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Finally in Section 9 we show how to compute the circuit series of a free product of graphs (an analogue of the free products of groups, *via* their Cayley graph), and in Section 10 do the same for direct products of graphs.

### 3. APPLICATIONS TO OTHER FIELDS

The original motivation for Formula 2.2 was its implication of a well-known result in the theory of random walks on discrete groups.

#### 3.1 APPLICATIONS TO RANDOM WALKS ON GROUPS

In this section we show how  $G$  is related to random walks and  $F$  to cogrowth. This will give a generalization of the main formula (1.1) to homogeneous spaces  $\Pi/\Xi$ , where  $\Xi$  does not have to be normal and  $\Pi$  is a free product of infinite-cyclic and order-two groups. For a survey on the topic of random walks see [MW89,Woe94].

Throughout this subsection we will have  $F(t) = F(0, t)$ . We recall the notion of growth of groups:

DEFINITION 3.1. Let  $\Gamma$  be a group generated by a finite symmetric set  $S$ . For a  $\gamma \in \Gamma$  define its *length*

$$|\gamma| = \min\{n \in \mathbf{N} : \gamma \in S^n\}.$$

The *growth series* of  $(\Gamma, S)$  is the formal power series

$$f_{(\Gamma, S)}(t) = \sum_{\gamma \in \Gamma} t^{|\gamma|} \in \mathbf{N}[[t]].$$

Expanding  $f_{(\Gamma, S)}(t) = \sum f_n t^n$ , the *growth* of  $(\Gamma, S)$  is

$$\alpha(\Gamma, S) = \limsup_{n \rightarrow \infty} \sqrt[n]{f_n}$$

(this supremum-limit is actually a limit and is smaller than  $|S| - 1$ ).

Let  $R$  be a subset of  $\Gamma$ . The *growth series* of  $R$  relative to  $(\Gamma, S)$  is the formal power series

$$f_{(\Gamma, S)}^R(t) = \sum_{\gamma \in R} t^{|\gamma|} \in \mathbf{N}[[t]].$$

Expanding  $f_{(\Gamma, S)}^R(t) = \sum f_n t^n$ , define the *growth* of  $R$  relative to  $(\Gamma, S)$  as

$$\alpha(R; \Gamma, S) = \limsup_{n \rightarrow \infty} \sqrt[n]{f_n}.$$

If  $X$  is a transitive right  $\Gamma$ -set, the *simple random walk* on  $(X, S)$  is the random walk of a point on  $X$ , having probability  $1/|S|$  of moving from its current position  $x$  to a neighbour  $x \cdot s$ , for all  $s \in S$ . Fix a point  $\star \in X$ , and let  $p_n$  be the probability that a walk starting at  $\star$  finish at  $\star$  after  $n$  moves. We define the *spectral radius* (which does not depend on the choice of  $\star$ ) of the random walk as

$$\nu(X, S) = \limsup_{n \rightarrow \infty} \sqrt[n]{p_n}.$$

A group  $\Pi$  is *quasi-free* if it is a free product of cyclic groups of order 2 and  $\infty$ . Equivalently, there exists a finite set  $S$  and an involution  $\bar{\cdot}: S \rightarrow S$  such that, as a monoid,

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle.$$

$\Pi$  is then said to be *quasi-free on  $S$* . All quasi-free groups on  $S$  have the same Cayley graph, which is a regular tree of degree  $|S|$ .

Every group  $\Gamma$  generated by a symmetric set  $S$  is a quotient of a quasi-free group in the following way: let  $\bar{\cdot}$  be an involution on  $S$  such that for all  $s \in S$  we have the equality  $\bar{s} = s^{-1}$  in  $\Gamma$ . Then  $\Gamma$  is a quotient of the quasi-free group  $\langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle$ .

The *cogrowth series* (respectively *cogrowth*) of  $(\Gamma, S)$  is defined as the growth series (respectively growth) of  $\ker(\pi: \Pi \rightarrow \Gamma)$  relative to  $(\Pi, S)$ , where  $\Pi$  is a quasi-free group on  $S$ .

