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Similarly, letting $\mathfrak{F}_{x,e,y}$ count the paths from x to y that start with the edge e,

$$\mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}): e^{\alpha} = x} \mathfrak{F}_{x,e,y},$$

$$\mathfrak{F}_{x,e,y} = e^{\ell} \left(\mathfrak{F}_{e^{\omega},y} + (u-1)\mathfrak{F}_{e^{\omega},\overline{e},y} \right),$$

$$\mathfrak{F}_{e^{\omega},\overline{e},y} = \overline{e}^{\ell} \left(\mathfrak{F}_{x,y} + (u-1)\mathfrak{F}_{x,e,y} \right);$$

these last two lines solve to

$$\mathfrak{F}_{x,e,y} = \left(1 - (u-1)^2 (e\overline{e})^\ell\right)^{-1} \left(e^\ell \mathfrak{F}_{e^\omega,y} + (u-1)(e\overline{e})^\ell \mathfrak{F}_{x,y}\right) ,$$

which we insert in the first line to obtain

$$K_x^{-1}\mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}): e^{\alpha} = x} \frac{e^{\ell}}{1 - (u - 1)^2 (e \,\overline{e})^{\ell}} K_{e^{\omega}} \cdot K_{e^{\omega}}^{-1} \mathfrak{F}_{e^{\omega},y} .$$

Thus if we let

(4.1)
$$e^{\ell'} = \frac{e^{\ell}}{1 - (u - 1)^2 (e \,\overline{e})^{\ell}} K_{e^{\omega}}, \qquad A' = \sum_{e \in E(\mathcal{X})} [e^{\ell'}]_{e^{\alpha}}^{e^{\omega}},$$

we obtain

(4.2)
$$(K_x^{-1} \mathfrak{F}_{x,y})_{x,y \in V(\mathcal{X})} = \frac{1}{1 - A'}$$

and the proof is finished in the case that \mathcal{X} is finite, because the matrix A' is precisely that obtained from A by substituting ℓ' for ℓ .

If \mathcal{X} has infinitely many vertices, we approximate it, using Lemma 3.7, by finite graphs. Denote by $\mathfrak{F}^n_{\star,\dagger}(\ell)$ and $\mathfrak{G}^n_{\star,\dagger}(\ell')$ the enriched path series and path series respectively in $\mathcal{B}(\star,n)$, and write

$$K_{\star} \cdot \mathfrak{F}(\ell) = \lim_{n \to \infty} \mathfrak{F}_{\star,\dagger}^n(\ell) = \lim_{n \to \infty} \mathfrak{G}_{\star,\dagger}^n(\ell') = \mathfrak{G}(\ell')$$

to complete the proof.

5. Graphs and matrices

Graphs can be studied through their *adjacency* and *incidence* matrices. We give here the relevant definitions and obtain an extension of a theorem by Hyman Bass [Bas92] on the Ihara-Selberg zeta function. We will use power series with coefficients in a matrix ring, and fractional expressions in matrices; by convention, we understand (X/Y) as $(X \cdot Y^{-1})$.

DEFINITION 5.1. Let \mathcal{X} be a finite graph. The *edge-adjacency* and *inversion* matrices of \mathcal{X} , respectively B and J, are $E(\mathcal{X}) \times E(\mathcal{X})$ matrices over \mathbf{Z} defined by

$$B_{e,f} = \left\{ egin{array}{ll} 1 & ext{if } e^{\omega} = f^{lpha} \\ 0 & ext{else}, \end{array}
ight. \quad J_{e,f} = \left\{ egin{array}{ll} 1 & ext{if } \overline{e} = f \\ 0 & ext{else}. \end{array}
ight.$$

The *vertex-adjacency* and *degree* matrices of \mathcal{X} , respectively A and D, are $V(\mathcal{X}) \times V(\mathcal{X})$ matrices over \mathbf{Z} defined by

$$A_{v,w} = |\{e \in E(\mathcal{X}) \mid e^{\alpha} = v \text{ and } e^{\omega} = w\}|, \quad D_{v,w} = \begin{cases} \deg(v) & \text{if } v = w, \\ 0 & \text{else.} \end{cases}$$

A *cycle* is the equivalence class of a circuit under cyclic permutation of its edges. A *proper cycle* is a cycle all of whose representatives are proper circuits. A cycle is *primitive* if none of its representatives can be written as π^k for some $k \geq 2$. The *cyclic bump count* $cbc(\pi)$ of a circuit $\pi = (\pi_1, \ldots, \pi_n)$ is

$$cbc(\pi) = |\{i = 1, ..., n \mid \pi_i = \overline{\pi_{i+1}}\}|,$$

where the edge π_{n+1} is understood to be π_1 .

