the set of paths of length  $\leq k$  in  $\mathcal{X}$ . Is there some simple relation between the path series of  $\mathcal{X}$  and of  $\mathcal{X}^{(k)}$ ? This could be useful for example to obtain asymptotics about the cogrowth of a group subject to enlargement of generating set [Cha93].

The equation (9.2) corresponds to Voiculescu's  $R$ -transform [Voi90]. His  $S$ -transform, in terms of graphs, corresponds to  $\mathcal{E} * \mathcal{F}$  with as edge set all sequences  $(e, f)$  and  $(f, e)$ , for  $e \in E(\mathcal{E})$  and  $f \in E(\mathcal{F})$ . Is there an analogue to Theorem 9.2 in this context?

Finally, (9.2) computes the circuit series of a free product in terms of the circuit series of the factors. A more complicated formula yields the path series of a free product in terms of the path series of the factors. Such considerations give another derivation of the results in Section 8.

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*Added in proof.* Recently Vaughan Jones has obtained very similar results in the context of planar algebras, for which some 'path' and 'proper path' series give the Hilbert-Poincaré series of a planar algebra over different subalgebras (see *Planar Algebras I*; preprint at <http://www.math.berkeley.edu/~vfr/plnalg1.ps>).