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## 11. FURTHER WORK

It was mentioned in Subsection 3.3 how the main result applies to languages. This is a special case of a much more general problem:

PROBLEM 11.1. *Given a language  $L$  and a set  $\mathcal{U}$  of words, define the desiccation  $L_{\mathcal{U}}$  of  $L$  as the set of words in  $L$  containing no  $u \in \mathcal{U}$  as a subword.*

*Give sufficient conditions on  $L$  and  $\mathcal{U}$  such that a formula exist relating  $\Theta(L)$  and  $\Theta(L_{\mathcal{U}})$ .*

The special case we studied in this paper is that of

$$\mathcal{U} = \{s\bar{s} \mid s \in S\}$$

and a sufficient condition is that  $L$  be saturated.

For general  $\mathcal{U}$  this is not always sufficient: let  $S = \{a, b\}$  and  $L = b^*(ab^*ab^*)^*$  be the set of words with an even number of  $a$ 's. Then if  $\mathcal{U} = \{a^2\}$  there are 7 desiccated words of length 5:

$$\{b^5, ab^3a, ab^2ab, abab^2, bab^2a, babab, b^2aba\}$$

and if  $\mathcal{U}' = \{b^2\}$  there are 6 desiccated words of length 5:

$$\{babab, ba^4, aba^3, a^2ba^2, a^3ba, a^4b\}.$$

The growth series of  $\mathcal{U}$  and  $\mathcal{U}'$  are the same, namely  $t^2$ , but the growth series of  $L_{\mathcal{U}}$  and  $L_{\mathcal{U}'}$  differ in their degree-5 coefficient.

We gave in Section 9 a formula relating the circuit series of a free product to the circuit series of its factors. There is a notion of *amalgamated product* of graphs, that is a direct analogue of the amalgamated product of groups.

PROBLEM 11.2. *What conditions on  $\mathcal{D}, \mathcal{E}, \mathcal{F}$  are sufficient so that*

$$\frac{1}{(zG_{\mathcal{X}})^{-1}} = \frac{1}{(zG_{\mathcal{E}})^{-1}} + \frac{1}{(zG_{\mathcal{F}})^{-1}} - \frac{1}{(zG_{\mathcal{D}})^{-1}}$$

*where  $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$  is an amalgamated product of  $\mathcal{E}$  and  $\mathcal{F}$  along  $\mathcal{D}$ ?*

The formula holds if  $\mathcal{D}$  is the trivial graph; but it cannot hold in general. If  $\mathcal{E} = \mathcal{F}$  is the "ladder graph" described in Section 7.4: the set of points  $(i, j)$  with  $i \in \mathbf{Z}$  and  $j \in \{0, 1\}$ , with edges connecting all pairs of vertices at Euclidean distance 1, and  $\mathcal{D}$  is  $\mathbf{Z}$ , embedded as a pole of the ladder, then

the amalgamated product  $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$  is isomorphic to  $\mathbf{Z}^2$ . The circuit series of  $\mathcal{D}$ ,  $\mathcal{E}$  and  $\mathcal{F}$  have been calculated explicitly and are algebraic. The circuit series of  $\mathcal{X}$  was shown in Section 10 to be transcendental; so there can exist no algebraic definition of  $G_{\mathcal{X}}$  in terms of  $G_{\mathcal{D}}$ ,  $G_{\mathcal{E}}$  and  $G_{\mathcal{F}}$ . However, there exists some relations between these series, as given by [Voi90, Theorem 5.5].

Given a graph  $\mathcal{X}$ , one can construct a graph  $\mathcal{X}^{(k)}$  on the same vertex set, and with edge set the set of paths of length  $\leq k$  in  $\mathcal{X}$ . Is there some simple relation between the path series of  $\mathcal{X}$  and of  $\mathcal{X}^{(k)}$ ? This could be useful for example to obtain asymptotics about the cogrowth of a group subject to enlargement of generating set [Cha93].

The equation (9.2) corresponds to Voiculescu's  $R$ -transform [Voi90]. His  $S$ -transform, in terms of graphs, corresponds to  $\mathcal{E} * \mathcal{F}$  with as edge set all sequences  $(e, f)$  and  $(f, e)$ , for  $e \in E(\mathcal{E})$  and  $f \in E(\mathcal{F})$ . Is there an analogue to Theorem 9.2 in this context?

Finally, (9.2) computes the circuit series of a free product in terms of the circuit series of the factors. A more complicated formula yields the path series of a free product in terms of the path series of the factors. Such considerations give another derivation of the results in Section 8.

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*Added in proof.* Recently Vaughan Jones has obtained very similar results in the context of planar algebras, for which some 'path' and 'proper path' series give the Hilbert-Poincaré series of a planar algebra over different subalgebras (see *Planar Algebras I*; preprint at <http://www.math.berkeley.edu/~vfr/plnalg1.ps>).