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giving a solution f that is algebraic of degree 4 over $\mathbf{Z}(t)$.

Then define the following series:

G counts the walks from 0 to 0 in $\mathbf{Z}$;

e counts the walks from 0 to 1 in $\mathbf{Z}$.

They satisfy the equations

$$G = 1 + 2(\uparrow f \downarrow G + \upuparrows g \downarrow G + \uparrow f \downdownarrows e + \upuparrows g \downdownarrows e + \uparrow g \downdownarrows G + \upuparrows h \downdownarrows G),$$
$$e = G \uparrow f + G \upuparrows f + G \upuparrows g$$

giving the solution

$$G = \frac{4 + 3t - 6t^2 - 10t(1+2t)\delta + 2t^2(3+8t)\delta^2 - 6t^4(1+t)\delta^3}{4 - 7t - 36t^2}$$

where δ is a root of the equation

$$1 - (2t+1)\delta + t(2+3t)\delta^2 - t^2(1+2t)\delta^3 + t^4\delta^4 = 0\,.$$

8. Cogrowth of non-free presentations

We perform here a computation extending the results of Section 3.1. The general setting, expressed in the language of group theory, is the following: let Π be a group generated by a finite set S and let $\Xi < \Pi$ be any subgroup. We consider the following generating series:

$$F(t) = \sum_{\gamma \in \Xi < \Pi} t^{|\gamma|},$$

$$G(t) = \sum_{\substack{\text{words } w \text{ in } S \\ \text{defining an element in } \Xi}} t^{|w|},$$

where $|\gamma|$ is the minimal length of γ in the generators S, and $|w|$ is the usual length of the word w. Is there some relation between these series? In case Π is quasi-free on S, the relation between F and G is given by Corollary 2.6. We consider two other examples: Π quasi-free but on a set smaller than S, and $\Pi = \mathbf{PSL}_2(\mathbf{Z})$.

8.1 Π QUASI-FREE

Let S, T be finite sets, and $\bar{\cdot}$ an involution on S. Consider the two presentations

$$\Pi = \langle S \mid s\bar{s} = 1 \ \forall s \in S \rangle ,$$
$$\Pi = \langle S \cup T \mid s\bar{s} = 1 \ \forall s \in S; \ t = 1 \ \forall t \in T \rangle .$$

Let $\Xi < \Pi$ be any subgroup, and let F' and G' be the generating series related to the first presentation. Clearly $F' = F$, as both series count the same objects in Π (regardless of Π's presentation); while

$$G(t) = \frac{G'\left(\frac{t}{1-|T|t}\right)}{1-|T|t} .$$

Indeed any word $w = w_1 \dots w_n$ in $S \cup T$ defining an element of Ξ can be uniquely decomposed as $w = t_0 s_1 t_1 \dots s_m t_m$, where $s_i \in S$, t_i are words in T for all i, and $s_1 \dots s_n$ defines an element of Ξ; moreover all choices of $s_1 \dots s_n$ defining an element of Ξ and words t_i in T give a distinct word w. It then suffices to note that the generating series for any of the t_i is $1/(1-|T|t)$.

Putting everything together, we obtain:

PROPOSITION 8.1. *Let Π be as above, $\Xi < \Pi$ a subgroup. Then*

$$\frac{F(t)}{1-t^2} = \frac{G\left(\frac{t}{1+|T|t+(|S|-1)t^2}\right)}{1+|T|t+(|S|-1)t^2} .$$

8.2 $\Pi = \mathbf{PSL}_2(\mathbf{Z})$

Let

$$\Pi = \mathbf{PSL}_2(\mathbf{Z}) = \langle a, b \mid a^2, b^3 \rangle ,$$

and let $\Xi < \Pi$ be any subgroup. We take $S = \{a, b, b^{-1}\}$.

We suppose Ξ is torsion-free, i.e. contains no element of the form waw^{-1} or $wb^{\pm 1}w^{-1}$. Let $\mathcal{X}$ be the Schreier graph of $(\Pi, \{a, b, b^{-1}\})$ relative to Ξ, as defined in Subsection 3.1; it is a trivalent graph whose vertex set is $\Xi \backslash \Pi$. Its vertices can be grouped in triples $w^{\Delta} = \{w, wb, wb^{-1}\}$ connected in triangles. Let $\mathcal{F}$ be the graph obtained from $\mathcal{X}$ by identifying each triple to a vertex. Explicitly,

