

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 45 (1999)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** COUNTING PATHS IN GRAPHS  
**Autor:** Bartholdi, Laurent  
**Kapitel:** 3.1 Applications to random walks on groups  
**DOI:** <https://doi.org/10.5169/seals-64442>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 15.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

Finally in Section 9 we show how to compute the circuit series of a free product of graphs (an analogue of the free products of groups, *via* their Cayley graph), and in Section 10 do the same for direct products of graphs.

### 3. APPLICATIONS TO OTHER FIELDS

The original motivation for Formula 2.2 was its implication of a well-known result in the theory of random walks on discrete groups.

#### 3.1 APPLICATIONS TO RANDOM WALKS ON GROUPS

In this section we show how  $G$  is related to random walks and  $F$  to cogrowth. This will give a generalization of the main formula (1.1) to homogeneous spaces  $\Pi/\Xi$ , where  $\Xi$  does not have to be normal and  $\Pi$  is a free product of infinite-cyclic and order-two groups. For a survey on the topic of random walks see [MW89, Woe94].

Throughout this subsection we will have  $F(t) = F(0, t)$ . We recall the notion of growth of groups:

DEFINITION 3.1. Let  $\Gamma$  be a group generated by a finite symmetric set  $S$ . For a  $\gamma \in \Gamma$  define its *length*

$$|\gamma| = \min\{n \in \mathbf{N} : \gamma \in S^n\}.$$

The *growth series* of  $(\Gamma, S)$  is the formal power series

$$f_{(\Gamma, S)}(t) = \sum_{\gamma \in \Gamma} t^{|\gamma|} \in \mathbf{N}[[t]].$$

Expanding  $f_{(\Gamma, S)}(t) = \sum f_n t^n$ , the *growth* of  $(\Gamma, S)$  is

$$\alpha(\Gamma, S) = \limsup_{n \rightarrow \infty} \sqrt[n]{f_n}$$

(this supremum-limit is actually a limit and is smaller than  $|S| - 1$ ).

Let  $R$  be a subset of  $\Gamma$ . The *growth series* of  $R$  relative to  $(\Gamma, S)$  is the formal power series

$$f_{(\Gamma, S)}^R(t) = \sum_{\gamma \in R} t^{|\gamma|} \in \mathbf{N}[[t]].$$

Expanding  $f_{(\Gamma, S)}^R(t) = \sum f_n^R t^n$ , define the *growth* of  $R$  relative to  $(\Gamma, S)$  as

$$\alpha(R; \Gamma, S) = \limsup_{n \rightarrow \infty} \sqrt[n]{f_n^R}.$$

If  $X$  is a transitive right  $\Gamma$ -set, the *simple random walk* on  $(X, S)$  is the random walk of a point on  $X$ , having probability  $1/|S|$  of moving from its current position  $x$  to a neighbour  $x \cdot s$ , for all  $s \in S$ . Fix a point  $\star \in X$ , and let  $p_n$  be the probability that a walk starting at  $\star$  finish at  $\star$  after  $n$  moves. We define the *spectral radius* (which does not depend on the choice of  $\star$ ) of the random walk as

$$\nu(X, S) = \limsup_{n \rightarrow \infty} \sqrt[n]{p_n}.$$

A group  $\Pi$  is *quasi-free* if it is a free product of cyclic groups of order 2 and  $\infty$ . Equivalently, there exists a finite set  $S$  and an involution  $\bar{\cdot}: S \rightarrow S$  such that, as a monoid,

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle.$$

$\Pi$  is then said to be *quasi-free on  $S$* . All quasi-free groups on  $S$  have the same Cayley graph, which is a regular tree of degree  $|S|$ .

Every group  $\Gamma$  generated by a symmetric set  $S$  is a quotient of a quasi-free group in the following way: let  $\bar{\cdot}$  be an involution on  $S$  such that for all  $s \in S$  we have the equality  $\bar{s} = s^{-1}$  in  $\Gamma$ . Then  $\Gamma$  is a quotient of the quasi-free group  $\langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle$ .

