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The first few values of F, where $\square$ stands for $1+(1-u^2)t^2$, are:

k	$F(u,t)$	k	$F(u,t)$
1	$\frac{1+(1-u)t}{1-(1+u)t}$	2	$\frac{\square}{1-(1+u)^2t^2}$
3	$\frac{(1+(1-u)t)(\square-t)}{(1-(1+u)t)(\square+t)}$	4	$\frac{\square^2-2t^2}{1-(1+u)^2t^2}$
5	$\frac{(1+(1-u)t)(\square^2-\square t-t^2)}{(1-(1+u)t)(\square^2+\square t+t^2)}$	6	$\frac{\square^2-3t^2}{(1-(1+u)^2t^2)(\square^2-t^2)}$
7	$\frac{(1+(1-u)t)(\square^3-\square^2t-2\square t^2+t^3)}{(1-(1+u)t)(\square^3+\square^2t-2\square t^2-t^3)}$	8	$\frac{\square^4-4\square^2t^2+2t^4}{(1-(1+u)^2t^2)(\square^2-2t^2)}$
9	$\frac{(1+(1-u)t)(\square-t)(\square^3-3\square t^2-t^3)}{(1-(1+u)t)(\square+t)(\square^3-3\square t^2+t^3)}$	10	$\frac{\square^4-5\square^2t^2+5t^4}{(1-(1+u)^2t^2)(\square^4-3\square^2t^2+t^4)}$

These rational expressions were computed and simplified using the computer algebra program *Maple*™.

7.3 Trees

Let $\mathcal{X}$ be the d-regular tree. Then

$$F(0,t)=1$$

as a tree has no proper circuit; while direct (i.e., without using Corollary 2.6) computation of G is more complicated. It was first performed by Kesten [Kes59]; here we will derive the extended circuit series $F(u,t)$ and also obtain the answer using Corollary 2.6.

Let $\mathcal{T}$ be a regular tree of degree d with a fixed root $\star$, and let $\mathcal{T}'$ be the connected component of $\star$ in the two-tree forest obtained by deleting in $\mathcal{T}$ an edge at $\star$. Let $F(u,t)$ and $F'(u,t)$ respectively count circuits at $\star$ in $\mathcal{T}$ and $\mathcal{T}'$. For instance if $d=2$ then F' counts circuits in $\mathbf{N}$ and F counts circuits in $\mathbf{Z}$. For a reason that will become clear below, we make the convention that the empty circuit is counted as '1' in F and as 'u' in F'. Then we have

$$F'=u+(d-1)\,tF't\frac{1}{1-(d-2+u)\,tF't},$$
$$F=1+dtF't\frac{1}{1-(d-1+u)\,tF't}.$$

Indeed a circuit in $\mathcal{T}'$ is either the empty circuit (counted as u), or a sequence of circuits composed of, first, a step in any of $d-1$ directions, then a 'subcircuit' not returning to $\star$, then a step back to $\star$, followed by a step in any of $d-1$ directions (counting an extra factor of u if it was the same as before), a subcircuit, etc. If the 'subcircuit' is the empty circuit, it contributes a bump, hence the convention on F'. Likewise, a circuit in $\mathcal{T}$ is either the empty circuit (now counted as 1) or a sequence of circuits in subtrees each isomorphic to $\mathcal{T}'$.

We solve these equations to

$$F'(1-u,t) = \frac{2(1-u)}{1-u(d-u)t^2+\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}},$$

$$F(1-u,t) = \frac{2(d-1)(1-u^2t^2)}{(d-2)(1+u(d-u)t^2)+d\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}}.$$

Using (2.3) and $F(0,t)=1$ we would obtain

$$G(t) = \frac{1+(d-1)\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2}{1-\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2},$$

or, after simplification,

$$G(t) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}},$$

which could have been obtained by setting $u=0$ in $F(1-u,t)$.

In particular if $d=2$, then $\mathcal{X}=C_\infty=\mathbf{Z}$ and

$$G(t) = \sum_{n\geq 0}\binom{2n}{n}t^{2n} = \frac{1}{\sqrt{1-4t^2}}.$$

Note that for all d the d-regular tree $\mathcal{X}$ is the Cayley graph of $\Gamma = (\mathbf{Z}/2\mathbf{Z})^{*d}$ with standard generating set. If d is even, $\mathcal{X}$ is also the Cayley graph of a free group of rank $d/2$ generated by a free set. We have thus computed the spectral radius of a random walk on a freely generated free group: it is, for $(\mathbf{Z}/2\mathbf{Z})^{*d}$ or for $\mathbf{F}_{d/2}$, equal to

$$\frac{2\sqrt{d-1}}{d}. \tag{7.1}$$

Remark that for $d=2$ the series $F(u,t)$ does have a simple series expansion. By direct expansion, we obtain the number of circuits of length $2n$ in $\mathbf{Z}$, with m local extrema, as

$$(t^{2n}u^m \mid F(u,t)) = \begin{cases} 2\binom{n-1}{\frac{m-1}{2}}^2 & \text{if } m\equiv 1\ [2],\\ 2\binom{n-1}{\frac{m}{2}}\binom{n-1}{\frac{m-2}{2}} & \text{if } m\equiv 0\ [2].\end{cases}$$