The matrices given above are related to paths in ${\mathcal X}$ as follows: Consider first the matrix

$$M = \mathbf{1} - (B - (1 - u)J)t.$$

Then the (e,f) coefficient of M^{-1} is precisely

$$\sum_{\pi:\pi_1=e,\pi^\omega=f^\alpha} u^{\mathrm{bc}(\pi f)} t^{|\pi|} .$$

This is clear because the series expansion of M^{-1} is the sum of sequences of (B-J)t (contributing edges with no bump) and Jut (contributing edges with bumps), with an extra factor of u in case the path ends in \overline{f} . As a consequence,

LEMMA 5.2. Let

$$X_E = \frac{\mathbf{1} + (1-u)Jt - M}{Mt} = \frac{B}{\mathbf{1} - (B - (1-u)J)t}$$
.

Then the (e,f) coefficient of X_E counts the non-trivial paths starting with e and ending at f^{α} , with t-weight shifted one down:

$$(X_E)_{e,f} = \sum_{\pi : \pi_1 = e, \ \pi^\omega = f^\alpha} u^{\mathrm{bc}(\pi)} t^{|\pi| - 1} .$$

Likewise, consider the matrix

$$P = \mathbf{1} - At + (1 - u)(D - (1 - u)\mathbf{1})t^{2}$$
.

The following lemma will be a consequence of the computations in the next section.

LEMMA 5.3. Let

$$X_V = \frac{(1 - (1 - u)^2 t^2) \mathbf{1} - P}{Pt} = \frac{A - (1 - u)Dt}{\mathbf{1} - At + (1 - u)(D - (1 - u)\mathbf{1})t^2}.$$

Then the (v, w) coefficient of X_V counts the non-trivial paths starting at v and ending at w, with t-weight shifted one down:

$$(X_V)_{v,w} = \sum_{\pi: \pi^{\alpha} = v, \ \pi^{\omega} = w} u^{\operatorname{bc}(\pi)} t^{|\pi| - 1}.$$

Proof. We will show the matrix $1 + X_V t$ has as (v, w) coefficient the enriched path series from v to w. By simple calculation

$$1 + X_V t = \frac{1 - (1 - u)^2 t^2}{1 - At + (1 - u)(D - (1 - u)\mathbf{1})t^2} = \frac{K^{-1}}{1 - A'},$$

where K and A' are given by

$$K = \frac{\mathbf{1} + (1 - u)(D - 1 + u)t^2}{1 - (1 - u)^2 t^2}, \qquad A' = \frac{AKt}{1 - (1 - u)^2 t^2}.$$

K is a diagonal matrix and the coefficient $K_{x,x}$ is precisely K_x for the length labelling, while the matrix A' is the matrix of (4.1) in the previous section. The result then follows from Equation (4.2).

In particular, the two matrices X_E and X_V have the same trace, as this trace counts all the non-trivial circuits π in \mathcal{X} , with weight $u^{\operatorname{bc}(\pi)}t^{|\pi|-1}$.

We now state and prove an extension of a theorem by Bass [Bas92, FZ98, Nor96]:

Theorem 5.4. Let C be a set of representatives of primitive cycles in \mathcal{X} , and form the zeta function of \mathcal{X}

$$\zeta(u,t) = \prod_{\gamma \in \mathcal{C}} \frac{1}{1 - u^{\operatorname{cbc}(\gamma)} t^{|\gamma|}}.$$

(The choice of representatives does not change the zeta function.) Then ζ^{-1} is a polynomial in u and t and can be expressed as

$$(5.1) \qquad \frac{1}{\zeta(u,t)} = \det M$$

(5.2)
$$\zeta(u,t) = (1 + (1-u)t)^n (1 - (1-u)^2 t^2)^{m-|V(\mathcal{X})|} \det P,$$

where

$$n = |\{e \in E(\mathcal{X}) \mid e = \overline{e}\}|, \qquad 2m = |\{e \in E(\mathcal{X}) \mid e \neq \overline{e}\}|.$$

