Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COUNTING PATHS IN GRAPHS
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Kapitel: 3.1 Applications to random walks on groups
DOI: https://doi.org/10.5169/seals-64442

Finally in Section 9 we show how to compute the circuit series of a free product of graphs (an analogue of the free products of groups, *via* their Cayley graph), and in Section 10 do the same for direct products of graphs.

3. Applications to other fields

The original motivation for Formula 2.2 was its implication of a well-known result in the theory of random walks on discrete groups.

3.1 Applications to random walks on groups

In this section we show how G is related to random walks and F to cogrowth. This will give a generalization of the main formula (1.1) to homogeneous spaces Π/Ξ, where Ξ does not have to be normal and Π is a free product of infinite-cyclic and order-two groups. For a survey on the topic of random walks see [MW89,Woe94].

Throughout this subsection we will have $F(t) = F(0,t)$. We recall the notion of growth of groups:

DEFINITION 3.1. Let Γ be a group generated by a finite symmetric set S. For a $\gamma \in \Gamma$ define its *length*

$$|\gamma| = \min\{n \in \mathbf{N} : \gamma \in S^n\} .$$

The *growth series* of (Γ, S) is the formal power series

$$f_{(\Gamma,S)}(t) = \sum_{\gamma\in\Gamma} t^{|\gamma|} \in \mathbf{N}[[t]] .$$

Expanding $f_{(\Gamma,S)}(t) = \sum f_n t^n$, the *growth* of (Γ, S) is

$$\alpha(\Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n}$$

(this supremum-limit is actually a limit and is smaller than $|S| - 1$).

Let R be a subset of Γ. The *growth series* of R relative to (Γ, S) is the formal power series

$$f^R_{(\Gamma,S)}(t) = \sum_{\gamma\in R} t^{|\gamma|} \in \mathbf{N}[[t]] .$$

Expanding $f^R_{(\Gamma,S)}(t) = \sum f_n t^n$, define the *growth* of R relative to (Γ, S) as

$$\alpha(R; \Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n} .$$

If X is a transitive right Γ-set, the *simple random walk* on (X,S) is the random walk of a point on X, having probability $1/|S|$ of moving from its current position x to a neighbour $x \cdot s$, for all $s \in S$. Fix a point $\star \in X$, and let p_n be the probability that a walk starting at $\star$ finish at $\star$ after n moves. We define the *spectral radius* (which does not depend on the choice of $\star$) of the random walk as

$$\nu(X,S) = \limsup_{n \to \infty} \sqrt[n]{p_n} \, .$$

A group Π is *quasi-free* if it is a free product of cyclic groups of order 2 and ∞. Equivalently, there exists a finite set S and an involution $\bar{\cdot}: S \to S$ such that, as a monoid,

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle \, .$$

Π is then said to be *quasi-free on* S. All quasi-free groups on S have the same Cayley graph, which is a regular tree of degree $|S|$.

Every group Γ generated by a symmetric set S is a quotient of a quasi-free group in the following way: let $\bar{\cdot}$ be an involution on S such that for all $s \in S$ we have the equality $\bar{s} = s^{-1}$ in Γ. Then Γ is a quotient of the quasi-free group $\langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle$.

The *cogrowth series* (respectively *cogrowth*) of (Γ, S) is defined as the growth series (respectively growth) of $\ker(\pi\colon \Pi \to \Gamma)$ relative to (Π, S), where Π is a quasi-free group on S.

Associated with a group Π generated by a set S and a subgroup Ξ of Π, there is a $|S|$-regular graph $\mathcal{X}$ on which Π acts, called the *Schreier graph* of (Π, S) relative to Ξ. It is given by $\mathcal{X} = (V, E)$, with

$$V = \Xi \backslash \Pi$$

and

$$E = V \times S \, , \quad (v,s)^\alpha = v \, , \quad (v,s)^\omega = vs \, , \quad \overline{(v,s)} = (vs, s^{-1}) \, ;$$

i.e. two cosets A, B are joined by at least one edge if and only if $AS \supset B$. (This is the Cayley graph of (Π, S) if $\Xi = 1$.) There is a circuit in $\mathcal{X}$ at every vertex $\Xi v \in \Xi \backslash \Pi$ such that $s \in v^{-1} \Xi v$ for some $s \in S$; and there is a multiple edge from Ξv to Ξw in $\mathcal{X}$ if there are $s, t \in v^{-1} \Xi w$ with $s \neq t \in S$.