The *cogrowth series* (respectively *cogrowth*) of  $(\Gamma, S)$  is defined as the growth series (respectively growth) of  $\ker(\pi: \Pi \rightarrow \Gamma)$  relative to  $(\Pi, S)$ , where  $\Pi$  is a quasi-free group on  $S$ .

Associated with a group  $\Pi$  generated by a set  $S$  and a subgroup  $\Xi$  of  $\Pi$ , there is a  $|S|$ -regular graph  $\mathcal{X}$  on which  $\Pi$  acts, called the *Schreier graph* of  $(\Pi, S)$  relative to  $\Xi$ . It is given by  $\mathcal{X} = (V, E)$ , with

$$V = \Xi \backslash \Pi$$

and

$$E = V \times S, \quad (v, s)^\alpha = v, \quad (v, s)^\omega = vs, \quad \overline{(v, s)} = (vs, s^{-1});$$

i.e. two cosets  $A, B$  are joined by at least one edge if and only if  $AS \supset B$ . (This is the Cayley graph of  $(\Pi, S)$  if  $\Xi = 1$ .) There is a circuit in  $\mathcal{X}$  at every vertex  $\Xi v \in \Xi \backslash \Pi$  such that  $s \in v^{-1}\Xi v$  for some  $s \in S$ ; and there is a multiple edge from  $\Xi v$  to  $\Xi w$  in  $\mathcal{X}$  if there are  $s, t \in v^{-1}\Xi w$  with  $s \neq t \in S$ .

COROLLARY 3.2 (of Corollary 2.6). *Let  $\Pi$  be a quasi-free group, presented as a monoid as*

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle.$$

*Let  $\Xi < \Pi$  be a subgroup of  $\Pi$ . Let  $\nu = \nu(\Xi \backslash \Pi, S)$  denote the spectral radius of the simple random walk on  $\Xi \backslash \Pi$  generated by  $S$ ; and  $\alpha = \alpha(\Xi; \Pi, S)$  denote the relative growth of  $\Xi$  in  $\Pi$ . Then we have*

$$(3.1) \quad \nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|} \left( \frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha} \right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{if } \alpha \leq \sqrt{|S|-1}. \end{cases}$$

*Proof.* Let  $\mathcal{X}$  be the Schreier graph of  $(\Pi, S)$  relative to  $\Xi$  defined above. Fix the endpoints  $\star = \dagger = \Xi$ , the coset of 1, and give  $\mathcal{X}$  the length labelling. Let  $G$  and  $F$  be the circuit and proper circuit series of  $\mathcal{X}$ . In this setting, expressing  $F(t) = \sum f_n t^n$  and  $G(t) = \sum g_n t^n$ , we see that  $|S|\nu$  is the growth rate  $\limsup \sqrt[n]{g_n}$  of circuits in  $\mathcal{X}$ , and  $\alpha$  the growth rate  $\limsup \sqrt[n]{f_n}$  of proper circuits in  $\mathcal{X}$ . As both  $F$  and  $G$  are power series with non-negative coefficients,  $1/(|S|\nu)$  is the radius of convergence of  $G$  and  $1/\alpha$  the radius of convergence of  $F$ . Let  $d = |S|$  and consider the function

$$(t)\phi = \frac{t}{1 + (d-1)t^2}.$$

This function is strictly increasing for  $0 \leq t < 1/\sqrt{d-1}$ , has a maximum at  $t = 1/\sqrt{d-1}$  with  $(t)\phi = 1/(2\sqrt{d-1})$ , and is strictly decreasing for  $t > 1/\sqrt{d-1}$ .