We may even look for a richer generating series than F: let

$$H(u,v,t)=\sum_{\pi:\text{ path starting at }\star} u^{\mathrm{bc}(\pi)}v^{\delta(\star,\pi_{|\pi|})}t^{|\pi|}\in \mathbf{N}[u,v][[t]]\,,$$

where δ denotes the graph distance. Then

$$\begin{aligned}H(1,v,t)&=F(1,t)+dF'tvF+dF'tv(d-1)F'tvF+\dots\\&=\frac{1+F'(1,t)tv}{1-(d-1)F'(1,t)tv}F(1,t)\,;\end{aligned}$$

and as H is a sum of series counting paths between fixed vertices we obtain $H(u,v,t)$ from $H(1,v,t)$ by extending (2.2) linearly:

$$\frac{H(1-u,v,t)}{1-u^2t^2}=\frac{H\left(1,v,\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}\,.$$

We could also have started by computing

$$H(0,v,t)=\frac{1+vt}{1-(d-1)vt}\,,$$

the growth series of all proper paths in $\mathcal{T}$, and using (2.3) and (2.5) obtain

$$H(1,v,t)=\frac{1+\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2t}\right)^2}{1-u^2\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2}\cdot H\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t},0,v\right),$$

$$H(u,v,t)=\frac{1-t^2u^2}{1+u(d-u)t^2}\cdot\frac{(d-1)(4t^2+\Box^2)}{4(d-1)^2t^2-u^2\Box^2}\cdot\frac{2(d-1)t+v\Box}{2t-v\Box}\,,$$

where $\Box=1+u(d-u)t^2-\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}$.

Recall that the growth series of a graph $\mathcal{X}$ at a base point $\star$ is the power series

$$P(t)=\sum_{v\in V(\mathcal{X})}t^{\delta(\star,v)}\,,$$

where δ denotes the distance in $\mathcal{X}$. The series H is very general in that it contains a lot of information on $\mathcal{T}$, namely

- $H(u,0,t)=F(u,t)$;
- $H(0,1,t)=\frac{1+t}{1-(d-1)t}=P(t)$ is the growth series of $\mathcal{T}$;
- $H(1,1,t)=1/(1-dt)$ is the growth series of all paths in $\mathcal{T}$.

(Note that these substitutions yield well-defined series because for any i there are only finitely many monomials having t-degree equal to i.)

We can also use this series H to compute the circuit series F_C of the cycle of length k, that was found in the previous section. Indeed the universal cover of a cycle is the regular tree $\mathcal{T}$ of degree 2, and circuits in C correspond bijectively to paths in $\mathcal{T}$ from $\star$ to any vertex at distance a multiple of k. We thus have

$$F_C(u,t) = \sum_{\zeta:\zeta^k=1} H(u,\zeta,t)$$

where the sum runs over all kth roots of unity and $d=2$ in H.

We consider next the following graphs: take a d-regular tree and fix a vertex $\star$. At $\star$, delete e vertices and replace them by e loops. Then clearly

$$F(0,t) = \frac{1+t}{1-(e-1)t},$$

as all the non-backtracking paths are constrained to the e loops. Using (2.3), we obtain after simplifications

$$G(t) = \frac{2(d-1)}{d+e-2-2e(d-1)t+(d-e)\sqrt{1-4(d-1)t^2}}. \tag{7.2}$$

The radius of convergence of G is

$$\min\left\{\frac{1}{2\sqrt{d-1}}, \frac{e-1}{d+e^2-2e}\right\}.$$

7.4 Tougher examples

In this subsection we outline the computations of F and G for more complicated graphs. They are only provided as examples and are logically independent from the remainder of the paper. The arguments will therefore be somewhat condensed.

First take for $\mathcal{X}$ the Cayley graph of $\Gamma = (\mathbf{Z}/2\mathbf{Z}) \times \mathbf{Z}$ with generators $(0,-1) =$ '$\downarrow$', $(0,1) =$ '$\uparrow$' and $(1,0) =$ '$\leftrightarrow$'. Geometrically, $\mathcal{X}$ is a doubly-infinite two-poled ladder.

In Subsection 7.3 we computed

$$F_{\mathbf{Z}}(u,t) = \frac{1-(1-u)^2t^2}{\sqrt{(1+(1-u^2)t^2)^2-4t^2}},$$

the growth of circuits restricted to one pole of the ladder. A circuit in $\mathcal{X}$ is a circuit in $\mathbf{Z}$, before and after each step ($\uparrow$ or $\downarrow$) of which we may switch to the other pole (with a $\leftrightarrow$) as many times as we wish, subject to the condition that the circuit finish at the same pole as it started. This last condition is expressed by the fact that the series we obtain must have only coefficients of even degree in t. Thus, letting $\text{even}(f) = \frac{f(t)+f(-t)}{2}$, we have