Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 45 (1999)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COUNTING PATHS IN GRAPHS

Autor: Bartholdi, Laurent

Kapitel: 5. Graphs and matrices

DOI: https://doi.org/10.5169/seals-64442

Similarly, letting $\mathfrak{F}_{x,e,y}$ count the paths from x to y that start with the edge e,

$$\begin{aligned}\mathfrak{F}_{x,y} &= \delta_{x,y} + \sum_{e\in E(\mathcal{X}):e^\alpha=x} \mathfrak{F}_{x,e,y}\,,\\ \mathfrak{F}_{x,e,y} &= e^\ell\left(\mathfrak{F}_{e^\omega,y} + (u-1)\mathfrak{F}_{e^\omega,\overline{e},y}\right)\,,\\ \mathfrak{F}_{e^\omega,\overline{e},y} &= \overline{e}^\ell\left(\mathfrak{F}_{x,y} + (u-1)\mathfrak{F}_{x,e,y}\right)\,;\end{aligned}$$

these last two lines solve to

$$\mathfrak{F}_{x,e,y} = \left(1-(u-1)^2(e\overline{e})^\ell\right)^{-1}\left(e^\ell\mathfrak{F}_{e^\omega,y} + (u-1)(e\overline{e})^\ell\mathfrak{F}_{x,y}\right)\,,$$

which we insert in the first line to obtain

$$K_x^{-1}\mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e\in E(\mathcal{X}):e^\alpha=x} \frac{e^\ell}{1-(u-1)^2(e\,\overline{e})^\ell}K_{e^\omega}\cdot K_{e^\omega}^{-1}\mathfrak{F}_{e^\omega,y}\,.$$

Thus if we let

$$e^{\ell'} = \frac{e^\ell}{1-(u-1)^2(e\,\overline{e})^\ell}K_{e^\omega}\,,\qquad A' = \sum_{e\in E(\mathcal{X})}[e^{\ell'}]_{e^\alpha}^{e^\omega}\,, \tag{4.1}$$

we obtain

$$(K_x^{-1}\mathfrak{F}_{x,y})_{x,y\in V(\mathcal{X})} = \frac{1}{1-A'} \tag{4.2}$$

and the proof is finished in the case that $\mathcal{X}$ is finite, because the matrix A' is precisely that obtained from A by substituting ℓ' for ℓ.

If $\mathcal{X}$ has infinitely many vertices, we approximate it, using Lemma 3.7, by finite graphs. Denote by $\mathfrak{F}^n_{\star,\dagger}(\ell)$ and $\mathfrak{G}^n_{\star,\dagger}(\ell')$ the enriched path series and path series respectively in $\mathcal{B}(\star,n)$, and write

$$K_\star\cdot\mathfrak{F}(\ell) = \lim_{n\to\infty}\mathfrak{F}^n_{\star,\dagger}(\ell) = \lim_{n\to\infty}\mathfrak{G}^n_{\star,\dagger}(\ell') = \mathfrak{G}(\ell')$$

to complete the proof.

5. Graphs and matrices

Graphs can be studied through their *adjacency* and *incidence* matrices. We give here the relevant definitions and obtain an extension of a theorem by Hyman Bass [Bas92] on the Ihara-Selberg zeta function. We will use power series with coefficients in a matrix ring, and fractional expressions in matrices; by convention, we understand 'X/Y' as '$X\cdot Y^{-1}$'.

DEFINITION 5.1. Let $\mathcal{X}$ be a finite graph. The *edge-adjacency* and *inversion* matrices of $\mathcal{X}$, respectively B and J, are $E(\mathcal{X}) \times E(\mathcal{X})$ matrices over $\mathbf{Z}$ defined by

$$B_{e,f} = \begin{cases} 1 & \text{if } e^\omega = f^\alpha \\ 0 & \text{else,} \end{cases} \qquad J_{e,f} = \begin{cases} 1 & \text{if } \overline{e} = f \\ 0 & \text{else.} \end{cases}$$

The *vertex-adjacency* and *degree* matrices of $\mathcal{X}$, respectively A and D, are $V(\mathcal{X}) \times V(\mathcal{X})$ matrices over $\mathbf{Z}$ defined by

$$A_{v,w} = \left|\{e \in E(\mathcal{X}) \mid e^\alpha = v \text{ and } e^\omega = w\}\right|, \quad D_{v,w} = \begin{cases} \deg(v) & \text{if } v = w, \\ 0 & \text{else.} \end{cases}$$

A *cycle* is the equivalence class of a circuit under cyclic permutation of its edges. A *proper cycle* is a cycle all of whose representatives are proper circuits. A cycle is *primitive* if none of its representatives can be written as π^k for some $k \geq 2$. The *cyclic bump count* $\operatorname{cbc}(\pi)$ of a circuit $\pi = (\pi_1, \ldots, \pi_n)$ is

$$\operatorname{cbc}(\pi) = \left|\{i = 1, \ldots, n \mid \pi_i = \overline{\pi_{i+1}}\}\right| ,$$

where the edge π_{n+1} is understood to be π_1.