Associated with a group  $\Pi$  generated by a set  $S$  and a subgroup  $\Xi$  of  $\Pi$ , there is a  $|S|$ -regular graph  $\mathcal{X}$  on which  $\Pi$  acts, called the *Schreier graph* of  $(\Pi, S)$  relative to  $\Xi$ . It is given by  $\mathcal{X} = (V, E)$ , with

$$V = \Xi \backslash \Pi$$

and

$$E = V \times S, \quad (v, s)^\alpha = v, \quad (v, s)^\omega = vs, \quad \overline{(v, s)} = (vs, s^{-1});$$

i.e. two cosets  $A, B$  are joined by at least one edge if and only if  $AS \supset B$ . (This is the Cayley graph of  $(\Pi, S)$  if  $\Xi = 1$ .) There is a circuit in  $\mathcal{X}$  at every vertex  $\Xi v \in \Xi \backslash \Pi$  such that  $s \in v^{-1}\Xi v$  for some  $s \in S$ ; and there is a multiple edge from  $\Xi v$  to  $\Xi w$  in  $\mathcal{X}$  if there are  $s, t \in v^{-1}\Xi w$  with  $s \neq t \in S$ .

COROLLARY 3.2 (of Corollary 2.6). *Let  $\Pi$  be a quasi-free group, presented as a monoid as*

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle .$$

*Let  $\Xi < \Pi$  be a subgroup of  $\Pi$ . Let  $\nu = \nu(\Xi \backslash \Pi, S)$  denote the spectral radius of the simple random walk on  $\Xi \backslash \Pi$  generated by  $S$ ; and  $\alpha = \alpha(\Xi; \Pi, S)$  denote the relative growth of  $\Xi$  in  $\Pi$ . Then we have*

$$(3.1) \quad \nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|} \left( \frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha} \right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{if } \alpha \leq \sqrt{|S|-1}. \end{cases}$$

*Proof.* Let  $\mathcal{X}$  be the Schreier graph of  $(\Pi, S)$  relative to  $\Xi$  defined above. Fix the endpoints  $\star = \dagger = \Xi$ , the coset of 1, and give  $\mathcal{X}$  the length labelling. Let  $G$  and  $F$  be the circuit and proper circuit series of  $\mathcal{X}$ . In this setting, expressing  $F(t) = \sum f_n t^n$  and  $G(t) = \sum g_n t^n$ , we see that  $|S|\nu$  is the growth rate  $\limsup \sqrt[n]{g_n}$  of circuits in  $\mathcal{X}$ , and  $\alpha$  the growth rate  $\limsup \sqrt[n]{f_n}$  of proper circuits in  $\mathcal{X}$ . As both  $F$  and  $G$  are power series with non-negative coefficients,  $1/(|S|\nu)$  is the radius of convergence of  $G$  and  $1/\alpha$  the radius of convergence of  $F$ . Let  $d = |S|$  and consider the function

$$(t)\phi = \frac{t}{1 + (d-1)t^2} .$$

This function is strictly increasing for  $0 \leq t < 1/\sqrt{d-1}$ , has a maximum at  $t = 1/\sqrt{d-1}$  with  $(t)\phi = 1/(2\sqrt{d-1})$ , and is strictly decreasing for  $t > 1/\sqrt{d-1}$ .

First we suppose that  $\alpha \geq \sqrt{d-1}$ , so  $\phi$  is monotonously increasing on  $[0, 1/\alpha]$ . We set  $u = 1$  in (2.2) and note that, for  $t < 1$ , it says that  $F$  has a singularity at  $t$  if and only if  $G$  has a singularity at  $(t)\phi$ . Now as  $1/\alpha$  is the singularity of  $F$  closest to 0, we conclude by monotonicity of  $\phi$  that the singularity of  $G$  closest to 0 is at  $(1/\alpha)\phi$ ; thus

$$\frac{1}{d\nu} = \frac{1/\alpha}{1 + (d-1)/\alpha^2} = (1/\alpha)\phi .$$

Suppose now that  $\alpha < \sqrt{d-1}$ . If  $d\nu < 2\sqrt{d-1}$ , the right-hand side of (2.2) would be bounded for all  $t \in \mathbf{R}$  while the left-hand side diverges at  $t = 1$ . If  $d\nu > 2\sqrt{d-1}$ , there would be a  $t \in [0, 1/\sqrt{d-1}[$  with  $(t)\phi = d\nu$ ; and  $F$  would have a singularity at  $t < 1/\alpha$ . The only case left is  $d\nu = 2\sqrt{d-1}$ .  $\square$

