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Note that $F(0,t) = F(t)$ and $F(1,t) = G(t)$. The following equation now holds:

$$\frac{F(1-u,t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}\ .$$

Setting $u = 1$ in this equation reduces it to (1.2).

A generalization of the Grigorchuk Formula in a completely different direction can be attempted: consider again a finitely generated group $\Gamma$, and an exact sequence

$$1 \longrightarrow \Xi \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow 1\,,$$

where this time $\Pi$ is not necessarily free. Assume $\Pi$ is generated as a monoid by a finite set $S$. Let again $g_n$ be the number of words of length $n$ in $\Pi$ evaluating to 1 in $\Gamma$, and let $f_n$ be the number of elements of $\Xi$ whose minimal-length representation as a word in $S$ has length $n$. Is there again a relation between the $f_n$ and the $g_n$ ? In Section 8 we derive such a relation when $\Pi$ is the modular group $\mathbf{PSL}_2(\mathbf{Z})$.

Again there is a combinatorial counterpart; rather than considering graphs one considers a locally finite cellular complex $\mathcal{K}$ such that all vertices have isomorphic neighbourhoods. As before, $g_n$ counts the number of paths of length $n$ in the 1-skeleton of $\mathcal{K}$ between two fixed vertices; and $f_n$ counts elements of the fundamental groupoid, i.e. homotopy classes of paths, between two fixed vertices whose minimal-length representation as a path in the 1-skeleton of $\mathcal{K}$ has length $n$. We obtain a relation between these numbers when $\mathcal{K}$ consists solely of triangles and arcs, with no two triangles nor two arcs meeting; these are precisely the complexes associated with quotients of the modular group.

The original motivation for our research was the study of cogrowth in group theory [Gri78a]; however, as it turned out, the more general problem in graph theory has applications to other domains of mathematics, like the Ihara-Selberg zeta function and its evaluation by Hyman Bass [Bas92].

## 2. Main result

Let $\mathcal{X}$ be a graph, that may have multiple edges and loops. We make the following typographical convention for the power series that will appear: a series in the formal variable $t$ is written $G(t)$, or $G$ for short, and $G(x)$ refers to the series $G$ with $x$ substituted for $t$. Functions are written on the right, with $(x)f$ or $x^f$ denoting $f$ evaluated at $x$.

We start by the precise definition of graph we will use:

DEFINITION 2.1 (Graphs). A *graph* $\mathcal{X}$ is a pair of sets $\mathcal{X} = (V, E)$ and maps

$$\alpha : E \to V, \qquad \omega : E \to V, \qquad \bar{\cdot} : E \to E$$

satisfying

$$\bar{\bar{e}} = e, \qquad \bar{e}^{\alpha} = e^{\omega}.$$

The graph $\mathcal{X}$ is said to be *finite* if both $E(\mathcal{X})$ and $V(\mathcal{X})$ are finite sets.

A *graph morphism* $\phi : \mathcal{G} \to \mathcal{H}$ is a pair of maps $(V(\phi), E(\phi))$ with $V(\phi) : V(\mathcal{G}) \to V(\mathcal{H})$ and $E(\phi) : E(\mathcal{G}) \to E(\mathcal{H})$ satisfying

$$\overline{eE(\phi)} = \bar{e}\, E(\phi), \qquad e^{\alpha} V(\phi) = (eE(\phi))^{\alpha}.$$

Given an edge $e \in E(\mathcal{X})$, we call $e^{\alpha}$ and $e^{\omega}$ $e$'s *source* and *destination*, respectively. We say two vertices $x, y$ are *adjacent*, and write $x \sim y$, if they are connected by an edge, i.e. if there exists an $e \in E(\mathcal{X})$ with $e^{\alpha} = x$ and $e^{\omega} = y$. We say two edges $e, f$ are *consecutive* if $e^{\omega} = f^{\alpha}$. A *loop* is an edge $e$ with $e^{\alpha} = e^{\omega}$.

The *degree* $\deg(x)$ of a vertex $x$ is the number of incident edges:

$$\deg(x) = \#\{e \in E(\mathcal{X}) \mid e^{\alpha} = x\} = \#\{e \in E(\mathcal{X}) \mid e^{\omega} = x\}.$$

If $\deg(x)$ is finite for all $x$, we say $\mathcal{X}$ is *locally finite*. If $\deg(x) = d$ for all vertices $x$, we say $\mathcal{X}$ is *$d$-regular*.

