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Finally in Section 9 we show how to compute the circuit series of a free product of graphs (an analogue of the free products of groups, *via* their Cayley graph), and in Section 10 do the same for direct products of graphs.

### 3. APPLICATIONS TO OTHER FIELDS

The original motivation for Formula 2.2 was its implication of a well-known result in the theory of random walks on discrete groups.

#### 3.1 APPLICATIONS TO RANDOM WALKS ON GROUPS

In this section we show how  $G$  is related to random walks and  $F$  to cogrowth. This will give a generalization of the main formula (1.1) to homogeneous spaces  $\Pi/\Xi$ , where  $\Xi$  does not have to be normal and  $\Pi$  is a free product of infinite-cyclic and order-two groups. For a survey on the topic of random walks see [MW89,Woe94].

Throughout this subsection we will have  $F(t) = F(0, t)$ . We recall the notion of growth of groups:

DEFINITION 3.1. Let  $\Gamma$  be a group generated by a finite symmetric set  $S$ . For a  $\gamma \in \Gamma$  define its *length*

$$|\gamma| = \min\{n \in \mathbf{N} : \gamma \in S^n\}.$$

The *growth series* of  $(\Gamma, S)$  is the formal power series

$$f_{(\Gamma, S)}(t) = \sum_{\gamma \in \Gamma} t^{|\gamma|} \in \mathbf{N}[[t]].$$

Expanding  $f_{(\Gamma, S)}(t) = \sum f_n t^n$ , the *growth* of  $(\Gamma, S)$  is

$$\alpha(\Gamma, S) = \limsup_{n \rightarrow \infty} \sqrt[n]{f_n}$$

(this supremum-limit is actually a limit and is smaller than  $|S| - 1$ ).

Let  $R$  be a subset of  $\Gamma$ . The *growth series* of  $R$  relative to  $(\Gamma, S)$  is the formal power series

$$f_{(\Gamma, S)}^R(t) = \sum_{\gamma \in R} t^{|\gamma|} \in \mathbf{N}[[t]].$$

Expanding  $f_{(\Gamma, S)}^R(t) = \sum f_n t^n$ , define the *growth* of  $R$  relative to  $(\Gamma, S)$  as

$$\alpha(R; \Gamma, S) = \limsup_{n \rightarrow \infty} \sqrt[n]{f_n}.$$

If  $X$  is a transitive right  $\Gamma$ -set, the *simple random walk* on  $(X, S)$  is the random walk of a point on  $X$ , having probability  $1/|S|$  of moving from its current position  $x$  to a neighbour  $x \cdot s$ , for all  $s \in S$ . Fix a point  $\star \in X$ , and let  $p_n$  be the probability that a walk starting at  $\star$  finish at  $\star$  after  $n$  moves. We define the *spectral radius* (which does not depend on the choice of  $\star$ ) of the random walk as

$$\nu(X, S) = \limsup_{n \rightarrow \infty} \sqrt[n]{p_n}.$$

A group  $\Pi$  is *quasi-free* if it is a free product of cyclic groups of order 2 and  $\infty$ . Equivalently, there exists a finite set  $S$  and an involution  $\bar{\cdot}: S \rightarrow S$  such that, as a monoid,

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle.$$

$\Pi$  is then said to be *quasi-free on  $S$* . All quasi-free groups on  $S$  have the same Cayley graph, which is a regular tree of degree  $|S|$ .

Every group  $\Gamma$  generated by a symmetric set  $S$  is a quotient of a quasi-free group in the following way: let  $\bar{\cdot}$  be an involution on  $S$  such that for all  $s \in S$  we have the equality  $\bar{s} = s^{-1}$  in  $\Gamma$ . Then  $\Gamma$  is a quotient of the quasi-free group  $\langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle$ .

The *cogrowth series* (respectively *cogrowth*) of  $(\Gamma, S)$  is defined as the growth series (respectively growth) of  $\ker(\pi: \Pi \rightarrow \Gamma)$  relative to  $(\Pi, S)$ , where  $\Pi$  is a quasi-free group on  $S$ .

