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Note that if  $\mathcal{X}$  is not quasi-transitive, a somewhat weaker result holds [Kit98, §7.1]: if  $\mathcal{X}$  is transient or null-recurrent then the common limsup is 0. If  $\mathcal{X}$  is positive-recurrent then the limsup's are normalized coefficients of  $\mathcal{X}$ 's Perron-Frobenius eigenvector. Lemma 3.9 is not true for arbitrary  $d$ -regular graphs: consider for instance the graph  $\mathcal{X}_3$  described above. Its circuit series  $G_3$ , given in (3.3), has radius of convergence  $1/\beta = 2/7$ , and one easily checks that all its coefficients  $g_n$  satisfy  $g_n/\beta^n \geq 1/2$ .

We obtain the following characterization of rational series:

**THEOREM 3.10.** *For regular quasi-transitive connected graphs  $\mathcal{X}$ , the following are equivalent:*

1.  $\mathcal{X}$  is finite;
2.  $G(t)$  is a rational function of  $t$ ;
3.  $F(t)$  is a rational function of  $t$ , and  $\mathcal{X}$  is not an infinite tree.

*Proof.* By Corollary 2.7, Statement 1 implies the other two. By Corollary 2.6, and a computation on trees done in Section 7.3 to deal with the case  $F(t) = 1$ , Statement 2 implies 3. It remains to show that Statement 3 implies 1.

Assume that  $F(t) = \sum f_n t^n$  is rational, not equal to 1. As the  $f_n$  are positive,  $F$  has a pole, of multiplicity  $m$ , at  $1/\alpha$ . There is then a constant  $a > 0$  such that  $f_n > a \binom{n}{m-1} \alpha^n$  for infinitely many values of  $n$  [GKP94, page 341]. It follows by Lemma 3.9 that  $m = 1$  and the graph  $\mathcal{X}$  is finite, of cardinality at most  $1/a$ .  $\square$

It is not known whether the same holds for regular, or even arbitrary connected graphs. Certainly an altogether different proof would be needed.

### 3.3 APPLICATION TO LANGUAGES

Let  $S$  be a finite set of cardinality  $d$  and let  $\bar{\cdot}$  be an involution on  $S$ . A *word* is an element  $w$  of the free monoid  $S^*$ . A *language* is a set  $L$  of words. The language  $L$  is called *saturated* if for any  $u, v \in S^*$  and  $s \in S$  we have

$$uv \in L \iff us\bar{s}v \in L;$$

that is to say,  $L$  is stable under insertion and deletion of subwords of the form  $s\bar{s}$ . The language  $L$  is called *desiccated* if no word in  $L$  contains a subword of the form  $s\bar{s}$ . Given a language  $L$  we may naturally construct its *saturation*

$\langle L \rangle$ , the smallest saturated language containing  $L$ , and its *desiccation*  $\widehat{L}$ , the largest desiccated language contained in  $L$ .

Let  $\Sigma$  be the monoid defined by generators  $S$  and relations  $s\bar{s} = 1$  for all  $s \in S$ :

$$(3.4) \quad \Sigma = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle.$$

This is a free product of free groups and order-two groups; if  $\bar{\cdot}$  is fixed-point-free,  $\Sigma$  is a free group. Write  $\phi$  for the canonical projection from  $S^*$  to  $\Sigma$ . Let  $\mathbf{k} = \mathbf{Z}[\Sigma]$  be its monoid ring. Then given a language  $L \subset S^*$  we may define its *growth series*  $\Theta(L)$  as

$$\Theta(L) = \sum_{w \in L} w^\phi t^{|w|} \in \mathbf{k}[[t]].$$

This notion of growth series with coefficients was introduced by Fabrice Liardet in his doctoral thesis [Lia96], where he studied *complete growth functions* of groups.

THEOREM 3.11. *For any language  $L$  there holds*

$$(3.5) \quad \frac{\Theta(\widehat{L})(t)}{1 - t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1 + (d-1)t^2}\right)}{1 + (d-1)t^2},$$

where  $d = |S|$ .

