

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 45 (1999)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** COUNTING PATHS IN GRAPHS  
**Autor:** Bartholdi, Laurent  
**Kapitel:** 3.2 The series F and G on their circle of convergence  
**DOI:** <https://doi.org/10.5169/seals-64442>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 17.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

### 3.2 THE SERIES $F$ AND $G$ ON THEIR CIRCLE OF CONVERGENCE

In this subsection we study the singularities the series  $F$  and  $G$  may have on their circle of convergence. The smallest positive real singularity has a special importance:

DEFINITION 3.4. For a series  $f(t)$  with positive coefficients, let  $\rho(f)$  denote its radius of convergence. Then  $f$  is  $\rho(f)$ -recurrent if

$$\lim_{t \rightarrow \rho(f)} f(t) = \infty .$$

Otherwise, it is  $\rho(f)$ -transient.

As typical examples,  $1/(\rho - t)$  is  $\rho$ -recurrent, as are all rational series;  $\sqrt{\rho - t}$  is  $\rho$ -transient, while  $1/\sqrt{\rho - t}$  is not.

To study the singularities of  $F$  or  $G$ , we may suppose that  $\star = \dagger$ ; indeed in was shown in [Kes59] and [Woe83, Lemma 1] that the singularities of  $F$  and  $G$  do not depend on the choice of  $\star$  and  $\dagger$ . We make that assumption for the remainder of the subsection. We will also suppose throughout that  $\mathcal{X}$  is  $d$ -regular, that the radius of convergence of  $F$  is  $1/\alpha$  and the radius of convergence of  $G$  is  $1/(d\nu) = 1/\beta$ .

DEFINITION 3.5. Let  $\mathcal{X}$  be a connected graph. A *proper cycle* in  $\mathcal{X}$  is a proper circuit  $(\pi_1, \dots, \pi_n)$  such that  $\overline{\pi_1} \neq \pi_n$ . The *proper period*  $p$  and *strong proper period*  $p_s$  are defined as follows:

$$p = \gcd\{n \mid \text{there exists a proper cycle } \pi \text{ in } \mathcal{X} \text{ with } |\pi| = n\},$$

$$p_s = \gcd\{n \mid \forall x \in V(\mathcal{X}) \text{ there exists} \\ \text{a proper cycle } \pi \text{ in } \mathcal{B}(x, n) \text{ with } |\pi| = n\},$$

where by convention the gcd of the empty set is  $\infty$ . The graph  $\mathcal{X}$  is *strongly properly periodic* if  $p = p_s$ .

The *period*  $q$  and *strong period*  $q_s$  of  $\mathcal{X}$  are defined analogously with ‘proper cycle’ replaced by ‘circuit’.  $\mathcal{X}$  is *strongly periodic* if  $q = q_s$ .

THEOREM 3.6 (Cartwright [Car92]). *Let  $\mathcal{X}$  have proper period  $p$  and strong proper period  $p_s$ . Then the singularities of  $F$  on its circle of convergence are among the*

$$\frac{e^{2i\pi k/p_s}}{\alpha}, \quad k = 1, \dots, p_s .$$

If moreover  $\mathcal{X}$  is strongly properly periodic, the singularities of  $F$  on its circle of convergence are precisely these numbers.

Let  $\mathcal{X}$  have period  $q$  and strong period  $q_s$ . Then the singularities of  $G$  on its circle of convergence are among the

$$\frac{e^{2i\pi k/q_s}}{\beta}, \quad k = 1, \dots, q_s.$$

If moreover  $\mathcal{X}$  is strongly periodic, the singularities of  $G$  on its circle of convergence are precisely these numbers.

If  $\mathcal{X}$  is connected and non-trivial, there is a path of even length at every vertex (a sequence of bumps, for instance). All graphs are then either 2-periodic (if they are bipartite) or 1-periodic. If there is a constant  $N$  such that for all  $x \in V(\mathcal{X})$  there is at  $x$  a circuit of odd length bounded by  $N$ , then  $\mathcal{X}$  is strongly 1-periodic; otherwise  $\mathcal{X}$  is strongly 2-periodic. The singularities of  $G$  on its circle of convergence are then at  $1/\beta$ , and also at  $-1/\beta$  if  $\mathcal{X}$  is strongly periodic with period 2.