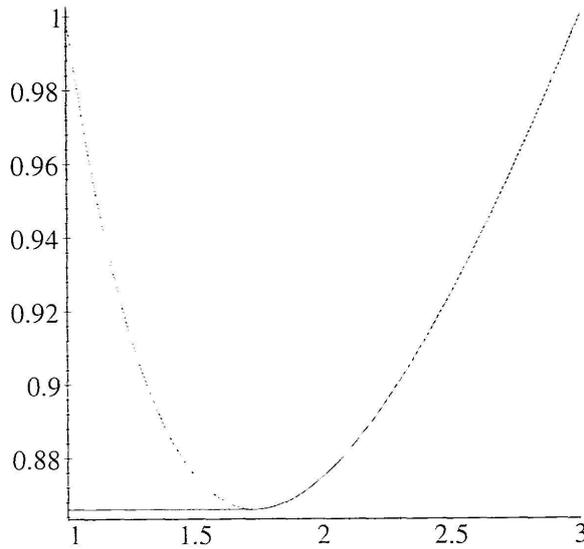


FIGURE 1

The function  $\alpha \mapsto \nu$  relating cogrowth and spectral radius (for  $d = 4$ )

COROLLARY 3.3 (Grigorchuk [Gri78b]). *Let  $\Gamma$  be a group generated by a symmetric finite set  $S$ , let  $\nu$  denote the spectral radius of the simple random walk on  $\Gamma$ , and let  $\alpha$  denote the cogrowth of  $(\Gamma, S)$ . Then*

$$(3.2) \quad \nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|} \left( \frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha} \right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{else.} \end{cases}$$

A variety of proofs exist for this result: the original [Gri78b] by Grigorchuk, one by Cohen [Coh82], an extension by Northshield to regular graphs [Nor92], a short proof by Szwarc [Szw89] using operator theory, one by Woess [Woe94], etc.

*Proof.* Present  $\Gamma$  as  $\Pi/\Xi$ , with  $\Pi$  a quasi-free group and  $\Xi$  the normal subgroup of  $\Pi$  generated by the relators in  $\Gamma$ , and apply Corollary 3.2.  $\square$

We note in passing that if  $\alpha < \sqrt{|S|-1}$ , then necessarily  $\alpha = 0$ . Equivalently, we will show that if  $\alpha < \sqrt{|S|-1}$ , then  $\Xi = 1$ , so the Cayley graph  $\mathcal{X}$  is a tree. Indeed, suppose  $\mathcal{X}$  is not a tree, so it contains a circuit  $\lambda$  at  $\star$ . As  $\mathcal{X}$  is transitive, there is a translate of  $\lambda$  at every vertex, which we will still write  $\lambda$ . There are at least  $|S|(|S|-1)^{t-2}(|S|-2)$  paths  $p$  of length  $t$  in  $\mathcal{X}$  starting at  $\star$  such that the circuit  $p\lambda\bar{p}$  is proper; thus

$$\alpha \geq \limsup_{t \rightarrow \infty} {}^{2t+|\lambda|} \sqrt{|S|(|S|-1)^{t-2}(|S|-2)} = \sqrt{|S|-1}.$$

In fact it is known that  $\alpha > \sqrt{|S|-1}$ ; see [Pas93].

### 3.2 THE SERIES $F$ AND $G$ ON THEIR CIRCLE OF CONVERGENCE

In this subsection we study the singularities the series  $F$  and  $G$  may have on their circle of convergence. The smallest positive real singularity has a special importance:

DEFINITION 3.4. For a series  $f(t)$  with positive coefficients, let  $\rho(f)$  denote its radius of convergence. Then  $f$  is  $\rho(f)$ -recurrent if

$$\lim_{t \rightarrow \rho(f)} f(t) = \infty .$$

Otherwise, it is  $\rho(f)$ -transient.

As typical examples,  $1/(\rho - t)$  is  $\rho$ -recurrent, as are all rational series;  $\sqrt{\rho - t}$  is  $\rho$ -transient, while  $1/\sqrt{\rho - t}$  is not.