The matrices given above are related to paths in $\mathcal{X}$ as follows: Consider first the matrix

$$M = \mathbf{1} - (B - (1 - u)J)t .$$

Then the (e,f) coefficient of M^{-1} is precisely

$$\sum_{\pi : \pi_1 = e, \pi^\omega = f^\alpha} u^{\operatorname{bc}(\pi f)} t^{|\pi|} .$$

This is clear because the series expansion of M^{-1} is the sum of sequences of $(B - J)t$ (contributing edges with no bump) and Jut (contributing edges with bumps), with an extra factor of u in case the path ends in $\overline{f}$. As a consequence,

LEMMA 5.2. *Let*

$$X_E = \frac{\mathbf{1} + (1 - u)Jt - M}{Mt} = \frac{B}{\mathbf{1} - (B - (1 - u)J)t} .$$

Then the (e,f) coefficient of X_E counts the non-trivial paths starting with e and ending at f^α, with t-weight shifted one down:

$$(X_E)_{e,f} = \sum_{\pi : \pi_1 = e,\ \pi^\omega = f^\alpha} u^{\operatorname{bc}(\pi)} t^{|\pi| - 1} .$$

Likewise, consider the matrix

$$P = \mathbf{1} - At + (1-u)(D - (1-u)\mathbf{1})t^2 .$$

The following lemma will be a consequence of the computations in the next section.

LEMMA 5.3. *Let*

$$X_V = \frac{(1-(1-u)^2t^2)\mathbf{1} - P}{Pt} = \frac{A - (1-u)Dt}{\mathbf{1} - At + (1-u)(D-(1-u)\mathbf{1})t^2} .$$

Then the (v, w) *coefficient of* X_V *counts the non-trivial paths starting at* v *and ending at* w, *with* t*-weight shifted one down:*

$$(X_V)_{v,w} = \sum_{\pi\,:\,\pi^\alpha = v,\ \pi^\omega = w} u^{\mathrm{bc}(\pi)} t^{|\pi|-1} .$$

Proof. We will show the matrix $\mathbf{1} + X_V t$ has as (v, w) coefficient the enriched path series from v to w. By simple calculation

$$\mathbf{1} + X_V t = \frac{\mathbf{1} - (1-u)^2t^2}{\mathbf{1} - At + (1-u)(D-(1-u)\mathbf{1})t^2} = \frac{K^{-1}}{\mathbf{1} - A'} ,$$

where K and A' are given by

$$K = \frac{\mathbf{1} + (1-u)(D - 1 + u)t^2}{1 - (1-u)^2t^2} , \qquad A' = \frac{AKt}{1-(1-u)^2t^2} .$$

K is a diagonal matrix and the coefficient $K_{x,x}$ is precisely K_x for the length labelling, while the matrix A' is the matrix of (4.1) in the previous section. The result then follows from Equation (4.2). □

In particular, the two matrices X_E and X_V have the same trace, as this trace counts all the non-trivial circuits π in $\mathcal{X}$, with weight $u^{\mathrm{bc}(\pi)}t^{|\pi|-1}$.

We now state and prove an extension of a theorem by Bass [Bas92, FZ98, Nor96]:

THEOREM 5.4. *Let* C *be a set of representatives of primitive cycles in* $\mathcal{X}$, *and form the zeta function of* $\mathcal{X}$

$$\zeta(u, t) = \prod_{\gamma \in C} \frac{1}{1 - u^{\mathrm{cbc}(\gamma)}t^{|\gamma|}} .$$

(The choice of representatives does not change the zeta function.) Then ζ^{-1} *is a polynomial in* u *and* t *and can be expressed as*

$$\frac{1}{\zeta(u,t)} = \det M \tag{5.1}$$

$$= (1+(1-u)t)^n(1-(1-u)^2t^2)^{m-|V(\mathcal{X})|}\det P\,, \tag{5.2}$$

where

$$n = |\{e \in E(\mathcal{X}) \mid e = \bar{e}\}|\,, \qquad 2m = |\{e \in E(\mathcal{X}) \mid e \neq \bar{e}\}|\,.$$