*Proof.* For any language there exists a unique minimal (possibly infinite) automaton recognising it ([Eil74, §III.5] is a good reference). Let  $\mathcal{X}$  be the minimal automaton recognising  $\langle L \rangle$ . Recall that this is a graph with an initial vertex  $\star$ , a set of terminal vertices  $T$  and a labelling  $\ell': E(\mathcal{X}) \rightarrow S$  of the graph's edges such that the number of paths labelled  $w$ , starting at  $\star$  and ending at a  $\tau \in T$  is 1 if  $w \in L$  and 0 otherwise. Extend the labelling  $\ell'$  to a labelling  $\ell: E(\mathcal{X}) \rightarrow \mathbf{k}[[t]]$  by

$$e^\ell = t \cdot (e^{\ell'})^\phi.$$

Because  $\langle L \rangle$  is saturated, and  $\mathcal{X}$  is minimal,  $(\bar{e})^\ell = \overline{e^\ell}$ ; then  $\widehat{L}$  is the set of labels on proper paths from  $\star$  to some  $\tau \in T$ . Choosing in turn all  $\tau \in T$  as  $\dagger$ , we obtain growth series  $F_\tau, G_\tau$  counting the formal sum of paths and proper paths from  $\star$  to  $\tau$ . It then suffices to write

$$\frac{\Theta(\widehat{L})(t)}{1 - t^2} = \frac{\sum_{\tau \in T} F_\tau(t)}{1 - t^2} = \frac{\sum_{\tau \in T} G_\tau\left(\frac{t}{1 + (d-1)t^2}\right)}{1 + (d-1)t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1 + (d-1)t^2}\right)}{1 + (d-1)t^2}. \quad \square$$

The following result is well-known:

**THEOREM 3.12** (Müller & Schupp [MS81, MS83]). *Let  $\Gamma$  be a finitely generated group, presented as a quotient  $\Sigma/\Xi$  with  $\Sigma$  as in (3.4). Then  $\Theta(\Xi)$  is an algebraic series (i.e. satisfies a polynomial equation over  $\mathbf{k}[t]$ ) if and only if  $\Sigma/\Xi$  is virtually free (i.e. has a normal subgroup of finite index that is free).*

It is not known whether there exists a non-virtually-free quasi-transitive graph whose circuit series (as defined in Corollary 2.6) is algebraic.

#### 4. FIRST PROOF OF THEOREM 2.4

We now prove Theorem 2.4 using linear algebra. We first assume the graph has a finite number of vertices, for the computations refer to  $\mathbf{k}$ -matrices and  $\mathbf{k}[[u]]$ -matrices indexed by the graph's vertices. This proof is hinted at in Godsil's book as an exercise [God93, page 72]; it was also suggested to the author by Gilles Robert.

For all pairs of vertices  $x, y \in V(\mathcal{X})$  let

$$\mathfrak{G}_{x,y}(\ell) = \sum_{\pi \in [x,y]} \pi^\ell, \quad \mathfrak{F}_{x,y}(\ell) = \sum_{\pi \in [x,y]} u^{\text{bc}(\pi)} \pi^\ell$$

be the path and enriched path series from  $x$  to  $y$ ; for ease of notation we will leave out the labelling  $\ell$  if it is obvious from the context. Let  $\delta_{x,y}$  denote the Kronecker delta, equal to 1 if  $x = y$  and 0 otherwise. For any  $v \in \mathbf{k}$ , let  $[v]_x^y$  denote the  $V(\mathcal{X}) \times V(\mathcal{X})$  matrix with zeros everywhere except at  $(x, y)$ , where it has value  $v$ . Then

$$\mathfrak{G}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}) : e^\alpha = x} e^\ell \mathfrak{G}_{e^\omega, y}$$

so that if

$$A = \sum_{e \in E(\mathcal{X})} [e^\ell]_{e^\alpha}^{e^\omega}$$

be the adjacency matrix of  $\mathcal{X}$ , with labellings, then we have

$$(\mathfrak{G}_{x,y})_{x,y \in V(\mathcal{X})} = \frac{1}{1 - A},$$

an equation holding between  $V(\mathcal{X}) \times V(\mathcal{X})$  matrices over  $\mathbf{k}$ .