

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COUNTING PATHS IN GRAPHS
Autor: Bartholdi, Laurent
Kapitel: 7.3 Trees
DOI: <https://doi.org/10.5169/seals-64442>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 18.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

The first few values of F , where \square stands for $1 + (1 - u^2)t^2$, are:

k	$F(u, t)$	k	$F(u, t)$
1	$\frac{1 + (1 - u)t}{1 - (1 + u)t}$	2	$\frac{\square}{1 - (1 + u)^2 t^2}$
3	$\frac{(1 + (1 - u)t)(\square - t)}{(1 - (1 + u)t)(\square + t)}$	4	$\frac{\square^2 - 2t^2}{1 - (1 + u)^2 t^2}$
5	$\frac{(1 + (1 - u)t)(\square^2 - \square t - t^2)}{(1 - (1 + u)t)(\square^2 + \square t + t^2)}$	6	$\frac{\square^2 - 3t^2}{(1 - (1 + u)^2 t^2)(\square^2 - t^2)}$
7	$\frac{(1 + (1 - u)t)(\square^3 - \square^2 t - 2\square t^2 + t^3)}{(1 - (1 + u)t)(\square^3 + \square^2 t - 2\square t^2 - t^3)}$	8	$\frac{\square^4 - 4\square^2 t^2 + 2t^4}{(1 - (1 + u)^2 t^2)(\square^2 - 2t^2)}$
9	$\frac{(1 + (1 - u)t)(\square - t)(\square^3 - 3\square t^2 - t^3)}{(1 - (1 + u)t)(\square + t)(\square^3 - 3\square t^2 + t^3)}$	10	$\frac{\square^4 - 5\square^2 t^2 + 5t^4}{(1 - (1 + u)^2 t^2)(\square^4 - 3\square^2 t^2 + t^4)}$

These rational expressions were computed and simplified using the computer algebra program *Maple*TM.

7.3 TREES

Let \mathcal{X} be the d -regular tree. Then

$$F(0, t) = 1$$

as a tree has no proper circuit; while direct (i.e., without using Corollary 2.6) computation of G is more complicated. It was first performed by Kesten [Kes59]; here we will derive the extended circuit series $F(u, t)$ and also obtain the answer using Corollary 2.6.

Let \mathcal{T} be a regular tree of degree d with a fixed root \star , and let \mathcal{T}' be the connected component of \star in the two-tree forest obtained by deleting in \mathcal{T} an edge at \star . Let $F(u, t)$ and $F'(u, t)$ respectively count circuits at \star in \mathcal{T} and \mathcal{T}' . For instance if $d = 2$ then F' counts circuits in \mathbf{N} and F counts circuits in \mathbf{Z} . For a reason that will become clear below, we make the convention that the empty circuit is counted as '1' in F and as 'u' in F' . Then we have

$$F' = u + (d - 1)tF't \frac{1}{1 - (d - 2 + u)tF't},$$

$$F = 1 + dtF't \frac{1}{1 - (d - 1 + u)tF't}.$$

Indeed a circuit in \mathcal{T}' is either the empty circuit (counted as u), or a sequence of circuits composed of, first, a step in any of $d - 1$ directions, then

a ‘subcircuit’ not returning to \star , then a step back to \star , followed by a step in any of $d - 1$ directions (counting an extra factor of u if it was the same as before), a subcircuit, etc. If the ‘subcircuit’ is the empty circuit, it contributes a bump, hence the convention on F' . Likewise, a circuit in \mathcal{T} is either the empty circuit (now counted as 1) or a sequence of circuits in subtrees each isomorphic to T' .

We solve these equations to

$$F'(1 - u, t) = \frac{2(1 - u)}{1 - u(d - u)t^2 + \sqrt{(1 + u(d - u)t^2)^2 - 4(d - 1)t^2}},$$

$$F(1 - u, t) = \frac{2(d - 1)(1 - u^2t^2)}{(d - 2)(1 + u(d - u)t^2) + d\sqrt{(1 + u(d - u)t^2)^2 - 4(d - 1)t^2}}.$$

Using (2.3) and $F(0, t) = 1$ we would obtain

$$G(t) = \frac{1 + (d - 1)\left(\frac{1 - \sqrt{1 - 4(d - 1)t^2}}{2(d - 1)t}\right)^2}{1 - \left(\frac{1 - \sqrt{1 - 4(d - 1)t^2}}{2(d - 1)t}\right)^2},$$

or, after simplification,

$$G(t) = \frac{2(d - 1)}{d - 2 + d\sqrt{1 - 4(d - 1)t^2}},$$

which could have been obtained by setting $u = 0$ in $F(1 - u, t)$.

In particular if $d = 2$, then $\mathcal{X} = C_\infty = \mathbf{Z}$ and

$$G(t) = \sum_{n \geq 0} \binom{2n}{n} t^{2n} = \frac{1}{\sqrt{1 - 4t^2}}.$$

Note that for all d the d -regular tree \mathcal{X} is the Cayley graph of $\Gamma = (\mathbf{Z}/2\mathbf{Z})^{*d}$ with standard generating set. If d is even, \mathcal{X} is also the Cayley graph of a free group of rank $d/2$ generated by a free set. We have thus computed the spectral radius of a random walk on a freely generated free group: it is, for $(\mathbf{Z}/2\mathbf{Z})^{*d}$ or for $\mathbf{F}_{d/2}$, equal to

$$(7.1) \quad \frac{2\sqrt{d - 1}}{d}.$$

Remark that for $d = 2$ the series $F(u, t)$ does have a simple series expansion. By direct expansion, we obtain the number of circuits of length $2n$ in \mathbf{Z} , with m local extrema, as

$$(t^{2n}u^m | F(u, t)) = \begin{cases} 2\binom{n-1}{\frac{m-1}{2}}^2 & \text{if } m \equiv 1 [2], \\ 2\binom{n-1}{\frac{m}{2}}\binom{n-1}{\frac{m-2}{2}} & \text{if } m \equiv 0 [2]. \end{cases}$$

