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Solving the system, we obtain

$$F(u, t) = \frac{1 + (1 - u)t}{1 - (v - 2 + u)t} \cdot \frac{1 - (v - 2)t + (1 - u)(v - 2 + u)t^2}{1 + t + (1 - u)(v - 2 + u)t^2}.$$

We then compute

$$G(t) = F(1, t) = \frac{1 - (v - 2)t}{(1 + t)(1 - (v - 1)t)},$$

$$F(0, t) = \frac{(1 + t)(1 - (v - 2)t + (v - 2)t^2)}{(1 - (v - 2)t)(1 + t + (v - 2)t^2)}.$$

7.2 CYCLES

Let $\mathcal{X} = C_k$, the cycle on k vertices. Here, as there are 2 proper circuits of length n for all n multiple of k (except 0), we have

$$F(0, t) = \frac{1 + t^k}{1 - t^k}.$$

Obtaining a closed form for G is much harder. The number of circuits of length n is

$$g_n = \sum_{i \in \mathbf{Z} : i \equiv 0 [k], i \equiv n [2]} \binom{n}{\frac{n+i}{2}},$$

from which, by [Gou72, 1.54], it follows that

$$G(t) = \frac{1}{k} \sum_{\zeta^k=1} \frac{1}{1 - (\zeta + \zeta^{-1})t} = \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{1 - 2 \cos\left(\frac{2\pi j}{k}\right)t}.$$

It is not at all obvious how to simplify the above expression. A closed-form answer can be obtained from (2.3), namely

$$G(t) = \frac{(2t)^2 + (1 - \sqrt{1 - 4t^2})^2}{(2t)^2 - (1 - \sqrt{1 - 4t^2})^2} \cdot \frac{(2t)^k + (1 - \sqrt{1 - 4t^2})^k}{(2t)^k - (1 - \sqrt{1 - 4t^2})^k},$$

or, expanding,

$$G(t) = \frac{(2t)^k + \sum_{m=0}^{k/2} (1 - 4t^2)^m \binom{k}{2m}}{\sum_{m=1}^{(k+1)/2} (1 - 4t^2)^m \binom{k}{2m-1}}.$$

However in general this fraction is not reduced. To obtain reduced fractions for $F(u, t)$ (and thus for $G(t)$), we have to consider separately the cases where k is odd or even.

For odd k , letting $k = 2\ell + 1$, we obtain

$$F(u, t) = \frac{1 + (1 - u)t}{1 - (1 + u)t} \cdot \frac{\sum_{m=0}^{\ell} \alpha_m^{\ell} (-t)^m (1 + (1 - u^2)t^2)^{\ell - m}}{\sum_{m=0}^{\ell} \alpha_m^{\ell} t^m (1 + (1 - u^2)t^2)^{\ell - m}},$$

$$G(t) = \frac{\sum_{m=0}^{\ell} \alpha_m^{\ell} (-t)^m}{(1 - 2t) \left(\sum_{m=0}^{\ell} \alpha_m^{\ell} t^m \right)},$$

where

$$\alpha_m^{\ell} = \begin{cases} (-1)^{\frac{m}{2}} \binom{\ell - \frac{m}{2}}{\frac{m}{2}} & \text{if } m \equiv 0 [2], \\ (-1)^{\frac{m-1}{2}} \binom{\ell - \frac{m+1}{2}}{\frac{m-1}{2}} & \text{if } m \equiv 1 [2]. \end{cases}$$

For even k , with $k = 2\ell$,

$$F(u, t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell - m} \binom{\ell - m}{m} (-t^2)^m (1 - (1 - u^2)t^2)^{\ell - 2m}}{(1 - (1 + u)^2 t^2) \left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell - 1 - m}{m} (-t^2)^m (1 - (1 - u^2)t^2)^{\ell - 1 - 2m} \right)},$$

$$G(t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell - m} \binom{\ell - m}{m} (-t^2)^m}{(1 - 4t^2) \left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell - 1 - m}{m} (-t^2)^m \right)},$$

expressed as reduced fractions.

The first few values of F , where \square stands for $1 + (1 - u^2)t^2$, are:

k	$F(u, t)$	k	$F(u, t)$
1	$\frac{1 + (1 - u)t}{1 - (1 + u)t}$	2	$\frac{\square}{1 - (1 + u)^2 t^2}$
3	$\frac{(1 + (1 - u)t)(\square - t)}{(1 - (1 + u)t)(\square + t)}$	4	$\frac{\square^2 - 2t^2}{1 - (1 + u)^2 t^2}$
5	$\frac{(1 + (1 - u)t)(\square^2 - \square t - t^2)}{(1 - (1 + u)t)(\square^2 + \square t + t^2)}$	6	$\frac{\square^2 - 3t^2}{(1 - (1 + u)^2 t^2)(\square^2 - t^2)}$
7	$\frac{(1 + (1 - u)t)(\square^3 - \square^2 t - 2\square t^2 + t^3)}{(1 - (1 + u)t)(\square^3 + \square^2 t - 2\square t^2 - t^3)}$	8	$\frac{\square^4 - 4\square^2 t^2 + 2t^4}{(1 - (1 + u)^2 t^2)(\square^2 - 2t^2)}$
9	$\frac{(1 + (1 - u)t)(\square - t)(\square^3 - 3\square t^2 - t^3)}{(1 - (1 + u)t)(\square + t)(\square^3 - 3\square t^2 + t^3)}$	10	$\frac{\square^4 - 5\square^2 t^2 + 5t^4}{(1 - (1 + u)^2 t^2)(\square^4 - 3\square^2 t^2 + t^4)}$

These rational expressions were computed and simplified using the computer algebra program *Maple*TM.

7.3 TREES

Let \mathcal{X} be the d -regular tree. Then

$$F(0, t) = 1$$

as a tree has no proper circuit; while direct (i.e., without using Corollary 2.6) computation of G is more complicated. It was first performed by Kesten [Kes59]; here we will derive the extended circuit series $F(u, t)$ and also obtain the answer using Corollary 2.6.

Let \mathcal{T} be a regular tree of degree d with a fixed root \star , and let \mathcal{T}' be the connected component of \star in the two-tree forest obtained by deleting in \mathcal{T} an edge at \star . Let $F(u, t)$ and $F'(u, t)$ respectively count circuits at \star in \mathcal{T} and \mathcal{T}' . For instance if $d = 2$ then F' counts circuits in \mathbf{N} and F counts circuits in \mathbf{Z} . For a reason that will become clear below, we make the convention that the empty circuit is counted as '1' in F and as 'u' in F' . Then we have

$$F' = u + (d - 1)tF't \frac{1}{1 - (d - 2 + u)tF't},$$

$$F = 1 + dtF't \frac{1}{1 - (d - 1 + u)tF't}.$$

Indeed a circuit in \mathcal{T}' is either the empty circuit (counted as u), or a sequence of circuits composed of, first, a step in any of $d - 1$ directions, then