COROLLARY 3.2 (of Corollary 2.6). *Let Π be a quasi-free group, presented as a monoid as*

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle .$$

Let $\Xi < \Pi$ be a subgroup of Π. Let $\nu = \nu(\Xi \backslash \Pi, S)$ denote the spectral radius of the simple random walk on $\Xi \backslash \Pi$ generated by S; and $\alpha = \alpha(\Xi; \Pi, S)$ denote the relative growth of Ξ in Π. Then we have

$$\nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|}\left(\frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha}\right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{if } \alpha \leq \sqrt{|S|-1}. \end{cases} \tag{3.1}$$

Proof. Let $\mathcal{X}$ be the Schreier graph of (Π, S) relative to Ξ defined above. Fix the endpoints $\star = \dagger = \Xi$, the coset of 1, and give $\mathcal{X}$ the length labelling. Let G and F be the circuit and proper circuit series of $\mathcal{X}$. In this setting, expressing $F(t) = \sum f_n t^n$ and $G(t) = \sum g_n t^n$, we see that $|S|\nu$ is the growth rate $\limsup \sqrt[n]{g_n}$ of circuits in $\mathcal{X}$, and α the growth rate $\limsup \sqrt[n]{f_n}$ of proper circuits in $\mathcal{X}$. As both F and G are power series with non-negative coefficients, $1/(|S|\nu)$ is the radius of convergence of G and $1/\alpha$ the radius of convergence of F. Let $d = |S|$ and consider the function

$$(t)\phi = \frac{t}{1 + (d-1)t^2} .$$

This function is strictly increasing for $0 \leq t < 1/\sqrt{d-1}$, has a maximum at $t = 1/\sqrt{d-1}$ with $(t)\phi = 1/(2\sqrt{d-1})$, and is strictly decreasing for $t > 1/\sqrt{d-1}$.

First we suppose that $\alpha \geq \sqrt{d-1}$, so ϕ is monotonously increasing on $[0, 1/\alpha]$. We set $u = 1$ in (2.2) and note that, for $t < 1$, it says that F has a singularity at t if and only if G has a singularity at $(t)\phi$. Now as $1/\alpha$ is the singularity of F closest to 0, we conclude by monotonicity of ϕ that the singularity of G closest to 0 is at $(1/\alpha)\phi$; thus

$$\frac{1}{d\nu} = \frac{1/\alpha}{1 + (d-1)/\alpha^2} = (1/\alpha)\phi .$$

Suppose now that $\alpha < \sqrt{d-1}$. If $d\nu < 2\sqrt{d-1}$, the right-hand side of (2.2) would be bounded for all $t \in \mathbf{R}$ while the left-hand side diverges at $t = 1$. If $d\nu > 2\sqrt{d-1}$, there would be a $t \in [0, 1/\sqrt{d-1}[$ with $(t)\phi = d\nu$; and F would have a singularity at $t < 1/\alpha$. The only case left is $d\nu = 2\sqrt{d-1}$. $\square$

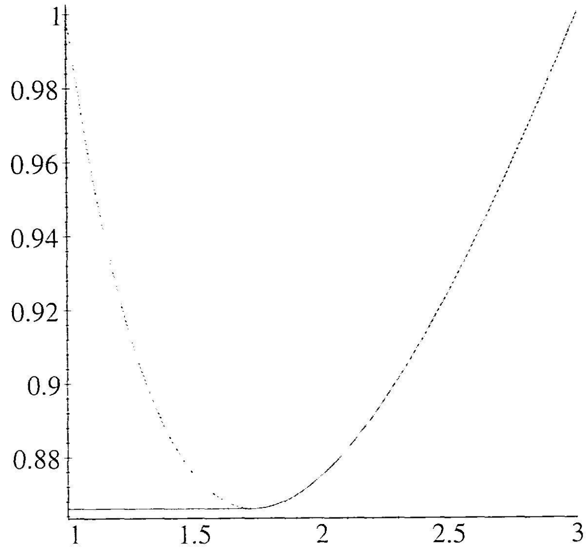

FIGURE 1

The function $\alpha \mapsto \nu$ relating cogrowth and spectral radius (for $d = 4$)

COROLLARY 3.3 (Grigorchuk [Gri78b]). *Let Γ be a group generated by a symmetric finite set S, let ν denote the spectral radius of the simple random walk on Γ, and let α denote the cogrowth of (Γ, S). Then*

$$\nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|}\left(\frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha}\right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{else}. \end{cases} \tag{3.2}$$

A variety of proofs exist for this result: the original [Gri78b] by Grigorchuk, one by Cohen [Coh82], an extension by Northshield to regular graphs [Nor92], a short proof by Szwarc [Szw89] using operator theory, one by Woess [Woe94], etc.

Proof. Present Γ as Π/Ξ, with Π a quasi-free group and Ξ the normal subgroup of Π generated by the relators in Γ, and apply Corollary 3.2. □

We note in passing that if $\alpha < \sqrt{|S|-1}$, then necessarily $\alpha = 0$. Equivalently, we will show that if $\alpha < \sqrt{|S|-1}$, then $\Xi = 1$, so the Cayley graph $\mathcal{X}$ is a tree. Indeed, suppose $\mathcal{X}$ is not a tree, so it contains a circuit λ at $\star$. As $\mathcal{X}$ is transitive, there is a translate of λ at every vertex, which we will still write λ. There are at least $|S|(|S|-1)^{t-2}(|S|-2)$ paths p of length t in $\mathcal{X}$ starting at $\star$ such that the circuit $p\lambda\overline{p}$ is proper; thus

$$\alpha \geq \limsup_{t\to\infty} \sqrt[2t+|\lambda|]{|S|(|S|-1)^{t-2}(|S|-2)} = \sqrt{|S|-1}\,.$$

In fact it is known that $\alpha > \sqrt{|S|-1}$; see [Pas93].