To study the singularities of  $F$  or  $G$ , we may suppose that  $\star = \dagger$ ; indeed in was shown in [Kes59] and [Woe83, Lemma 1] that the singularities of  $F$  and  $G$  do not depend on the choice of  $\star$  and  $\dagger$ . We make that assumption for the remainder of the subsection. We will also suppose throughout that  $\mathcal{X}$  is  $d$ -regular, that the radius of convergence of  $F$  is  $1/\alpha$  and the radius of convergence of  $G$  is  $1/(d\nu) = 1/\beta$ .

DEFINITION 3.5. Let  $\mathcal{X}$  be a connected graph. A *proper cycle* in  $\mathcal{X}$  is a proper circuit  $(\pi_1, \dots, \pi_n)$  such that  $\overline{\pi_1} \neq \pi_n$ . The *proper period*  $p$  and *strong proper period*  $p_s$  are defined as follows:

$$p = \gcd\{n \mid \text{there exists a proper cycle } \pi \text{ in } \mathcal{X} \text{ with } |\pi| = n\},$$

$$p_s = \gcd\{n \mid \forall x \in V(\mathcal{X}) \text{ there exists} \\ \text{a proper cycle } \pi \text{ in } \mathcal{B}(x, n) \text{ with } |\pi| = n\},$$

where by convention the gcd of the empty set is  $\infty$ . The graph  $\mathcal{X}$  is *strongly properly periodic* if  $p = p_s$ .

The *period*  $q$  and *strong period*  $q_s$  of  $\mathcal{X}$  are defined analogously with ‘proper cycle’ replaced by ‘circuit’.  $\mathcal{X}$  is *strongly periodic* if  $q = q_s$ .

THEOREM 3.6 (Cartwright [Car92]). *Let  $\mathcal{X}$  have proper period  $p$  and strong proper period  $p_s$ . Then the singularities of  $F$  on its circle of convergence are among the*

$$\frac{e^{2i\pi k/p_s}}{\alpha}, \quad k = 1, \dots, p_s .$$

If moreover  $\mathcal{X}$  is strongly properly periodic, the singularities of  $F$  on its circle of convergence are precisely these numbers.

Let  $\mathcal{X}$  have period  $q$  and strong period  $q_s$ . Then the singularities of  $G$  on its circle of convergence are among the

$$\frac{e^{2i\pi k/q_s}}{\beta}, \quad k = 1, \dots, q_s.$$

If moreover  $\mathcal{X}$  is strongly periodic, the singularities of  $G$  on its circle of convergence are precisely these numbers.

If  $\mathcal{X}$  is connected and non-trivial, there is a path of even length at every vertex (a sequence of bumps, for instance). All graphs are then either 2-periodic (if they are bipartite) or 1-periodic. If there is a constant  $N$  such that for all  $x \in V(\mathcal{X})$  there is at  $x$  a circuit of odd length bounded by  $N$ , then  $\mathcal{X}$  is strongly 1-periodic; otherwise  $\mathcal{X}$  is strongly 2-periodic. The singularities of  $G$  on its circle of convergence are then at  $1/\beta$ , and also at  $-1/\beta$  if  $\mathcal{X}$  is strongly periodic with period 2.

If  $\mathcal{X}$  is not strongly periodic, there may be one or two singularities on  $G$ 's circle of convergence; consider for instance the 4-regular tree, and at a vertex  $\star$  delete two or three edges replacing them by loops. The resulting graphs  $\mathcal{X}_2$  and  $\mathcal{X}_3$  are still 4-regular and their circuit series, as computed using (7.2), are respectively

$$(3.3) \quad \begin{aligned} G_2(t) &= \frac{3}{2 - 6t + \sqrt{1 - 12t^2}}, \\ G_3(t) &= \frac{6}{5 - 18t + \sqrt{1 - 12t^2}}. \end{aligned}$$

$G_2$  has singularities at  $\pm 1/\sqrt{12}$  on its circle of convergence, while  $G_3$  has only  $2/7$  as singularity on its circle of convergence.