Associated with a group  $\Pi$  generated by a set  $S$  and a subgroup  $\Xi$  of  $\Pi$ , there is a  $|S|$ -regular graph  $\mathcal{X}$  on which  $\Pi$  acts, called the *Schreier graph* of  $(\Pi, S)$  relative to  $\Xi$ . It is given by  $\mathcal{X} = (V, E)$ , with

$$V = \Xi \backslash \Pi$$

and

$$E = V \times S, \quad (v, s)^\alpha = v, \quad (v, s)^\omega = vs, \quad \overline{(v, s)} = (vs, s^{-1});$$

i.e. two cosets  $A, B$  are joined by at least one edge if and only if  $AS \supset B$ . (This is the Cayley graph of  $(\Pi, S)$  if  $\Xi = 1$ .) There is a circuit in  $\mathcal{X}$  at every vertex  $\Xi v \in \Xi \backslash \Pi$  such that  $s \in v^{-1}\Xi v$  for some  $s \in S$ ; and there is a multiple edge from  $\Xi v$  to  $\Xi w$  in  $\mathcal{X}$  if there are  $s, t \in v^{-1}\Xi w$  with  $s \neq t \in S$ .

COROLLARY 3.2 (of Corollary 2.6). *Let  $\Pi$  be a quasi-free group, presented as a monoid as*

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle .$$

*Let  $\Xi < \Pi$  be a subgroup of  $\Pi$ . Let  $\nu = \nu(\Xi \backslash \Pi, S)$  denote the spectral radius of the simple random walk on  $\Xi \backslash \Pi$  generated by  $S$ ; and  $\alpha = \alpha(\Xi; \Pi, S)$  denote the relative growth of  $\Xi$  in  $\Pi$ . Then we have*

$$(3.1) \quad \nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|} \left( \frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha} \right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{if } \alpha \leq \sqrt{|S|-1}. \end{cases}$$

*Proof.* Let  $\mathcal{X}$  be the Schreier graph of  $(\Pi, S)$  relative to  $\Xi$  defined above. Fix the endpoints  $\star = \dagger = \Xi$ , the coset of 1, and give  $\mathcal{X}$  the length labelling. Let  $G$  and  $F$  be the circuit and proper circuit series of  $\mathcal{X}$ . In this setting, expressing  $F(t) = \sum f_n t^n$  and  $G(t) = \sum g_n t^n$ , we see that  $|S|\nu$  is the growth rate  $\limsup \sqrt[n]{g_n}$  of circuits in  $\mathcal{X}$ , and  $\alpha$  the growth rate  $\limsup \sqrt[n]{f_n}$  of proper circuits in  $\mathcal{X}$ . As both  $F$  and  $G$  are power series with non-negative coefficients,  $1/(|S|\nu)$  is the radius of convergence of  $G$  and  $1/\alpha$  the radius of convergence of  $F$ . Let  $d = |S|$  and consider the function

$$(t)\phi = \frac{t}{1 + (d-1)t^2} .$$

This function is strictly increasing for  $0 \leq t < 1/\sqrt{d-1}$ , has a maximum at  $t = 1/\sqrt{d-1}$  with  $(t)\phi = 1/(2\sqrt{d-1})$ , and is strictly decreasing for  $t > 1/\sqrt{d-1}$ .