$$V(\mathcal{F}) = \{w^{\Delta} : w \in V(\mathcal{X})\},$$
$$E(\mathcal{F}) = \{(v^{\Delta}, (va)^{\Delta}) : v \in V(\mathcal{X})\};$$

the involution on $E(\mathcal{F})$ is the switch $(A,B) \mapsto (B,A)$ and the extremity functions $E(\mathcal{F}) \to V(\mathcal{F})$ are the natural projections. Note that $\mathcal{F}$ is a 3-regular graph (for instance, 1^{Δ} is connected to a^{Δ}, $(ba)^{\Delta}$ and $(b^{-1}a)^{\Delta}$). In case $\Xi = 1$, it is the 3-regular tree. By construction we have a 3-to-1 map $\Delta\colon V(\mathcal{X}) \to V(\mathcal{F})$. We fix an origin $\star = 1^{\Delta}$ in $\mathcal{F}$, and let $F_{\mathcal{F}}(u,t)$ be the circuit series of $(\mathcal{F}, \star)$.

Let $\mathcal{E}$ be a triangle, $G_{\mathcal{E}}(t)$ count the circuits at a fixed vertex of $\mathcal{E}$ and $G_{\mathcal{E}}^{\neq}(t)$ count paths between two fixed distinct vertices of $\mathcal{E}$. These series were computed in Section 7.1, with $G_{\mathcal{E}}^{\neq}(t) = F'(1,t) + F''(1,t)$.

Circuits at $\star$ in $\mathcal{X}$ can be projected to circuits at $\star$ in $\mathcal{F}$ simply by deleting all edges of type $(w, wb^{\pm 1})$ and projecting the other edges through Δ. Conversely, circuits in $\mathcal{F}$ can be lifted to $\mathcal{X}$ by lifting the edges through Δ^{-1}, and connecting them in $\mathcal{X}$ with arbitrary paths remaining inside the triples: to lift the path $\pi = (\pi_1, \dots, \pi_n)$ from $\mathcal{F}$ to $\mathcal{X}$, choose edges $\rho_1, \dots, \rho_n$ with $(\rho_i^{\alpha})^{\Delta} = \pi_i^{\alpha}$ and $(\rho_i^{\omega})^{\Delta} = \pi_i^{\omega}$ for all $i \in \{1, \dots, n\}$, and choose, for all $i \in \{0, \dots, n\}$, paths τ_i from ρ_i^{ω} to ρ_{i+1}^{α} remaining inside $(\rho_i^{\omega})^{\Delta}$, where by convention $\rho_0^{\omega} = \rho_{n+1}^{\alpha} = \star$. Then the lift corresponding to these choices is

$$\tau_0 \cdot \rho_1 \cdot \tau_1 \cdot \dots \cdot \rho_n \cdot \tau_n \,. \tag{8.1}$$

Furthermore all circuits at $\star$ in $\mathcal{X}$ can be obtained this way.

Define $\overline{G}$ as the series counting paths that start and finish at a vertex in the same triple as $\star$. It can be obtained using (8.1) by letting ρ range over all paths in $\mathcal{F}$, and for each choice of ρ and for each $i \in \{1, \dots, n-1\}$ letting τ_i range over $G_{\mathcal{E}}$ or $G_{\mathcal{E}}^{\neq}$ depending on whether ρ has or not a bump at i, and letting τ_0 and τ_n range over all paths inside the triple $\star^{\Delta}$. In equations, this relation is expressed as

$$\overline{G}(t) = \left(\frac{1}{1-2t}\right)^2 / G_{\mathcal{E}}(t) \cdot F_{\mathcal{F}}\big(G_{\mathcal{E}}^{\neq}(t)/G_{\mathcal{E}}(t), tG_{\mathcal{E}}(t)\big)\,.$$

Now the series G we wish to obtain is approximately $\overline{G}(t)/9$: for any choice of $x, y \in \star^{\Delta}$ there are approximately the same number of long enough paths from x to y.

A summand of $F(t)$ is the unique lifting of a summand of $F_{\mathcal{F}}(0,t)$, but is twice longer in $\mathcal{X}$ than in $\mathcal{F}$.

