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7. EXAMPLES

We give here examples of regular graphs and when possible compute independently the series F and G . In some cases it will be easier to compute F , while in others it will be simpler to compute G first. In all cases, once one of F and G has been computed, the other one can be obtained using Corollary 2.6.

In all the examples the graphs are vertex transitive, so the choice of \star is unimportant. To simplify the computations we choose $\dagger = \star$ and the length labelling.

7.1 COMPLETE GRAPHS

Let $\mathcal{X} = K_v$, the complete graph on $v \geq 3$ vertices. Its degree is $d = v - 1$. To compute F and G , choose three distinct vertices $\star, \$, \#$ (the choice is unimportant as K_v is three-transitive). Define growth series

$F(u, t)$ the growth series of circuits based at \star ;

$F'(u, t)$ the growth series of paths π from $\$$ to \star with $\pi_1^\omega = \#$;

$F''(u, t)$ the growth series of paths π from $\$$ to \star with $\pi_1^\omega = \star$.

Then

$$\begin{aligned} F &= 1 + (v - 1)t \left[(v - 2)F' + uF'' \right], \\ F' &= t \left[F'' + (v - 3 + u)F' \right], \\ F'' &= t \left[1 + (F - 1) \frac{v - 2 + u}{v - 1} \right]. \end{aligned}$$

Indeed the first line states that a circuit at \star is either the trivial circuit at \star , or a choice of one of $v - 1$ edges to another point (call it $\$$), followed by a path from $\$$ to \star ; this path can first go to any vertex of the $v - 2$ vertices (say $\#$) different from \star and $\$$, and thus contribute F' , or go back to \star and contribute F'' and a bump.

The second equation says that a path from $\$$ to \star starting by going to $\#$ can either continue to \star , contributing F' , go to any of the $v - 3$ other vertices contributing F' , or come back to $\$$, contributing F'' and a bump.

The third line says that a path from $\$$ to \star starting by going to \star continues as a circuit at \star ; but if the circuit is non-trivial, then one out of $v - 1$ times a bump will be contributed.

Solving the system, we obtain

$$F(u, t) = \frac{1 + (1 - u)t}{1 - (v - 2 + u)t} \cdot \frac{1 - (v - 2)t + (1 - u)(v - 2 + u)t^2}{1 + t + (1 - u)(v - 2 + u)t^2}.$$

We then compute

$$G(t) = F(1, t) = \frac{1 - (v - 2)t}{(1 + t)(1 - (v - 1)t)},$$

$$F(0, t) = \frac{(1 + t)(1 - (v - 2)t + (v - 2)t^2)}{(1 - (v - 2)t)(1 + t + (v - 2)t^2)}.$$

7.2 CYCLES

Let $\mathcal{X} = C_k$, the cycle on k vertices. Here, as there are 2 proper circuits of length n for all n multiple of k (except 0), we have

$$F(0, t) = \frac{1 + t^k}{1 - t^k}.$$

Obtaining a closed form for G is much harder. The number of circuits of length n is

$$g_n = \sum_{i \in \mathbf{Z} : i \equiv 0 [k], i \equiv n [2]} \binom{n}{\frac{n+i}{2}},$$

from which, by [Gou72, 1.54], it follows that

$$G(t) = \frac{1}{k} \sum_{\zeta^k=1} \frac{1}{1 - (\zeta + \zeta^{-1})t} = \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{1 - 2 \cos\left(\frac{2\pi j}{k}\right)t}.$$

It is not at all obvious how to simplify the above expression. A closed-form answer can be obtained from (2.3), namely

$$G(t) = \frac{(2t)^2 + (1 - \sqrt{1 - 4t^2})^2}{(2t)^2 - (1 - \sqrt{1 - 4t^2})^2} \cdot \frac{(2t)^k + (1 - \sqrt{1 - 4t^2})^k}{(2t)^k - (1 - \sqrt{1 - 4t^2})^k},$$

or, expanding,

$$G(t) = \frac{(2t)^k + \sum_{m=0}^{k/2} (1 - 4t^2)^m \binom{k}{2m}}{\sum_{m=1}^{(k+1)/2} (1 - 4t^2)^m \binom{k}{2m-1}}.$$

However in general this fraction is not reduced. To obtain reduced fractions for $F(u, t)$ (and thus for $G(t)$), we have to consider separately the cases where k is odd or even.