First we suppose that  $\alpha \geq \sqrt{d-1}$ , so  $\phi$  is monotonously increasing on  $[0, 1/\alpha]$ . We set  $u = 1$  in (2.2) and note that, for  $t < 1$ , it says that  $F$  has a singularity at  $t$  if and only if  $G$  has a singularity at  $(t)\phi$ . Now as  $1/\alpha$  is the singularity of  $F$  closest to 0, we conclude by monotonicity of  $\phi$  that the singularity of  $G$  closest to 0 is at  $(1/\alpha)\phi$ ; thus

$$\frac{1}{d\nu} = \frac{1/\alpha}{1 + (d-1)/\alpha^2} = (1/\alpha)\phi .$$

Suppose now that  $\alpha < \sqrt{d-1}$ . If  $d\nu < 2\sqrt{d-1}$ , the right-hand side of (2.2) would be bounded for all  $t \in \mathbf{R}$  while the left-hand side diverges at  $t = 1$ . If  $d\nu > 2\sqrt{d-1}$ , there would be a  $t \in [0, 1/\sqrt{d-1}[$  with  $(t)\phi = d\nu$ ; and  $F$  would have a singularity at  $t < 1/\alpha$ . The only case left is  $d\nu = 2\sqrt{d-1}$ .  $\square$

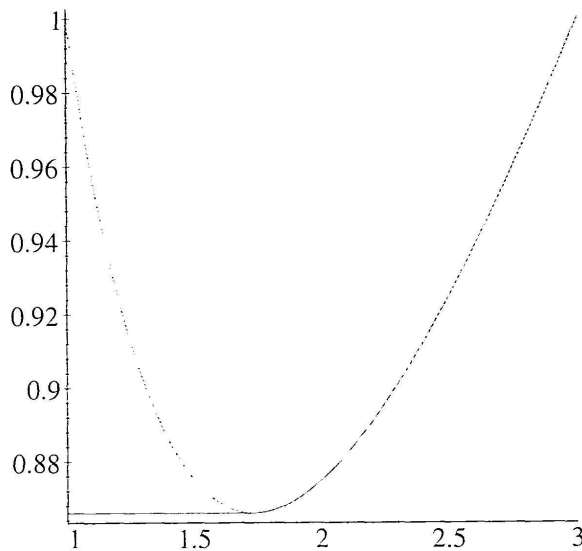


FIGURE 1

The function  $\alpha \mapsto \nu$  relating cogrowth and spectral radius (for  $d = 4$ )

COROLLARY 3.3 (Grigorchuk [Gri78b]). *Let  $\Gamma$  be a group generated by a symmetric finite set  $S$ , let  $\nu$  denote the spectral radius of the simple random walk on  $\Gamma$ , and let  $\alpha$  denote the cogrowth of  $(\Gamma, S)$ . Then*

$$(3.2) \quad \nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|} \left( \frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha} \right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{else.} \end{cases}$$

A variety of proofs exist for this result: the original [Gri78b] by Grigorchuk, one by Cohen [Coh82], an extension by Northshield to regular graphs [Nor92], a short proof by Szwarc [Szw89] using operator theory, one by Woess [Woe94], etc.

*Proof.* Present  $\Gamma$  as  $\Pi/\Xi$ , with  $\Pi$  a quasi-free group and  $\Xi$  the normal subgroup of  $\Pi$  generated by the relators in  $\Gamma$ , and apply Corollary 3.2.  $\square$

We note in passing that if  $\alpha < \sqrt{|S|-1}$ , then necessarily  $\alpha = 0$ . Equivalently, we will show that if  $\alpha < \sqrt{|S|-1}$ , then  $\Xi = 1$ , so the Cayley graph  $\mathcal{X}$  is a tree. Indeed, suppose  $\mathcal{X}$  is not a tree, so it contains a circuit  $\lambda$  at  $\star$ . As  $\mathcal{X}$  is transitive, there is a translate of  $\lambda$  at every vertex, which we will still write  $\lambda$ . There are at least  $|S|(|S|-1)^{t-2}(|S|-2)$  paths  $p$  of length  $t$  in  $\mathcal{X}$  starting at  $\star$  such that the circuit  $p\lambda\bar{p}$  is proper; thus

$$\alpha \geq \limsup_{t \rightarrow \infty} {}^{2t+|\lambda|} \sqrt{|S|(|S|-1)^{t-2}(|S|-2)} = \sqrt{|S|-1}.$$

In fact it is known that  $\alpha > \sqrt{|S|-1}$ ; see [Pas93].