First we suppose that  $\alpha \geq \sqrt{d-1}$ , so  $\phi$  is monotonously increasing on  $[0, 1/\alpha]$ . We set  $u = 1$  in (2.2) and note that, for  $t < 1$ , it says that  $F$  has a singularity at  $t$  if and only if  $G$  has a singularity at  $(t)\phi$ . Now as  $1/\alpha$  is the singularity of  $F$  closest to 0, we conclude by monotonicity of  $\phi$  that the singularity of  $G$  closest to 0 is at  $(1/\alpha)\phi$ ; thus

$$\frac{1}{d\nu} = \frac{1/\alpha}{1 + (d-1)/\alpha^2} = (1/\alpha)\phi.$$

Suppose now that  $\alpha < \sqrt{d-1}$ . If  $d\nu < 2\sqrt{d-1}$ , the right-hand side of (2.2) would be bounded for all  $t \in \mathbf{R}$  while the left-hand side diverges at  $t = 1$ . If  $d\nu > 2\sqrt{d-1}$ , there would be a  $t \in [0, 1/\sqrt{d-1}[$  with  $(t)\phi = d\nu$ ; and  $F$  would have a singularity at  $t < 1/\alpha$ . The only case left is  $d\nu = 2\sqrt{d-1}$ .  $\square$

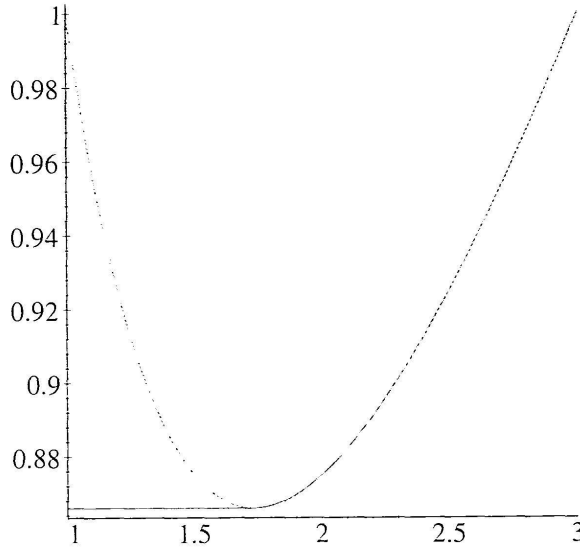


FIGURE 1

The function  $\alpha \mapsto \nu$  relating cogrowth and spectral radius (for  $d = 4$ )

**COROLLARY 3.3** (Grigorchuk [Gri78b]). *Let  $\Gamma$  be a group generated by a symmetric finite set  $S$ , let  $\nu$  denote the spectral radius of the simple random walk on  $\Gamma$ , and let  $\alpha$  denote the cogrowth of  $(\Gamma, S)$ . Then*

$$(3.2) \quad \nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|} \left( \frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha} \right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{else.} \end{cases}$$

A variety of proofs exist for this result: the original [Gri78b] by Grigorchuk, one by Cohen [Coh82], an extension by Northshield to regular graphs [Nor92], a short proof by Szwarc [Szw89] using operator theory, one by Woess [Woe94], etc.

*Proof.* Present  $\Gamma$  as  $\Pi/\Xi$ , with  $\Pi$  a quasi-free group and  $\Xi$  the normal subgroup of  $\Pi$  generated by the relators in  $\Gamma$ , and apply Corollary 3.2.  $\square$

We note in passing that if  $\alpha < \sqrt{|S|-1}$ , then necessarily  $\alpha = 0$ . Equivalently, we will show that if  $\alpha < \sqrt{|S|-1}$ , then  $\Xi = 1$ , so the Cayley graph  $\mathcal{X}$  is a tree. Indeed, suppose  $\mathcal{X}$  is not a tree, so it contains a circuit  $\lambda$  at  $\star$ . As  $\mathcal{X}$  is transitive, there is a translate of  $\lambda$  at every vertex, which we will still write  $\lambda$ . There are at least  $|S|(|S|-1)^{t-2}(|S|-2)$  paths  $p$  of length  $t$  in  $\mathcal{X}$  starting at  $\star$  such that the circuit  $p\lambda\bar{p}$  is proper; thus

$$\alpha \geq \limsup_{t \rightarrow \infty} \sqrt[2t+|\lambda|]{|S|(|S|-1)^{t-2}(|S|-2)} = \sqrt{|S|-1}.$$

In fact it is known that  $\alpha > \sqrt{|S|-1}$ ; see [Pas93].