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The following result is well-known:

THEOREM 3.12 (Müller & Schupp [MS81, MS83]). *Let $\Gamma$ be a finitely generated group, presented as a quotient $\Sigma/\Xi$ with $\Sigma$ as in (3.4). Then $\Theta(\Xi)$ is an algebraic series (i.e. satisfies a polynomial equation over $\mathbf{k}[t]$) if and only if $\Sigma/\Xi$ is virtually free (i.e. has a normal subgroup of finite index that is free).*

It is not known whether there exists a non-virtually-free quasi-transitive graph whose circuit series (as defined in Corollary 2.6) is algebraic.

## 4. First proof of Theorem 2.4

We now prove Theorem 2.4 using linear algebra. We first assume the graph has a finite number of vertices, for the computations refer to $\mathbf{k}$-matrices and $\mathbf{k}[[u]]$-matrices indexed by the graph's vertices. This proof is hinted at in Godsil's book as an exercise [God93, page 72]; it was also suggested to the author by Gilles Robert.

For all pairs of vertices $x, y \in V(\mathcal{X})$ let

$$\mathfrak{G}_{x,y}(\ell) = \sum_{\pi\in[x,y]} \pi^{\ell}, \qquad \mathfrak{F}_{x,y}(\ell) = \sum_{\pi\in[x,y]} u^{\mathrm{bc}(\pi)}\pi^{\ell}$$

be the path and enriched path series from $x$ to $y$; for ease of notation we will leave out the labelling $\ell$ if it is obvious from the context. Let $\delta_{x,y}$ denote the Kronecker delta, equal to 1 if $x = y$ and 0 otherwise. For any $v \in \mathbf{k}$, let $[v]_x^y$ denote the $V(\mathcal{X}) \times V(\mathcal{X})$ matrix with zeros everywhere except at $(x, y)$, where it has value $v$. Then

$$\mathfrak{G}_{x,y} = \delta_{x,y} + \sum_{e\in E(\mathcal{X})\,:\,e^{\alpha}=x} e^{\ell}\mathfrak{G}_{e^{\omega},y}$$

so that if

$$A = \sum_{e\in E(\mathcal{X})} [e^{\ell}]_{e^{\alpha}}^{e^{\omega}}$$

be the adjacency matrix of $\mathcal{X}$, with labellings, then we have

$$(\mathfrak{G}_{x,y})_{x,y\in V(\mathcal{X})} = \frac{1}{1-A},$$

an equation holding between $V(\mathcal{X}) \times V(\mathcal{X})$ matrices over $\mathbf{k}$.

Similarly, letting $\mathfrak{F}_{x,e,y}$ count the paths from $x$ to $y$ that start with the edge $e$,

$$\mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e\in E(\mathcal{X}):e^\alpha=x} \mathfrak{F}_{x,e,y}\,,$$
$$\mathfrak{F}_{x,e,y} = e^\ell\left(\mathfrak{F}_{e^\omega,y} + (u-1)\mathfrak{F}_{e^\omega,\overline{e},y}\right)\,,$$
$$\mathfrak{F}_{e^\omega,\overline{e},y} = \overline{e}^\ell\left(\mathfrak{F}_{x,y} + (u-1)\mathfrak{F}_{x,e,y}\right)\,;$$

these last two lines solve to

$$\mathfrak{F}_{x,e,y} = \left(1-(u-1)^2(e\overline{e})^\ell\right)^{-1}\left(e^\ell\mathfrak{F}_{e^\omega,y} + (u-1)(e\overline{e})^\ell\mathfrak{F}_{x,y}\right)\,,$$

which we insert in the first line to obtain

$$K_x^{-1}\mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e\in E(\mathcal{X}):e^\alpha=x} \frac{e^\ell}{1-(u-1)^2(e\,\overline{e})^\ell}K_{e^\omega}\cdot K_{e^\omega}^{-1}\mathfrak{F}_{e^\omega,y}\,.$$

Thus if we let

$$e^{\ell'} = \frac{e^\ell}{1-(u-1)^2(e\,\overline{e})^\ell}K_{e^\omega}\,,\qquad A' = \sum_{e\in E(\mathcal{X})}[e^{\ell'}]_{e^\alpha}^{e^\omega}\,, \tag{4.1}$$

we obtain

$$(K_x^{-1}\mathfrak{F}_{x,y})_{x,y\in V(\mathcal{X})} = \frac{1}{1-A'} \tag{4.2}$$

and the proof is finished in the case that $\mathcal{X}$ is finite, because the matrix $A'$ is precisely that obtained from $A$ by substituting $\ell'$ for $\ell$.

If $\mathcal{X}$ has infinitely many vertices, we approximate it, using Lemma 3.7, by finite graphs. Denote by $\mathfrak{F}^n_{\star,\dagger}(\ell)$ and $\mathfrak{G}^n_{\star,\dagger}(\ell')$ the enriched path series and path series respectively in $\mathcal{B}(\star,n)$, and write

$$K_\star\cdot\mathfrak{F}(\ell) = \lim_{n\to\infty}\mathfrak{F}^n_{\star,\dagger}(\ell) = \lim_{n\to\infty}\mathfrak{G}^n_{\star,\dagger}(\ell') = \mathfrak{G}(\ell')$$

to complete the proof.

## 5. Graphs and matrices

Graphs can be studied through their *adjacency* and *incidence* matrices. We give here the relevant definitions and obtain an extension of a theorem by Hyman Bass [Bas92] on the Ihara-Selberg zeta function. We will use power series with coefficients in a matrix ring, and fractional expressions in matrices; by convention, we understand '$X/Y$' as '$X\cdot Y^{-1}$'.