Note that the involution $e \mapsto \bar{e}$ may have fixed points. Even though the edges of $\mathcal{X}$ are individually oriented, the graph $\mathcal{X}$ itself should be viewed as an non-oriented graph. In case $\bar{\cdot}$ has no fixed point, $\mathcal{X}$ can be viewed as a geometric graph.

DEFINITION 2.2 (Paths). A *path* in $\mathcal{X}$ is a sequence $\pi$,

$$\pi = (v_0, e_1, v_1, e_2, \dots, e_n, v_n)$$

of edges and vertices of $\mathcal{X}$, with $e_i^{\alpha} = v_{i-1}$ and $e_i^{\omega} = v_i$ for all $i \in \{1, \dots, n\}$ and $n \geq 0$. The *length* of the path $\pi$ is the number $n$ of edges in $\pi$. The *start* of the path $\pi$ is $\pi^{\alpha} = v_0$, and its *end* is $\pi^{\omega} = v_n$. If $\pi^{\alpha} = \pi^{\omega}$, the path $\pi$ is called a *circuit* at $\pi^{\alpha}$. In most cases, we will omit the $v_i$ from the description of paths; they are necessary only if $|\pi| = 0$, in which case a starting vertex must be specified. We extend the involution $\bar{\cdot}$ from edges to paths by setting

$$\bar{\pi} = (v_n, \overline{e_n}, \dots, v_1, \overline{e_1}, v_0)$$

(note that $\overline{\pi}$ is a path from $\pi^\omega$ to $\pi^\alpha$).

We denote by $E^*(\mathcal{X})$ the set of paths, with a partially defined multiplication given by concatenation: if $\pi$ and $\rho$ be two paths with $\pi^\omega = \rho^\alpha$, their *product* is defined as $\pi\rho = (\pi_1, \dots, \pi_{|\pi|}, \rho_1, \dots, \rho_{|\rho|})$. For two vertices $x, y \in V(\mathcal{X})$ we denote by $[x, y]$ the set of paths from $x$ to $y$. We turn $V(\mathcal{X})$ into a metric space by defining for vertices $x, y \in V(\mathcal{X})$ their *distance*

$$\delta(x,y) = \min_{\pi\in[x,y]} |\pi| .$$

The *ball* of radius $n$ at a vertex $x \in V(\mathcal{X})$ is the subgraph $\mathcal{B}(x, n)$ of $\mathcal{X}$ with vertex set

$$V(\mathcal{B}(x,n)) = \{y \in V(\mathcal{X}) \ : \ \delta(x,y) \le n\}$$

and edge set

$$E(\mathcal{B}(x,n)) = \{e \in E(\mathcal{X}) \ : \ e^\alpha \in V(\mathcal{B}(x,n))\} .$$

We define $\alpha_{\mathcal{B}}(e) = \alpha(e)$,

$$\overline{e}^{\mathcal{B}} = \begin{cases} \overline{e} & \text{if } e^\omega \in E(\mathcal{B}) \\ e & \text{else ,} \end{cases}$$

and $\omega(e) = \alpha(\overline{e})$.

This definition amounts to "wrapping around disconnected edges". It has the advantage of preserving the degrees of vertices.

DEFINITION 2.3 (Bumps, Labellings). We say a path $\pi$ has a *bump at* $i$ if $\pi_i = \overline{\pi_{i+1}}$; if the location of the bump is unimportant we will just say $\pi$ has a bump. The *bump count* $\mathrm{bc}(\pi)$ of a path $\pi$ is the number of bumps in $\pi$. A *proper path* in $\mathcal{X}$ is a path $\pi$ with no bumps.

Let $\mathbf{k}$ be a ring. A $\mathbf{k}$-*labelling* of the graph $\mathcal{X}$ is a map

$$\ell \colon E(\mathcal{X}) \to \mathbf{k} .$$

The simplest examples of labellings are:

- the *trivial labelling*, given by $\mathbf{k} = \mathbf{Z}$ and $e^\ell = 1$ for all $e \in E(\mathcal{X})$;
- the *length labelling*, given by $\mathbf{k} = \mathbf{Z}[[t]]$ and $e^\ell = t$ for all $e \in E(\mathcal{X})$.