We may even look for a richer generating series than F : let

$$H(u, v, t) = \sum_{\pi: \text{ path starting at } \star} u^{bc(\pi)} v^{\delta(\star, \pi|\pi|)} t^{|\pi|} \in \mathbf{N}[u, v][[t]],$$

where δ denotes the graph distance. Then

$$\begin{aligned} H(1, v, t) &= F(1, t) + dF'tvF + dF'tv(d-1)F'tvF + \dots \\ &= \frac{1 + F'(1, t)tv}{1 - (d-1)F'(1, t)tv} F(1, t); \end{aligned}$$

and as H is a sum of series counting paths between fixed vertices we obtain $H(u, v, t)$ from $H(1, v, t)$ by extending (2.2) linearly:

$$\frac{H(1-u, v, t)}{1-u^2t^2} = \frac{H\left(1, v, \frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}.$$

We could also have started by computing

$$H(0, v, t) = \frac{1+vt}{1-(d-1)vt},$$

the growth series of all proper paths in \mathcal{T} , and using (2.3) and (2.5) obtain

$$\begin{aligned} H(1, v, t) &= \frac{1 + \left(\frac{1 - \sqrt{1-4(d-1)t^2}}{2t}\right)^2}{1 - u^2\left(\frac{1 - \sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2} \cdot H\left(\frac{1 - \sqrt{1-4(d-1)t^2}}{2(d-1)t}, 0, v\right), \\ H(u, v, t) &= \frac{1 - t^2u^2}{1 + u(d-u)t^2} \cdot \frac{(d-1)(4t^2 + \square^2)}{4(d-1)^2t^2 - u^2\square^2} \cdot \frac{2(d-1)t + v\square}{2t - v\square}, \end{aligned}$$

where $\square = 1 + u(d-u)t^2 - \sqrt{(1 + u(d-u)t^2)^2 - 4(d-1)t^2}$.

Recall that the growth series of a graph \mathcal{X} at a base point \star is the power series

$$P(t) = \sum_{v \in V(\mathcal{X})} t^{\delta(\star, v)},$$

where δ denotes the distance in \mathcal{X} . The series H is very general in that it contains a lot of information on \mathcal{T} , namely

- $H(u, 0, t) = F(u, t)$;
- $H(0, 1, t) = \frac{1+t}{1-(d-1)t} = P(t)$ is the growth series of \mathcal{T} ;
- $H(1, 1, t) = 1/(1-dt)$ is the growth series of all paths in \mathcal{T} .

(Note that these substitutions yield well-defined series because for any i there are only finitely many monomials having t -degree equal to i .)

We can also use this series H to compute the circuit series F_C of the cycle of length k , that was found in the previous section. Indeed the universal cover

of a cycle is the regular tree \mathcal{T} of degree 2, and circuits in C correspond bijectively to paths in \mathcal{T} from \star to any vertex at distance a multiple of k . We thus have

$$F_C(u, t) = \sum_{\zeta: \zeta^k=1} H(u, \zeta, t)$$

where the sum runs over all k th roots of unity and $d = 2$ in H .

We consider next the following graphs: take a d -regular tree and fix a vertex \star . At \star , delete e vertices and replace them by e loops. Then clearly

$$F(0, t) = \frac{1 + t}{1 - (e - 1)t},$$

as all the non-backtracking paths are constrained to the e loops. Using (2.3), we obtain after simplifications

$$(7.2) \quad G(t) = \frac{2(d - 1)}{d + e - 2 - 2e(d - 1)t + (d - e)\sqrt{1 - 4(d - 1)t^2}}.$$

The radius of convergence of G is

$$\min\left\{\frac{1}{2\sqrt{d - 1}}, \frac{e - 1}{d + e^2 - 2e}\right\}.$$

7.4 TOUGHER EXAMPLES

In this subsection we outline the computations of F and G for more complicated graphs. They are only provided as examples and are logically independent from the remainder of the paper. The arguments will therefore be somewhat condensed.

First take for \mathcal{X} the Cayley graph of $\Gamma = (\mathbf{Z}/2\mathbf{Z}) \times \mathbf{Z}$ with generators $(0, -1) = \text{‘}\downarrow\text{’}$, $(0, 1) = \text{‘}\uparrow\text{’}$ and $(1, 0) = \text{‘}\leftrightarrow\text{’}$. Geometrically, \mathcal{X} is a doubly-infinite two-poled ladder.

In Subsection 7.3 we computed

$$F_{\mathbf{Z}}(u, t) = \frac{1 - (1 - u)^2 t^2}{\sqrt{(1 + (1 - u^2)t^2)^2 - 4t^2}},$$

the growth of circuits restricted to one pole of the ladder. A circuit in \mathcal{X} is a circuit in \mathbf{Z} , before and after each step (\uparrow or \downarrow) of which we may switch to the other pole (with a \leftrightarrow) as many times as we wish, subject to the condition that the circuit finish at the same pole as it started. This last condition is expressed by the fact that the series we obtain must have only coefficients of even degree in t . Thus, letting $\text{even}(f) = \frac{f(t) + f(-t)}{2}$, we have