Following the proof of Corollary 3.2 above, we see that if  $\beta < d$  the singularities of  $F$  on its circle of convergence are in bijection with those of  $G$ , so are at  $1/\alpha$  and possibly  $-1/\alpha$ , if  $\mathcal{X}$  is strongly two-periodic. If  $\beta = d$ , though,  $\mathcal{X}$  can have any strong proper period; consider for example the cycles on length  $k$  studied in Section 7.2: they are strongly properly  $k$ -periodic.

The forthcoming simple result shows how  $\mathcal{X}$  can be approximated by finite subgraphs.

LEMMA 3.7. Let  $\mathcal{X}$  be a graph and  $x, y$  two vertices in  $\mathcal{X}$ . Let  $\mathfrak{G}_{x,y}$  and  $\mathfrak{F}_{x,y}$  be the path series and enriched path series respectively from  $x$  to  $y$  in  $\mathcal{X}$ , and let  $\mathfrak{G}_{x,y}^n$  and  $\mathfrak{F}_{x,y}^n$  be the path series and enriched path series respectively from  $x$  to  $y$  in the ball  $\mathcal{B}(x, n)$  (they are 0 if  $y \notin \mathcal{B}(x, n)$ ). Then

$$\lim_{n \rightarrow \infty} \mathfrak{G}_{x,y}^n = \mathfrak{G}_{x,y}, \quad \lim_{n \rightarrow \infty} \mathfrak{F}_{x,y}^n = \mathfrak{F}_{x,y}.$$

*Proof.* Recall that  $\lim \mathfrak{G}_{x,y}^n = \mathfrak{G}_{x,y}$  means that both terms are sums of paths, say  $A_n$  and  $A$ , such that the minimal length of paths in the symmetric difference  $A_n \triangle A$  tends to infinity. Now the difference between  $\mathfrak{G}_{x,y}^n$  and  $\mathfrak{G}_{x,y}$  consists only of paths in  $\mathcal{X}$  that exit  $\mathcal{B}(x, n)$ , and thus have length at least  $2n - \delta(x, y) \rightarrow \infty$ . The same argument holds for  $\mathfrak{F}$ .  $\square$

DEFINITION 3.8. The graph  $\mathcal{X}$  is *quasi-transitive* if  $\text{Aut}(\mathcal{X})$  acts with finitely many orbits.

LEMMA 3.9. Let  $\mathcal{X}$  be a regular quasi-transitive connected graph with distinguished vertex  $\star$ , and let  $f_n$  and  $g_n$  denote respectively the number of proper circuits and circuits at  $\star$  of length  $n$ . Then

$$\limsup_{n \rightarrow \infty} g_n / \beta^n = \limsup_{n \rightarrow \infty} f_n / \alpha^n = \begin{cases} 1/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite and has odd circuits;} \\ 2/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite} \\ & \text{and has only even circuits;} \\ 0 & \text{if } \mathcal{X} \text{ is infinite.} \end{cases}$$

*Proof.* If  $\mathcal{X}$  is finite, then  $\beta = d$ , the degree of  $\mathcal{X}$ ; after a large even number of steps, a random walk starting at  $\star$  will be uniformly distributed over  $\mathcal{X}$  (or over the vertices at even distance of  $\star$ , in case all circuits have even length). A long enough walk then has probability  $1/|\mathcal{X}|$  (or  $2/|\mathcal{X}|$  if all circuits have even length) of being a circuit.

If  $\mathcal{X}$  is infinite, we consider two cases. If  $G(1/\beta) < \infty$ , i.e.  $G$  is  $1/\beta$ -transient, the general term  $g_n/\beta^n$  of the series  $G(1/\beta)$  tends to 0. If  $G$  is  $1/\beta$ -recurrent, then, as  $\mathcal{X}$  is quasi-transitive,  $\beta = d$  by [Woe98, Theorem 7.7]. We then approximate  $\mathcal{X}$  by the sequence of its balls of radius  $R$ , by Lemma 3.7:

$$\lim_{n \rightarrow \infty} \frac{g_n}{\beta^n} = \lim_{R, n \rightarrow \infty} \frac{g_{R,n}}{d^n} = \lim_{R \rightarrow \infty} \frac{(1 \text{ or } 2)}{|\mathcal{B}(\star, R)|} = 0,$$

where we expand the circuit series of  $\mathcal{B}(\star, R)$  as  $\sum g_{R,n} t^n$ .

