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Proof. This follows from

$$\begin{aligned} 1/\rho(G_{\mathcal{X}}) &= \lim_{t\to\infty} \frac{1}{(tG_{\mathcal{X}})^{-1}} \\ &= \lim_{t\to\infty} \frac{1}{(tG_{\mathcal{E}})^{-1}} + \frac{1}{(tG_{\mathcal{F}})^{-1}} - \frac{1}{t} \qquad \text{by (9.2)} \\ &= 1/\rho(G_{\mathcal{E}}) + 1/\rho(G_{\mathcal{F}}) - 0\,. \qquad \square \end{aligned}$$

Note that the corollary does not extend to non-recurrent series; for instance, it fails if $\mathcal{E} = \mathcal{F} = \mathbf{Z}$. Indeed then

$$G_{\mathcal{E}} = G_{\mathcal{F}} = \frac{1}{\sqrt{1-4t^2}}\,, \qquad \rho(G_{\mathcal{E}}) = \rho(G_{\mathcal{F}}) = 1/4\,,$$

$$G_{\mathcal{X}} = \frac{3}{1+2\sqrt{1-12t^2}}\,, \qquad \rho(G_{\mathcal{X}}) = 1/\sqrt{12}\,.$$

10. Direct products of graphs

There are two natural definitions for *direct products* of graphs; they correspond to direct products of groups with generating set either the union or cartesian product of the generating sets of the factors. A general treatment of products of graphs can be found in [CDS79, pages 65 and 203].

DEFINITION 10.1. If S is a set, the *stationing graph* on S is the graph $\mathcal{X} = \Sigma_S$ with $V(\mathcal{X}) = E(\mathcal{X}) = S$, where for the edges $s^\alpha = s^\omega = \bar{s} = s$ hold.

LEMMA 10.2. *Let $\mathcal{X}$ be a graph, and $\mathcal{E} = \mathcal{X} \sqcup \Sigma_{\mathcal{X}}$ be the graph obtained by adding a loop to every vertex in $\mathcal{X}$. Let $G_{\mathcal{X}}$ and $G_{\mathcal{E}}$ be the growth functions for circuits in $\mathcal{X}$ and $\mathcal{E}$ respectively. Then we have*

$$G_{\mathcal{E}}(t) = \frac{1}{1-t} G_{\mathcal{X}}\left(\frac{t}{1-t}\right).$$

DEFINITION 10.3 (First Product). Let $\mathcal{E}$ and $\mathcal{F}$ be two graphs. Their *direct product* $\mathcal{X} = \mathcal{E} \times \mathcal{F}$ is defined by

$$V(\mathcal{X}) = V(\mathcal{E}) \times V(\mathcal{F})$$

and

$$E(\mathcal{X}) = (E(\mathcal{E}) \times \Sigma_{\mathcal{F}}) \sqcup (\Sigma_{\mathcal{E}} \times E(\mathcal{F}))\,.$$

Note that if the graphs $\mathcal{E}$ and $\mathcal{F}$ have respectively adjacency matrices E and F, then their product has adjacency matrix $E \otimes \mathbf{1} + \mathbf{1} \otimes F$.

In that case we have

$$G_{\mathcal{X}} = \frac{1}{2i\pi} \oint_{\mathbf{S}^1} \frac{G_{\mathcal{E}}((1+u)t)\, G_{\mathcal{F}}\left((1+u^{-1})t\right)}{u} du .$$

This is a simple application of the Laplace transform, that converts an exponential generating function into an ordinary one and vice versa [AS70, 29.3.3]. Indeed, if we had considered exponential generating functions, the formula would simply have been $G_{\mathcal{X}} = G_{\mathcal{E}} G_{\mathcal{F}}$, as is well known (see [Wil90] or [Sta78, page 102]).

As an example, let $\mathcal{E} = \mathcal{F} = \mathbf{Z}$, so $G_{\mathcal{E}} = G_{\mathcal{F}} = \frac{1}{\sqrt{1-t^2}}$. Then

$$\begin{aligned} G_{\mathcal{X}} &= \frac{1}{2i\pi} \oint_{\mathbf{S}^1} \frac{du}{\sqrt{(1-4(1+u)^2t^2)(u^2-4(1+u)^2t^2)}} \\ &= \frac{2}{\pi} K(16t^2) = F\left(\begin{matrix} 1/2 & & 1/2 \\ & 1 & \end{matrix} \,\middle|\, 16t^2 \right) \end{aligned}$$

where K is the complete elliptic function and F the hypergeometric series. These functions are known to be transcendental; thus the circuit series of $\mathbf{Z}^2$ is transcendental. This result appears in [GH97]. Numerical evidence suggests the growth function for $\mathbf{Z}^3$ is not even hypergeometric.

DEFINITION 10.4 (Second Product). Let $\mathcal{E}$ and $\mathcal{F}$ be two graphs, and suppose that for every vertex in $\mathcal{E}$ and $\mathcal{F}$ there is a loop at it. Then their *direct product* $\mathcal{X} = \mathcal{E} \times \mathcal{F}$ is defined by

$$V(\mathcal{X}) = V(\mathcal{E}) \times V(\mathcal{F})$$

and

$$E(\mathcal{X}) = E(\mathcal{E}) \times E(\mathcal{F}) .$$

Note that if the graphs $\mathcal{E}$ and $\mathcal{F}$ have respectively adjacency matrices E and F, then their product has adjacency matrix $E \otimes F$.

In that case we have, again using Laplace transformations

$$G_{\mathcal{X}}(t) = \frac{1}{2i\pi} \oint_{\mathbf{S}^1} \frac{G_{\mathcal{E}}(u) G_{\mathcal{F}}(t/u)}{u} du .$$

Note that with both definitions of products it is possible that the growth function for circuits in the product be transcendental even if the growth functions for circuits in the factors are algebraic.