If  $\mathcal{X}$  is not strongly periodic, there may be one or two singularities on  $G$ 's circle of convergence; consider for instance the 4-regular tree, and at a vertex  $\star$  delete two or three edges replacing them by loops. The resulting graphs  $\mathcal{X}_2$  and  $\mathcal{X}_3$  are still 4-regular and their circuit series, as computed using (7.2), are respectively

$$(3.3) \quad \begin{aligned} G_2(t) &= \frac{3}{2 - 6t + \sqrt{1 - 12t^2}}, \\ G_3(t) &= \frac{6}{5 - 18t + \sqrt{1 - 12t^2}}. \end{aligned}$$

$G_2$  has singularities at  $\pm 1/\sqrt{12}$  on its circle of convergence, while  $G_3$  has only  $2/7$  as singularity on its circle of convergence.

Following the proof of Corollary 3.2 above, we see that if  $\beta < d$  the singularities of  $F$  on its circle of convergence are in bijection with those of  $G$ , so are at  $1/\alpha$  and possibly  $-1/\alpha$ , if  $\mathcal{X}$  is strongly two-periodic. If  $\beta = d$ , though,  $\mathcal{X}$  can have any strong proper period; consider for example the cycles on length  $k$  studied in Section 7.2: they are strongly properly  $k$ -periodic.

The forthcoming simple result shows how  $\mathcal{X}$  can be approximated by finite subgraphs.

LEMMA 3.7. Let  $\mathcal{X}$  be a graph and  $x, y$  two vertices in  $\mathcal{X}$ . Let  $\mathfrak{G}_{x,y}$  and  $\mathfrak{F}_{x,y}$  be the path series and enriched path series respectively from  $x$  to  $y$  in  $\mathcal{X}$ , and let  $\mathfrak{G}_{x,y}^n$  and  $\mathfrak{F}_{x,y}^n$  be the path series and enriched path series respectively from  $x$  to  $y$  in the ball  $\mathcal{B}(x, n)$  (they are 0 if  $y \notin \mathcal{B}(x, n)$ ). Then

$$\lim_{n \rightarrow \infty} \mathfrak{G}_{x,y}^n = \mathfrak{G}_{x,y}, \quad \lim_{n \rightarrow \infty} \mathfrak{F}_{x,y}^n = \mathfrak{F}_{x,y}.$$

*Proof.* Recall that  $\lim \mathfrak{G}_{x,y}^n = \mathfrak{G}_{x,y}$  means that both terms are sums of paths, say  $A_n$  and  $A$ , such that the minimal length of paths in the symmetric difference  $A_n \triangle A$  tends to infinity. Now the difference between  $\mathfrak{G}_{x,y}^n$  and  $\mathfrak{G}_{x,y}$  consists only of paths in  $\mathcal{X}$  that exit  $\mathcal{B}(x, n)$ , and thus have length at least  $2n - \delta(x, y) \rightarrow \infty$ . The same argument holds for  $\mathfrak{F}$ .  $\square$

DEFINITION 3.8. The graph  $\mathcal{X}$  is *quasi-transitive* if  $\text{Aut}(\mathcal{X})$  acts with finitely many orbits.

LEMMA 3.9. Let  $\mathcal{X}$  be a regular quasi-transitive connected graph with distinguished vertex  $\star$ , and let  $f_n$  and  $g_n$  denote respectively the number of proper circuits and circuits at  $\star$  of length  $n$ . Then

$$\limsup_{n \rightarrow \infty} g_n / \beta^n = \limsup_{n \rightarrow \infty} f_n / \alpha^n = \begin{cases} 1/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite and has odd circuits;} \\ 2/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite} \\ & \text{and has only even circuits;} \\ 0 & \text{if } \mathcal{X} \text{ is infinite.} \end{cases}$$

*Proof.* If  $\mathcal{X}$  is finite, then  $\beta = d$ , the degree of  $\mathcal{X}$ ; after a large even number of steps, a random walk starting at  $\star$  will be uniformly distributed over  $\mathcal{X}$  (or over the vertices at even distance of  $\star$ , in case all circuits have even length). A long enough walk then has probability  $1/|\mathcal{X}|$  (or  $2/|\mathcal{X}|$  if all circuits have even length) of being a circuit.