The special case u = n = 0 of this result was stated and proved in the given sources. We will prove the general statement, using a result of Shimson Amitsur:

THEOREM 5.5 (Amitsur [Ami80,RS87]). Let X_1, \ldots, X_k be square matrices of the same dimension over an arbitrary ring. Let S contain one representative up to cyclic permutation of words over the alphabet $\{1, \ldots, k\}$ that are primitive, i.e. such that none of their cyclic permutations are proper powers of a word (S is infinite as soon as k > 1). For $p = i_1 \ldots i_n \in S$ set $X_p = X_{i_1} \ldots X_{i_n}$. Then

$$\det(\mathbf{1} - (X_1 + \dots + X_k)t) = \prod_{p \in S} \det(\mathbf{1} - X_p t^{|p|}),$$

considered as an equality of power series in t over the matrix ring.

The equality (5.1) then follows; indeed, for all edges $e \in E(\mathcal{X})$ let X_e be the $E(\mathcal{X}) \times E(\mathcal{X})$ matrix whose e-th row is the e-th row of B - (1 - u)J, and whose other rows are 0. Then clearly $\mathbf{1} - \sum_{e \in E(\mathcal{X})} X_e t = M$ and, for any sequence of edges π ,

$$\det(\mathbf{1} - X_{\pi}t^{|\pi|}) = \begin{cases} 1 - u^{\operatorname{cbc}(\pi)}t^{|\pi|} & \text{if } \pi \text{ is a circuit,} \\ 1 & \text{else,} \end{cases}$$

so equality of $\zeta(u,t)$ and det M follows from Amitsur's Theorem.

To prove (5.2), we use the following result, whose proof relies on Newton's formulas relating the trace of powers of X and the characteristic polynomial of X:

PROPOSITION 5.6 ([Ami80, Equation 4.4)]. Let X be a power series in t over a matrix ring, such that X(0) = 1. Then

$$\det X = \exp\left(-\int \operatorname{tr}\left(\frac{1-X}{Xt}\right)dt\right) ,$$

where the integration is the formal linear operation on power series that maps t^r to $t^{r+1}/(r+1)$.

We then have, using Lemmas 5.2 and 5.3,

$$\frac{\det M}{(1+(1-u)t)^n(1-(1-u)^2t^2)^m} = \det \frac{M}{1+(1-u)Jt}$$

$$= \exp\left(-\int \operatorname{tr} \frac{1+(1-u)Jt-M}{Mt}dt\right)$$

$$= \exp\left(-\int \operatorname{series counting non-trivial circuits, } dt\right)$$

$$= \exp\left(-\int \operatorname{tr} \frac{(1-(1-u)^2t^2)1-P}{Pt}dt\right)$$

$$= \det \frac{P}{1-(1-u)^2t^2} = \frac{\det P}{(1-(1-u)^2t^2)^{|V(\chi)|}}.$$

6. SECOND PROOF OF THEOREM 2.4

Let $P = [\star, \dagger]$ be the set of paths in \mathcal{X} from \star to \dagger . As we shall apply the principle of inclusion-exclusion [Wil90], it will be helpful to compute in $\Pi = \mathbf{Z}[[P]]$, the **Z**-module of functions from the set of paths to **Z**. We embed subsets of P in Π by mapping a subset to its characteristic function:

$$P \supset A \mapsto \chi_A$$
, with $(\pi)\chi_A = \begin{cases} 1 & \text{if } \pi \in A, \\ 0 & \text{otherwise.} \end{cases}$

Let \mathcal{B} be the subset of bounded non-negative elements of Π (i.e. functions f such that there is a constant N with $0 \le (\pi)f < N$ for all paths π). If ℓ is a complete labelling of \mathcal{X} , there is an induced labelling $\ell_* : \mathcal{B} \to \mathbf{k}$ given by

$$(f)\ell_* = \sum_{\pi \in P} (\pi) f \pi^{\ell} .$$

Note that the sum, although infinite, defines an element of ${\bf k}$ due to the fact that ℓ is complete.