The same proof holds for the  $f_n$ . Its particular case where  $\mathcal{X}$  is a Cayley graph appears in [Woe83].  $\square$

Note that if  $\mathcal{X}$  is not quasi-transitive, a somewhat weaker result holds [Kit98, §7.1]: if  $\mathcal{X}$  is transient or null-recurrent then the common limsup is 0. If  $\mathcal{X}$  is positive-recurrent then the limsups are normalized coefficients of  $\mathcal{X}$ 's Perron-Frobenius eigenvector. Lemma 3.9 is not true for arbitrary  $d$ -regular graphs: consider for instance the graph  $\mathcal{X}_3$  described above. Its circuit series  $G_3$ , given in (3.3), has radius of convergence  $1/\beta = 2/7$ , and one easily checks that all its coefficients  $g_n$  satisfy  $g_n/\beta^n \geq 1/2$ .

We obtain the following characterization of rational series:

**THEOREM 3.10.** *For regular quasi-transitive connected graphs  $\mathcal{X}$ , the following are equivalent:*

1.  $\mathcal{X}$  is finite;
2.  $G(t)$  is a rational function of  $t$ ;
3.  $F(t)$  is a rational function of  $t$ , and  $\mathcal{X}$  is not an infinite tree.

*Proof.* By Corollary 2.7, Statement 1 implies the other two. By Corollary 2.6, and a computation on trees done in Section 7.3 to deal with the case  $F(t) = 1$ , Statement 2 implies 3. It remains to show that Statement 3 implies 1.

Assume that  $F(t) = \sum f_n t^n$  is rational, not equal to 1. As the  $f_n$  are positive,  $F$  has a pole, of multiplicity  $m$ , at  $1/\alpha$ . There is then a constant  $a > 0$  such that  $f_n > a \binom{n}{m-1} \alpha^n$  for infinitely many values of  $n$  [GKP94, page 341]. It follows by Lemma 3.9 that  $m = 1$  and the graph  $\mathcal{X}$  is finite, of cardinality at most  $1/a$ .  $\square$

It is not known whether the same holds for regular, or even arbitrary connected graphs. Certainly an altogether different proof would be needed.

### 3.3 APPLICATION TO LANGUAGES

Let  $S$  be a finite set of cardinality  $d$  and let  $\bar{\cdot}$  be an involution on  $S$ . A *word* is an element  $w$  of the free monoid  $S^*$ . A *language* is a set  $L$  of words. The language  $L$  is called *saturated* if for any  $u, v \in S^*$  and  $s \in S$  we have

$$uv \in L \iff us\bar{s}v \in L;$$

that is to say,  $L$  is stable under insertion and deletion of subwords of the form  $s\bar{s}$ . The language  $L$  is called *desiccated* if no word in  $L$  contains a subword of the form  $s\bar{s}$ . Given a language  $L$  we may naturally construct its *saturation*

$\langle L \rangle$ , the smallest saturated language containing  $L$ , and its *desiccation*  $\widehat{L}$ , the largest desiccated language contained in  $L$ .

Let  $\Sigma$  be the monoid defined by generators  $S$  and relations  $s\bar{s} = 1$  for all  $s \in S$ :

$$(3.4) \quad \Sigma = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle.$$

This is a free product of free groups and order-two groups; if  $\bar{\cdot}$  is fixed-point-free,  $\Sigma$  is a free group. Write  $\phi$  for the canonical projection from  $S^*$  to  $\Sigma$ . Let  $\mathbf{k} = \mathbf{Z}[\Sigma]$  be its monoid ring. Then given a language  $L \subset S^*$  we may define its *growth series*  $\Theta(L)$  as

$$\Theta(L) = \sum_{w \in L} w^\phi t^{|w|} \in \mathbf{k}[[t]].$$

This notion of growth series with coefficients was introduced by Fabrice Liardet in his doctoral thesis [Lia96], where he studied *complete growth functions* of groups.

**THEOREM 3.11.** *For any language  $L$  there holds*

$$(3.5) \quad \frac{\Theta(\widehat{L})(t)}{1 - t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1 + (d-1)t^2}\right)}{1 + (d-1)t^2},$$

where  $d = |S|$ .