A $\mathbf{k}$-labelling $\ell$ of $\mathcal{X}$ induces a map, still written $\ell\colon E^*(\mathcal{X}) \to \mathbf{k}$, by setting

$$\pi^\ell = \pi_1^\ell \pi_2^\ell \cdot \ldots \cdot \pi_n^\ell \, .$$

The labelling $\ell\colon E^*(\mathcal{X}) \to \mathbf{k}$ is *complete* [Eil74, § VI.2] if for any vertex $x$ of $\mathcal{X}$ and any set $A$ of paths in $\mathcal{X}$ starting at $x$ there is an element $(A)\Sigma$ of $\mathbf{k}$, and this function $\Sigma$ satisfies

$$(\{\pi\})\Sigma = \pi^\ell \, , \qquad (A \sqcup B)\Sigma = (A)\Sigma + (B)\Sigma$$

for all paths $\pi$ and disjoint sets $A$ and $B$ ($\sqcup$ denotes disjoint union). If $A$ is infinite, it is customary, though abusive, to write $(A)\Sigma$ as $\sum_{\pi \in A} \pi^\ell$.

If $\mathbf{k}$ is a topological ring ($\mathbf{R}$, $\mathbf{C}$, the formal power series ring $\mathbf{Z}[[t]]$, etc.), completion of $\ell$ implies that $\pi^\ell \to 0$ when $|\pi| \to \infty$, but the converse does not hold. The completeness condition becomes that

$$(A)\Sigma = \lim_{B \subset A,\ |B| < \infty} \sum_{\pi \in B} \pi^\ell$$

be a well-defined element of $\mathbf{k}$ for all $A$; i.e., the limit exists. Generally, we define the following topology on $\mathbf{k}$: a sequence $(A_i)\Sigma \in \mathbf{k}$ converges to 0 if and only if $\min_{\pi \in A_i} |\pi|$ tends to infinity.

In the sequel of this paper all labellings will be assumed to be complete. The length labelling is complete for locally finite graphs; more generally, $\ell$ is complete when $\mathbf{k}$ is a discretely valued ring, $e^\ell$ has a positive valuation for all edges $e$, and $\mathcal{X}$ is locally finite. An arbitrary ring $\mathbf{k}$ may be embedded in $\mathbf{k}' = \mathbf{k}[[t]]$, where $t$ has valuation 1 and $\mathbf{k}$ has valuation 0; if $\ell\colon E(\mathcal{X}) \to \mathbf{k}$ is a labelling, we define $\ell'\colon E(\mathcal{X}) \to \mathbf{k}'$ by $e^{\ell'} = te^\ell$; and $\ell'$ will be complete as soon as $\mathcal{X}$ is locally finite. In particular the length labelling is obtained from the trivial labelling through this construction. In all the examples we consider the labelling is defined in this manner.

Throughout the paper we shall assume a graph $\mathcal{X}$ and two vertices $\star, \dagger \in V(\mathcal{X})$ have been fixed. We wish to enumerate the paths, counting their number of bumps, from $\star$ to $\dagger$ in $\mathcal{X}$. For a given complete edge-labelling $\ell$, consider the series

$$\mathfrak{G}(\ell) = \sum_{\pi \in [\star, \dagger]} \pi^\ell \in \mathbf{k} \, , \qquad \mathfrak{F}(\ell) = \sum_{\pi \in [\star, \dagger]} u^{\mathrm{bc}(\pi)} \pi^\ell \in \mathbf{k}[[u]] \, .$$

Note that in general $\mathfrak{G}$ and $\mathfrak{F}$ also depend on the choice of $\star$ and $\dagger$.

For all vertices $x \in V(\mathcal{X})$ let

$$K_x = \Big(1 - \sum_{e \in E(\mathcal{X})\colon e^\alpha = x} \frac{(u-1)(e\overline{e})^\ell}{1-(u-1)^2(e\,\overline{e})^\ell}\Big)^{-1} \in \mathbf{k}[[u]]\,.$$

(A combinatorial interpretation of these $K_x$ will be given in Section 6.) Let $\ell$ be a complete $\mathbf{k}$-labelling of $\mathcal{X}$, and define a new labelling $\ell'\colon E(\mathcal{X}) \to \mathbf{k}[[u]]$ by

$$e^{\ell'} = \frac{1}{1-(u-1)^2(e\,\overline{e})^\ell} e^\ell K_{e^\omega}\,.$$

Then our main result is the following:

THEOREM 2.4. *With the definitions of $\ell'$ and $K_x$ given above, $\ell'$ is a complete labelling and we have in* $\mathbf{k}[[u]]$ *the equality*

$$\mathfrak{F}(\ell) = K_\star \cdot \mathfrak{G}(\ell')\,. \tag{2.1}$$

We now explicit the definitions and main result for the length labelling on a locally finite graph.

