Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COUNTING PATHS IN GRAPHS
Autor: Bartholdi, Laurent
Kapitel: 7.1 COMPLETE GRAPHS
DOI: https://doi.org/10.5169/seals-64442

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Siehe Rechtliche Hinweise.

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. Voir Informations légales.

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. See Legal notice.

Download PDF: 18.05.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

7. EXAMPLES

We give here examples of regular graphs and when possible compute independently the series F and G. In some cases it will be easier to compute F, while in others it will be simpler to compute G first. In all cases, once one of F and G has been computed, the other one can be obtained using Corollary 2.6.

In all the examples the graphs are vertex transitive, so the choice of $\star$ is unimportant. To simplify the computations we choose $\dagger = \star$ and the length labelling.

7.1 COMPLETE GRAPHS

Let $\mathcal{X} = K_v$, the complete graph on $v \geq 3$ vertices. Its degree is $d = v-1$. To compute F and G, choose three distinct vertices $\star, \$, \#$ (the choice is unimportant as K_v is three-transitive). Define growth series

$F(u,t)$ the growth series of circuits based at $\star$;
$F'(u,t)$ the growth series of paths π from \$ to $\star$ with $\pi_1^\omega = \#$;
$F''(u,t)$ the growth series of paths π from \$ to $\star$ with $\pi_1^\omega = \star$.

Then

$$\begin{aligned} F &= 1 + (v-1)t\Big[(v-2)F' + uF''\Big], \\ F' &= t\Big[F'' + (v-3+u)F'\Big], \\ F'' &= t\Big[1 + (F-1)\frac{v-2+u}{v-1}\Big]. \end{aligned}$$

Indeed the first line states that a circuit at $\star$ is either the trivial circuit at $\star$, or a choice of one of $v-1$ edges to another point (call it \$), followed by a path from \$ to $\star$; this path can first go to any vertex of the $v-2$ vertices (say (#)) different from $\star$ and \$, and thus contribute F', or go back to $\star$ and contribute F'' and a bump.

The second equation says that a path from \$ to $\star$ starting by going to # can either continue to $\star$, contributing F', go to any of the $v-3$ other vertices contributing F', or come back to \$, contributing F' and a bump.

The third line says that a path from \$ to $\star$ starting by going to $\star$ continues as a circuit at $\star$; but if the circuit is non-trivial, then one out of $v-1$ times a bump will be contributed.

Solving the system, we obtain

$$F(u,t)=\frac{1+(1-u)t}{1-(v-2+u)t}\cdot\frac{1-(v-2)t+(1-u)(v-2+u)t^2}{1+t+(1-u)(v-2+u)t^2}.$$

We then compute

$$G(t)=F(1,t)=\frac{1-(v-2)t}{(1+t)(1-(v-1)t)},$$

$$F(0,t)=\frac{(1+t)(1-(v-2)t+(v-2)t^2)}{(1-(v-2)t)(1+t+(v-2)t^2)}.$$

7.2 Cycles

Let $\mathcal{X}=C_k$, the cycle on k vertices. Here, as there are 2 proper circuits of length n for all n multiple of k (except 0), we have

$$F(0,t)=\frac{1+t^k}{1-t^k}.$$

Obtaining a closed form for G is much harder. The number of circuits of length n is

$$g_n=\sum_{i\in\mathbf{Z}:\,i\equiv 0\ [k],\ i\equiv n\ [2]}\binom{n}{\frac{n+i}{2}},$$

from which, by [Gou72, 1.54], it follows that

$$G(t)=\frac{1}{k}\sum_{\zeta^k=1}\frac{1}{1-\left(\zeta+\zeta^{-1}\right)t}=\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{1-2\cos\left(\frac{2\pi j}{k}\right)t}.$$

It is not at all obvious how to simplify the above expression. A closed-form answer can be obtained from (2.3), namely

$$G(t)=\frac{(2t)^2+\left(1-\sqrt{1-4t^2}\right)^2}{(2t)^2-\left(1-\sqrt{1-4t^2}\right)^2}\cdot\frac{(2t)^k+\left(1-\sqrt{1-4t^2}\right)^k}{(2t)^k-\left(1-\sqrt{1-4t^2}\right)^k},$$

or, expanding,

$$G(t)=\frac{(2t)^k+\sum_{m=0}^{k/2}\left(1-4t^2\right)^m\binom{k}{2m}}{\sum_{m=1}^{(k+1)/2}\left(1-4t^2\right)^m\binom{k}{2m-1}}.$$

However in general this fraction is not reduced. To obtain reduced fractions for $F(u,t)$ (and thus for $G(t)$), we have to consider separately the cases where k is odd or even.