DEFINITION 8.2. Two series $A(t)$, $B(t)$ are *equivalent*, written $A \sim B$, if they have the same radius of convergence ρ, and there exists a constant K such that

$$\frac{1}{K} < A(t)/B(t) < K \text{ as } t \to \rho .$$

Then the remarks of the previous paragraph can be written as

$$F(t) \sim F_{\mathcal{F}}(0, t^2),$$
$$G(t) \sim F_{\mathcal{F}}(G_{\mathcal{E}}^{\neq}(t)/G_{\mathcal{E}}(t), tG_{\mathcal{E}}(t)) .$$

Letting $G_{\mathcal{F}}$ be the circuit series of $\mathcal{F}$, we use Corollary 2.6 to obtain

$$G_{\mathcal{E}}(t)^{\neq} = \frac{t}{1-t-2t^2}, \qquad G_{\mathcal{E}}(t) = \frac{1-t}{1-t-2t^2},$$
$$F(t) \sim G_{\mathcal{F}}\Big(\frac{t^2}{1+2t^4}\Big), \qquad G(t) \sim G_{\mathcal{F}}\Big(\frac{t^2}{1-t-3t^2}\Big),$$

so

$$F(t) \sim G\Big(\frac{t\sqrt{4+13t^2-8t^4}-t^2}{2(1+t^2)(1+2t^2)}\Big) .$$

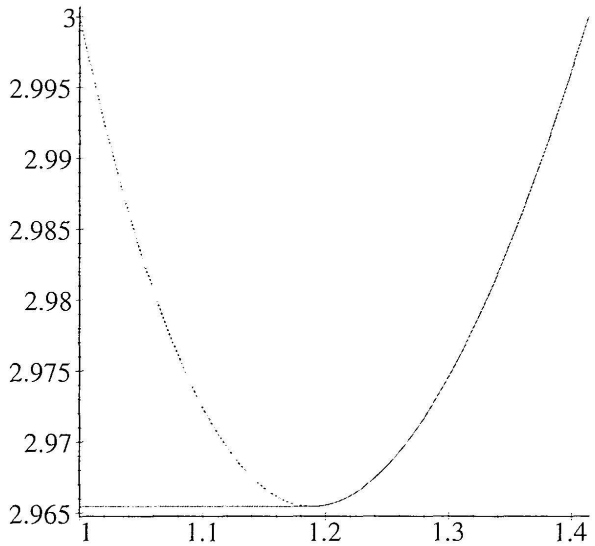

FIGURE 2

The function $\alpha \mapsto \nu$ relating cogrowth and spectral radius, for subgroups of $\mathbf{PSL}_2(\mathbf{Z})$

Let $\mathcal{X}$ be a simplicial complex such that at each vertex an edge and a (filled-in) triangle meet; choose a base point $\star$ in $\mathcal{X}$. Say a circuit in the 1-skeleton of $\mathcal{X}$ is *reduced* if it contains no bump nor two successive edges in the same triangle; thus reduced circuits are in bijection with homotopy classes in $\pi_1(\mathcal{X}, \star)$. Let $F(t)$ be the proper circuit series and $G(t)$ the circuit series of $\mathcal{X}$. Let

$$(t)\phi = \frac{t\sqrt{4 + 13t^2 + 8t^4} - t^2}{2(1+t^2)(1+2t^2)} .$$

We have proved the following theorem and corollary, similar to those in Section 3.1:

THEOREM 8.3. $F(t) \sim G\big((t)\phi\big)$.

COROLLARY 8.4. *Let* Ξ *be a subgroup of* $\Pi = \mathbf{PSL}_2(\mathbf{Z})$ *; let* ν *be the spectral radius of the simple random walk on* $\Xi\backslash\Pi$*, and* α *the "cogrowth" rate of* $\Xi\backslash\Pi$*. Then provided that* $\alpha \in [\sqrt{\rho}, \rho]$*, where* ρ *is the word growth of* Π*, namely* $\sqrt{2}$*, we have*

$$1/\nu = (1/\alpha)\phi, \quad so \quad \nu = \frac{1}{2}\sqrt{8\alpha^{-2} + 13 + 4\alpha^2} + \frac{1}{2} .$$

Proof. The function ϕ is monotonously increasing between 0 and $1/\sqrt[4]{2}$, where it reaches its maximum. The same argument applies as that given in the proof of Corollary 3.2. □

We now state the same results for an arbitrary virtually free group with an appropriate generating system. Let Π be a virtually free group, such that there is a split exact sequence

$$1 \longrightarrow \Sigma \longrightarrow \Pi \overset{\pi}{\rightleftarrows} \Upsilon \longrightarrow 1$$

where Υ is a finite group and Σ has a presentation

$$\Sigma = \langle s \in S \mid s\bar{s} = 1 \quad \forall s \in S \rangle .$$

We assume further that Π is generated by a set $T = T' \sqcup T''$ with T'' in bijection through π with $\Upsilon \setminus \{1\}$, T' mapping through π to $\{1\}$, and $T' \times (T'' \cup \{1\})$ in bijection with S through $(t, p) \mapsto p^{-1}tp$.