If  $\mathcal{X}$  is infinite, we consider two cases. If  $G(1/\beta) < \infty$ , i.e.  $G$  is  $1/\beta$ -transient, the general term  $g_n/\beta^n$  of the series  $G(1/\beta)$  tends to 0. If  $G$  is  $1/\beta$ -recurrent, then, as  $\mathcal{X}$  is quasi-transitive,  $\beta = d$  by [Woe98, Theorem 7.7]. We then approximate  $\mathcal{X}$  by the sequence of its balls of radius  $R$ , by Lemma 3.7:

$$\lim_{n \rightarrow \infty} \frac{g_n}{\beta^n} = \lim_{R, n \rightarrow \infty} \frac{g_{R,n}}{d^n} = \lim_{R \rightarrow \infty} \frac{(1 \text{ or } 2)}{|\mathcal{B}(\star, R)|} = 0,$$

where we expand the circuit series of  $\mathcal{B}(\star, R)$  as  $\sum g_{R,n} t^n$ .

The same proof holds for the  $f_n$ . Its particular case where  $\mathcal{X}$  is a Cayley graph appears in [Woe83].  $\square$

Note that if  $\mathcal{X}$  is not quasi-transitive, a somewhat weaker result holds [Kit98, §7.1]: if  $\mathcal{X}$  is transient or null-recurrent then the common limsup is 0. If  $\mathcal{X}$  is positive-recurrent then the limsups are normalized coefficients of  $\mathcal{X}$ 's Perron-Frobenius eigenvector. Lemma 3.9 is not true for arbitrary  $d$ -regular graphs: consider for instance the graph  $\mathcal{X}_3$  described above. Its circuit series  $G_3$ , given in (3.3), has radius of convergence  $1/\beta = 2/7$ , and one easily checks that all its coefficients  $g_n$  satisfy  $g_n/\beta^n \geq 1/2$ .

We obtain the following characterization of rational series:

**THEOREM 3.10.** *For regular quasi-transitive connected graphs  $\mathcal{X}$ , the following are equivalent:*

1.  $\mathcal{X}$  is finite;
2.  $G(t)$  is a rational function of  $t$ ;
3.  $F(t)$  is a rational function of  $t$ , and  $\mathcal{X}$  is not an infinite tree.

*Proof.* By Corollary 2.7, Statement 1 implies the other two. By Corollary 2.6, and a computation on trees done in Section 7.3 to deal with the case  $F(t) = 1$ , Statement 2 implies 3. It remains to show that Statement 3 implies 1.

Assume that  $F(t) = \sum f_n t^n$  is rational, not equal to 1. As the  $f_n$  are positive,  $F$  has a pole, of multiplicity  $m$ , at  $1/\alpha$ . There is then a constant  $a > 0$  such that  $f_n > a \binom{n}{m-1} \alpha^n$  for infinitely many values of  $n$  [GKP94, page 341]. It follows by Lemma 3.9 that  $m = 1$  and the graph  $\mathcal{X}$  is finite, of cardinality at most  $1/a$ .  $\square$

It is not known whether the same holds for regular, or even arbitrary connected graphs. Certainly an altogether different proof would be needed.

### 3.3 APPLICATION TO LANGUAGES

Let  $S$  be a finite set of cardinality  $d$  and let  $\bar{\cdot}$  be an involution on  $S$ . A *word* is an element  $w$  of the free monoid  $S^*$ . A *language* is a set  $L$  of words. The language  $L$  is called *saturated* if for any  $u, v \in S^*$  and  $s \in S$  we have

$$uv \in L \iff us\bar{v} \in L;$$

that is to say,  $L$  is stable under insertion and deletion of subwords of the form  $s\bar{s}$ . The language  $L$  is called *desiccated* if no word in  $L$  contains a subword of the form  $s\bar{s}$ . Given a language  $L$  we may naturally construct its *saturation*