DEFINITION 2.5 (Path Series). The integer-valued series

$$G(t) = \sum_{\pi \in [\star,\dagger]} t^{|\pi|} \in \mathbf{N}[[t]]$$

is called the *path series* of $(\mathcal{X},\star,\dagger)$. The series

$$F(u,t) = \sum_{\pi \in [\star,\dagger]} u^{\mathrm{bc}(\pi)} t^{|\pi|} \in \mathbf{N}[u][[t]] \subset \mathbf{N}[[u,t]]$$

is called the *enriched path series* of $(\mathcal{X},\star,\dagger)$. Its specialization $F(0,t)$ is called the *proper path series* of $(\mathcal{X},\star,\dagger)$.

In case $\star = \dagger$, we will call $G$ the *circuit series* of $(\mathcal{X},\star)$ and $F$ the *enriched circuit series* of $(\mathcal{X},\star)$. The circuit series is often called the *Green function* of the graph $\mathcal{X}$.

Note that $F(u,t)$ lies in $\mathbf{N}[u][[t]]$ because the number of bumps on a path is smaller than its length, so all monomials in the sum have a $u$-degree smaller than their $t$-degree; hence for any fixed $t$-degree there are only finitely many monomials with same $t$-degree, because $\mathcal{X}$ is locally finite.

Expressed in terms of length labellings, our main theorem then gives the following result:

COROLLARY 2.6. *Suppose $\mathcal{X}$ is a $d$-regular graph. Then we have*

$$\frac{F(1-u,t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}. \tag{2.2}$$

*Proof.* Because $\mathcal{X}$ is regular, the $K_x$ defined above do not depend on $x$ and are all equal to

$$K = \left(1 - d\frac{(u-1)t^2}{1-(u-1)^2t^2}\right)^{-1} = \frac{1-(1-u)^2t^2}{1+(d-1+u)(1-u)t^2};$$

thus Theorem 2.4 reads

$$\mathfrak{F}(e \mapsto t) = K \cdot \mathfrak{G}\left(e \mapsto \frac{tK}{1-(1-u)^2t^2} =: \circledast\right).$$

Now writing $\mathfrak{F}(e \mapsto t) = F(u,t)$ and $\mathfrak{G}(e \mapsto \circledast) = G(\circledast)$ completes the proof. $\square$

The special case $u = 1$ of this formula appears as an exercise in [God93, page 72].

The meaning of the corollary is that, for regular graphs, the richer two-variable generating series $F(u,t)$ can be recovered from the simpler $G(t)$. Conversely, $G$ can be recovered from $F$, for instance because $G(t) = F(1,t)$. Remember it is valid to substitute 1 for $u$ in $F$, because for any fixed $t$-degree only finitely many monomials with that $t$-degree occur in $F$. In fact, much more is true, as we have the equality

$$G(z) = \frac{1 + \frac{\left(1-\sqrt{1-4(1-u)(d-1+u)z^2}\right)^2}{4(1-u)(d-1+u)z^2}}{1 - \frac{\left(1-\sqrt{1-4(1-u)(d-1+u)z^2}\right)^2}{4(d-1+u)^2z^2}} F\left(u, \frac{1-\sqrt{1-4(1-u)(d-1+u)z^2}}{2(1-u)(d-1+u)z}\right),$$

or after simplification

$$G(z) = \frac{2(d-1+u)}{d-2+u+(d+u)\sqrt{1-4(1-u)(d-1+u)z^2}} \times F\left(u, \frac{1-\sqrt{1-4(1-u)(d-1+u)z^2}}{2(1-u)(d-1+u)z}\right) \tag{2.3}$$

where both sides are to be understood as power series in $\mathbf{N}[[u,z]]$ that actually reside in $\mathbf{N}[[z]]$. Then for any value (say, in $\mathbf{C}$) of $u$ we obtain an expression of $G$ in terms of $F(u,-)$. Of particular interest is the case $u=0$, where (2.3) specializes to

$$(2.4)\qquad G(z)=\frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)z^2}}F\Big(0,\frac{1-\sqrt{1-4(d-1)z^2}}{2(d-1)z}\Big).$$

This equation appears in a slightly different form in [Gri78a].