The special case $u = n = 0$ of this result was stated and proved in the given sources. We will prove the general statement, using a result of Shimson Amitsur:

THEOREM 5.5 (Amitsur [Ami80,RS87]). *Let* $X_1, \dots, X_k$ *be square matrices of the same dimension over an arbitrary ring. Let* S *contain one representative up to cyclic permutation of words over the alphabet* $\{1, \dots, k\}$ *that are primitive, i.e. such that none of their cyclic permutations are proper powers of a word (*S *is infinite as soon as* $k > 1$*). For* $p = i_1 \dots i_n \in S$ *set* $X_p = X_{i_1} \dots X_{i_n}$*. Then*

$$\det(\mathbf{1} - (X_1 + \cdots + X_k)t) = \prod_{p \in S} \det(\mathbf{1} - X_p t^{|p|})\,,$$

considered as an equality of power series in t *over the matrix ring.*

The equality (5.1) then follows; indeed, for all edges $e \in E(\mathcal{X})$ let X_e be the $E(\mathcal{X}) \times E(\mathcal{X})$ matrix whose e-th row is the e-th row of $B - (1-u)J$, and whose other rows are 0. Then clearly $\mathbf{1} - \sum_{e \in E(\mathcal{X})} X_e t = M$ and, for any sequence of edges π,

$$\det(\mathbf{1} - X_\pi t^{|\pi|}) = \begin{cases} 1 - u^{\mathrm{cbc}(\pi)} t^{|\pi|} & \text{if } \pi \text{ is a circuit,} \\ 1 & \text{else,} \end{cases}$$

so equality of $\zeta(u,t)$ and $\det M$ follows from Amitsur's Theorem.

To prove (5.2), we use the following result, whose proof relies on Newton's formulas relating the trace of powers of X and the characteristic polynomial of X:

PROPOSITION 5.6 ([Ami80, Equation 4.4)]. *Let X be a power series in t over a matrix ring, such that $X(0) = \mathbf{1}$. Then*

$$\det X = \exp\Big(-\int \operatorname{tr}\Big(\frac{\mathbf{1} - X}{Xt}\Big)dt\Big) ,$$

where the integration is the formal linear operation on power series that maps t^r to $t^{r+1}/(r+1)$.

We then have, using Lemmas 5.2 and 5.3,

$$\begin{aligned}
\frac{\det M}{(1+(1-u)t)^n(1-(1-u)^2t^2)^m} &= \det \frac{M}{\mathbf{1}+(1-u)Jt} \\
&= \exp\Big(-\int \operatorname{tr} \frac{\mathbf{1}+(1-u)Jt - M}{Mt} dt\Big) \\
&= \exp\Big(-\int \begin{matrix}\text{series counting non-trivial circuits,} \\ \text{length shifted down by one}\end{matrix} \; dt\Big) \\
&= \exp\Big(-\int \operatorname{tr} \frac{(1-(1-u)^2t^2)\mathbf{1} - P}{Pt} dt\Big) \\
&= \det \frac{P}{1-(1-u)^2t^2} = \frac{\det P}{(1-(1-u)^2t^2)^{|V(\mathcal{X})|}} .
\end{aligned}$$

6. SECOND PROOF OF THEOREM 2.4

Let $P = [\star, \dagger]$ be the set of paths in $\mathcal{X}$ from $\star$ to $\dagger$. As we shall apply the principle of inclusion-exclusion [Wil90], it will be helpful to compute in $\Pi = \mathbf{Z}[[P]]$, the $\mathbf{Z}$-module of functions from the set of paths to $\mathbf{Z}$. We embed subsets of P in Π by mapping a subset to its characteristic function:

$$P \supset A \mapsto \chi_A , \quad \text{with } (\pi)\chi_A = \begin{cases} 1 & \text{if } \pi \in A , \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{B}$ be the subset of bounded non-negative elements of Π (i.e. functions f such that there is a constant N with $0 \le (\pi)f < N$ for all paths π). If ℓ is a complete labelling of $\mathcal{X}$, there is an induced labelling $\ell_* \colon \mathcal{B} \to \mathbf{k}$ given by

$$(f)\ell_* = \sum_{\pi \in P} (\pi) f \pi^\ell .$$

Note that the sum, although infinite, defines an element of $\mathbf{k}$ due to the fact that ℓ is complete.