*Proof.* For any language there exists a unique minimal (possibly infinite) automaton recognising it ([Eil74, §III.5] is a good reference). Let  $\mathcal{X}$  be the minimal automaton recognising  $\langle L \rangle$ . Recall that this is a graph with an initial vertex  $\star$ , a set of terminal vertices  $T$  and a labelling  $\ell' : E(\mathcal{X}) \rightarrow S$  of the graph's edges such that the number of paths labelled  $w$ , starting at  $\star$  and ending at a  $\tau \in T$  is 1 if  $w \in L$  and 0 otherwise. Extend the labelling  $\ell'$  to a labelling  $\ell : E(\mathcal{X}) \rightarrow \mathbf{k}[[t]]$  by

$$e^\ell = t \cdot (e^{\ell'})^\phi.$$

Because  $\langle L \rangle$  is saturated, and  $\mathcal{X}$  is minimal,  $(\bar{e})^\ell = \overline{e^\ell}$ ; then  $\widehat{L}$  is the set of labels on proper paths from  $\star$  to some  $\tau \in T$ . Choosing in turn all  $\tau \in T$  as  $\dagger$ , we obtain growth series  $F_\tau, G_\tau$  counting the formal sum of paths and proper paths from  $\star$  to  $\tau$ . It then suffices to write

$$\frac{\Theta(\widehat{L})(t)}{1 - t^2} = \frac{\sum_{\tau \in T} F_\tau(t)}{1 - t^2} = \frac{\sum_{\tau \in T} G_\tau\left(\frac{t}{1 + (d-1)t^2}\right)}{1 + (d-1)t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1 + (d-1)t^2}\right)}{1 + (d-1)t^2}. \quad \square$$

The following result is well-known:

**THEOREM 3.12** (Müller & Schupp [MS81, MS83]). *Let  $\Gamma$  be a finitely generated group, presented as a quotient  $\Sigma/\Xi$  with  $\Sigma$  as in (3.4). Then  $\Theta(\Xi)$  is an algebraic series (i.e. satisfies a polynomial equation over  $\mathbf{k}[t]$ ) if and only if  $\Sigma/\Xi$  is virtually free (i.e. has a normal subgroup of finite index that is free).*

It is not known whether there exists a non-virtually-free quasi-transitive graph whose circuit series (as defined in Corollary 2.6) is algebraic.

#### 4. FIRST PROOF OF THEOREM 2.4

We now prove Theorem 2.4 using linear algebra. We first assume the graph has a finite number of vertices, for the computations refer to  $\mathbf{k}$ -matrices and  $\mathbf{k}[[u]]$ -matrices indexed by the graph's vertices. This proof is hinted at in Godsil's book as an exercise [God93, page 72]; it was also suggested to the author by Gilles Robert.

For all pairs of vertices  $x, y \in V(\mathcal{X})$  let

$$\mathfrak{G}_{x,y}(\ell) = \sum_{\pi \in [x,y]} \pi^\ell, \quad \mathfrak{F}_{x,y}(\ell) = \sum_{\pi \in [x,y]} u^{\text{bc}(\pi)} \pi^\ell$$

be the path and enriched path series from  $x$  to  $y$ ; for ease of notation we will leave out the labelling  $\ell$  if it is obvious from the context. Let  $\delta_{x,y}$  denote the Kronecker delta, equal to 1 if  $x = y$  and 0 otherwise. For any  $v \in \mathbf{k}$ , let  $[v]_x^y$  denote the  $V(\mathcal{X}) \times V(\mathcal{X})$  matrix with zeros everywhere except at  $(x, y)$ , where it has value  $v$ . Then

$$\mathfrak{G}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}) : e^\alpha = x} e^\ell \mathfrak{G}_{e^\omega, y}$$

so that if

$$A = \sum_{e \in E(\mathcal{X})} [e^\ell]_{e^\alpha}^{e^\omega}$$

be the adjacency matrix of  $\mathcal{X}$ , with labellings, then we have

$$(\mathfrak{G}_{x,y})_{x,y \in V(\mathcal{X})} = \frac{1}{1 - A},$$

an equation holding between  $V(\mathcal{X}) \times V(\mathcal{X})$  matrices over  $\mathbf{k}$ .