For example, consider $\Pi = \mathbf{PSL}_2(\mathbf{Z}) = \langle a, b, b^{-1} \rangle$. Take $T' = \{a\}$ and $T'' = \{b, b^{-1}\}$, take $\Upsilon = \langle b, b^{-1} \rangle$ and $\Sigma = \langle a, bab^{-1}, b^{-1}ab \rangle$. Then the hypotheses are satisfied.

With these hypotheses, the Cayley graph $\mathcal{X}$ of Π is a collection of complete graphs of size $|\Upsilon|$, with at each vertex $|T'|$ edges leaving to other complete graphs, and such that if each of these complete graphs is shrunk to a point the resulting graph is a tree. The following theorem is then a straightforward generalization of the argument given for $\mathbf{PSL}_2(\mathbf{Z})$.

THEOREM 8.5. *With the notation introduced above, let Ξ be any subgroup of Π not intersecting $\{t^\gamma \mid t \in T, \gamma \in \Pi\}$ and let $F(t)$, $G(t)$ be the "cogrowth" series and circuit series of $\Xi\backslash\Pi$. Let $\mathcal{E}$ be the complete graph on $|\Upsilon|$ vertices and let $G_{\mathcal{E}}(t)$, $G^{\neq}_{\mathcal{E}}(t)$ count the circuits and the non-closing paths respectively in $\mathcal{E}$. Define the function ϕ by*

$$\left(\frac{t^2}{1+(|S|-1)t^4}\right)\phi = \frac{tG_{\mathcal{E}}}{1+(G_{\mathcal{E}}-G^{\neq}_{\mathcal{E}})((|S|-1)G_{\mathcal{E}}+G^{\neq}_{\mathcal{E}})t^2}\,.$$

Then we have

$$F(t) \sim G((t)\phi)\,.$$

9. FREE PRODUCTS OF GRAPHS

We give here a general construction combining two pointed graphs and show how to compute the generating functions for circuits in the "product" in terms of the generating functions for circuits in the factors.

DEFINITION 9.1 (Free Product, [Que94, Definition 4.8]). Let $(\mathcal{E},\star)$ and $(\mathcal{F},\star)$ be two connected pointed graphs. Their *free product* $\mathcal{E}*\mathcal{F}$ is the graph constructed as follows: start with copies of $\mathcal{E}$ and $\mathcal{F}$ identified at $\star$; at each vertex v apart from $\star$ in $\mathcal{E}$, respectively $\mathcal{F}$, glue a copy of $\mathcal{F}$, respectively $\mathcal{E}$, by identifying v and the $\star$ of the copy. Repeat the process, each time glueing $\mathcal{E}$'s and $\mathcal{F}$'s to the new vertices.

If (E,S), (F,T) are two groups with fixed generators whose Cayley graphs are $\mathcal{E}$ and $\mathcal{F}$ respectively, then $\mathcal{E}*\mathcal{F}$ is the Cayley graph of $(E*F, S\sqcup T)$.

We now compute the circuit series of $\mathcal{E}*\mathcal{F}$ in terms of the circuit series of $\mathcal{E}$ and $\mathcal{F}$. Let $G_{\mathcal{E}}$, $G_{\mathcal{F}}$ and $G_{\mathcal{X}}$ be the generating functions counting circuits in $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{X}=\mathcal{E}*\mathcal{F}$ respectively. We will use the following description: given a circuit at $\star$ in $\mathcal{X}$, it can be decomposed as a product of circuits never passing through $\star$. Each of these circuits, in turn, starts either in the $\mathcal{E}$ or the $\mathcal{F}$ copy at $\star$. Say one starts in $\mathcal{E}$; it can then be expressed as a circuit in $\mathcal{E}$ never passing through $\star$, and such that at all vertices, except the first and last, a circuit starting in $\mathcal{F}$ has been inserted. Moreover, any choice of such circuits satisfying these conditions will give a circuit at $\star$ in $\mathcal{X}$, and different choices will yield different circuits.

Let $H_{\mathcal{E}}$ (respectively $H_{\mathcal{F}}$) be the generating function counting non-trivial circuits in $\mathcal{E}$ (respectively $\mathcal{F}$) never passing through $\star$. Obviously