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3.2 THE SERIES F AND G ON THEIR CIRCLE OF CONVERGENCE

In this subsection we study the singularities the series F and G may have on their circle of convergence. The smallest positive real singularity has a special importance:

DEFINITION 3.4. For a series $f(t)$ with positive coefficients, let $\rho(f)$ denote its radius of convergence. Then f is $\rho(f)$ -recurrent if

$$\lim_{t \rightarrow \rho(f)} f(t) = \infty.$$

Otherwise, it is $\rho(f)$ -transient.

As typical examples, $1/(\rho - t)$ is ρ -recurrent, as are all rational series; $\sqrt{\rho - t}$ is ρ -transient, while $1/\sqrt{\rho - t}$ is not.

To study the singularities of F or G , we may suppose that $\star = \dagger$; indeed in was shown in [Kes59] and [Woe83, Lemma 1] that the singularities of F and G do not depend on the choice of \star and \dagger . We make that assumption for the remainder of the subsection. We will also suppose throughout that \mathcal{X} is d -regular, that the radius of convergence of F is $1/\alpha$ and the radius of convergence of G is $1/(d\nu) = 1/\beta$.

DEFINITION 3.5. Let \mathcal{X} be a connected graph. A *proper cycle* in \mathcal{X} is a proper circuit (π_1, \dots, π_n) such that $\overline{\pi_1} \neq \pi_n$. The *proper period* p and *strong proper period* p_s are defined as follows:

$$p = \gcd\{n \mid \text{there exists a proper cycle } \pi \text{ in } \mathcal{X} \text{ with } |\pi| = n\},$$

$$p_s = \gcd\{n \mid \forall x \in V(\mathcal{X}) \text{ there exists} \\ \text{a proper cycle } \pi \text{ in } \mathcal{B}(x, n) \text{ with } |\pi| = n\},$$

where by convention the gcd of the empty set is ∞ . The graph \mathcal{X} is *strongly properly periodic* if $p = p_s$.

The *period* q and *strong period* q_s of \mathcal{X} are defined analogously with ‘proper cycle’ replaced by ‘circuit’. \mathcal{X} is *strongly periodic* if $q = q_s$.

THEOREM 3.6 (Cartwright [Car92]). Let \mathcal{X} have proper period p and strong proper period p_s . Then the singularities of F on its circle of convergence are among the

$$\frac{e^{2i\pi k/p_s}}{\alpha}, \quad k = 1, \dots, p_s.$$

If moreover \mathcal{X} is strongly properly periodic, the singularities of F on its circle of convergence are precisely these numbers.

Let \mathcal{X} have period q and strong period q_s . Then the singularities of G on its circle of convergence are among the

$$\frac{e^{2i\pi k/q_s}}{\beta}, \quad k = 1, \dots, q_s.$$

If moreover \mathcal{X} is strongly periodic, the singularities of G on its circle of convergence are precisely these numbers.

If \mathcal{X} is connected and non-trivial, there is a path of even length at every vertex (a sequence of bumps, for instance). All graphs are then either 2-periodic (if they are bipartite) or 1-periodic. If there is a constant N such that for all $x \in V(\mathcal{X})$ there is at x a circuit of odd length bounded by N , then \mathcal{X} is strongly 1-periodic; otherwise \mathcal{X} is strongly 2-periodic. The singularities of G on its circle of convergence are then at $1/\beta$, and also at $-1/\beta$ if \mathcal{X} is strongly periodic with period 2.

If \mathcal{X} is not strongly periodic, there may be one or two singularities on G 's circle of convergence; consider for instance the 4-regular tree, and at a vertex \star delete two or three edges replacing them by loops. The resulting graphs \mathcal{X}_2 and \mathcal{X}_3 are still 4-regular and their circuit series, as computed using (7.2), are respectively

$$(3.3) \quad \begin{aligned} G_2(t) &= \frac{3}{2 - 6t + \sqrt{1 - 12t^2}}, \\ G_3(t) &= \frac{6}{5 - 18t + \sqrt{1 - 12t^2}}. \end{aligned}$$

G_2 has singularities at $\pm 1/\sqrt{12}$ on its circle of convergence, while G_3 has only $2/7$ as singularity on its circle of convergence.

Following the proof of Corollary 3.2 above, we see that if $\beta < d$ the singularities of F on its circle of convergence are in bijection with those of G , so are at $1/\alpha$ and possibly $-1/\alpha$, if \mathcal{X} is strongly two-periodic. If $\beta = d$, though, \mathcal{X} can have any strong proper period; consider for example the cycles on length k studied in Section 7.2: they are strongly properly k -periodic.

The forthcoming simple result shows how \mathcal{X} can be approximated by finite subgraphs.