Similarly, we have in $\mathbf{N}[[t,z,z^{-1}]]$ the equality

$$(2.5)\quad G(z)=\frac{2}{2-d^2tz+dz\sqrt{d^2t^2+4-4t/z}} \times F\Big(1-\frac{dt-\sqrt{d^2t^2+4-4t/z}}{2t},t\Big).$$

Beware though that (2.5) holds for formal variables $z$ and $t$; if we were to substitute a real number for $t$, then the resulting series $G(z)$ would converge absolutely for $\frac{t}{1+(d-1)t^2}\leq z\leq t<\rho$, where $\rho$ is the radius of convergence of $F(1,-)=G$, and in particular not in a neighbourhood of $0$.

The equalities (2.3) and (2.5) are easily derived from (2.2) by setting $z=\frac{t}{1+u(d-u)t^2}$ and solving for $t$ and $u$.

COROLLARY 2.7. *In the setting described above:*

- *If $\mathcal{X}$ is finite, then both $F$ and $G$ are rational series.*
- *If $G$ is rational, then $F$ is rational too.*
- *$F$ is algebraic if and only if $G$ is algebraic.*

The proofs are immediate and follow from the explicit form of (2.2). The converse of the first statement of the preceding corollary will be proved in Section 3.2. The last statement appears in [Gri78a] and [GH97].

In the following section we draw some applications to other fields: group theory and language theory. We give applications of Theorem 2.4 and Corollary 2.6 to some examples of graphs in Section 7, and a derivation of a "cogrowth formula" (as that of Subsection 3.1) for a non-free presentation in Section 8.

We give two proofs of the main result in Sections 4 and 6. The first one, shorter, uses a reduction to finite graphs and their adjacency matrices. The second one is combinatorial and uses the inclusion-exclusion principle. Using the first proof, we obtain in Section 5 an extension of a result by Yasutaka Ihara.

Finally in Section 9 we show how to compute the circuit series of a free product of graphs (an analogue of the free products of groups, *via* their Cayley graph), and in Section 10 do the same for direct products of graphs.

## 3. Applications to other fields

The original motivation for Formula 2.2 was its implication of a well-known result in the theory of random walks on discrete groups.

### 3.1 Applications to random walks on groups

In this section we show how $G$ is related to random walks and $F$ to cogrowth. This will give a generalization of the main formula (1.1) to homogeneous spaces $\Pi/\Xi$, where $\Xi$ does not have to be normal and $\Pi$ is a free product of infinite-cyclic and order-two groups. For a survey on the topic of random walks see [MW89,Woe94].

Throughout this subsection we will have $F(t) = F(0,t)$. We recall the notion of growth of groups:

Definition 3.1. Let $\Gamma$ be a group generated by a finite symmetric set $S$. For a $\gamma \in \Gamma$ define its *length*

$$|\gamma| = \min\{n \in \mathbf{N} : \gamma \in S^n\} .$$

The *growth series* of $(\Gamma, S)$ is the formal power series

$$f_{(\Gamma,S)}(t) = \sum_{\gamma\in\Gamma} t^{|\gamma|} \in \mathbf{N}[[t]] .$$

Expanding $f_{(\Gamma,S)}(t) = \sum f_n t^n$, the *growth* of $(\Gamma, S)$ is

$$\alpha(\Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n}$$

(this supremum-limit is actually a limit and is smaller than $|S| - 1$).

Let $R$ be a subset of $\Gamma$. The *growth series* of $R$ relative to $(\Gamma, S)$ is the formal power series

$$f^R_{(\Gamma,S)}(t) = \sum_{\gamma\in R} t^{|\gamma|} \in \mathbf{N}[[t]] .$$

Expanding $f^R_{(\Gamma,S)}(t) = \sum f_n t^n$, define the *growth* of $R$ relative to $(\Gamma, S)$ as

$$\alpha(R; \Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n} .$$