LEMMA 3.7. Let \mathcal{X} be a graph and x, y two vertices in \mathcal{X} . Let $\mathfrak{G}_{x,y}$ and $\mathfrak{F}_{x,y}$ be the path series and enriched path series respectively from x to y in \mathcal{X} , and let $\mathfrak{G}_{x,y}^n$ and $\mathfrak{F}_{x,y}^n$ be the path series and enriched path series respectively from x to y in the ball $\mathcal{B}(x, n)$ (they are 0 if $y \notin \mathcal{B}(x, n)$). Then

$$\lim_{n \rightarrow \infty} \mathfrak{G}_{x,y}^n = \mathfrak{G}_{x,y}, \quad \lim_{n \rightarrow \infty} \mathfrak{F}_{x,y}^n = \mathfrak{F}_{x,y}.$$

Proof. Recall that $\lim_{n \rightarrow \infty} \mathfrak{G}_{x,y}^n = \mathfrak{G}_{x,y}$ means that both terms are sums of paths, say A_n and A , such that the minimal length of paths in the symmetric difference $A_n \triangle A$ tends to infinity. Now the difference between $\mathfrak{G}_{x,y}^n$ and $\mathfrak{G}_{x,y}$ consists only of paths in \mathcal{X} that exit $\mathcal{B}(x, n)$, and thus have length at least $2n - \delta(x, y) \rightarrow \infty$. The same argument holds for \mathfrak{F} . \square

DEFINITION 3.8. The graph \mathcal{X} is *quasi-transitive* if $\text{Aut}(\mathcal{X})$ acts with finitely many orbits.

LEMMA 3.9. Let \mathcal{X} be a regular quasi-transitive connected graph with distinguished vertex \star , and let f_n and g_n denote respectively the number of proper circuits and circuits at \star of length n . Then

$$\limsup_{n \rightarrow \infty} g_n / \beta^n = \limsup_{n \rightarrow \infty} f_n / \alpha^n = \begin{cases} 1/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite and has odd circuits;} \\ 2/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite} \\ & \text{and has only even circuits;} \\ 0 & \text{if } \mathcal{X} \text{ is infinite.} \end{cases}$$

Proof. If \mathcal{X} is finite, then $\beta = d$, the degree of \mathcal{X} ; after a large even number of steps, a random walk starting at \star will be uniformly distributed over \mathcal{X} (or over the vertices at even distance of \star , in case all circuits have even length). A long enough walk then has probability $1/|\mathcal{X}|$ (or $2/|\mathcal{X}|$ if all circuits have even length) of being a circuit.

If \mathcal{X} is infinite, we consider two cases. If $G(1/\beta) < \infty$, i.e. G is $1/\beta$ -transient, the general term g_n/β^n of the series $G(1/\beta)$ tends to 0. If G is $1/\beta$ -recurrent, then, as \mathcal{X} is quasi-transitive, $\beta = d$ by [Woe98, Theorem 7.7]. We then approximate \mathcal{X} by the sequence of its balls of radius R , by Lemma 3.7:

$$\lim_{n \rightarrow \infty} \frac{g_n}{\beta^n} = \lim_{R, n \rightarrow \infty} \frac{g_{R,n}}{d^n} = \lim_{R \rightarrow \infty} \frac{(1 \text{ or } 2)}{|\mathcal{B}(\star, R)|} = 0,$$

where we expand the circuit series of $\mathcal{B}(\star, R)$ as $\sum g_{R,n} t^n$.

The same proof holds for the f_n . Its particular case where \mathcal{X} is a Cayley graph appears in [Woe83]. \square

Note that if \mathcal{X} is not quasi-transitive, a somewhat weaker result holds [Kit98, §7.1]: if \mathcal{X} is transient or null-recurrent then the common limsup is 0. If \mathcal{X} is positive-recurrent then the limsup is a normalized coefficient of \mathcal{X} 's Perron-Frobenius eigenvector. Lemma 3.9 is not true for arbitrary d -regular graphs: consider for instance the graph \mathcal{X}_3 described above. Its circuit series G_3 , given in (3.3), has radius of convergence $1/\beta = 2/7$, and one easily checks that all its coefficients g_n satisfy $g_n/\beta^n \geq 1/2$.

We obtain the following characterization of rational series:

THEOREM 3.10. *For regular quasi-transitive connected graphs \mathcal{X} , the following are equivalent:*

1. \mathcal{X} is finite;
2. $G(t)$ is a rational function of t ;
3. $F(t)$ is a rational function of t , and \mathcal{X} is not an infinite tree.

Proof. By Corollary 2.7, Statement 1 implies the other two. By Corollary 2.6, and a computation on trees done in Section 7.3 to deal with the case $F(t) = 1$, Statement 2 implies 3. It remains to show that Statement 3 implies 1.

Assume that $F(t) = \sum f_n t^n$ is rational, not equal to 1. As the f_n are positive, F has a pole, of multiplicity m , at $1/\alpha$. There is then a constant $a > 0$ such that $f_n > a \binom{n}{m-1} \alpha^n$ for infinitely many values of n [GKP94, page 341]. It follows by Lemma 3.9 that $m = 1$ and the graph \mathcal{X} is finite, of cardinality at most $1/a$. \square

It is not known whether the same holds for regular, or even arbitrary connected graphs. Certainly an altogether different proof would be needed.

3.3 APPLICATION TO LANGUAGES

Let S be a finite set of cardinality d and let $\bar{\cdot}$ be an involution on S . A *word* is an element w of the free monoid S^* . A *language* is a set L of words. The language L is called *saturated* if for any $u, v \in S^*$ and $s \in S$ we have

$$uv \in L \iff us\bar{s}v \in L;$$

that is to say, L is stable under insertion and deletion of subwords of the form $s\bar{s}$. The language L is called *desiccated* if no word in L contains a subword of the form $s\bar{s}$. Given a language L we may naturally construct its *saturation*