

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON GROUPS ACTING ON NONPOSITIVELY CURVED CUBICAL COMPLEXES
Autor: BALLMANN, Werner / WITKOWSKI, Jacek
Kapitel: 5. Parallel transport in a cubical manifold and the proof of theorem 3
DOI: <https://doi.org/10.5169/seals-64441>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

the sequence of curves $r \circ \gamma_m$ converges locally uniformly. By Proposition 3.4, the corresponding subsequence of the sequence of unit speed geodesics γ_m converges locally uniformly. By definition, this means that the corresponding subsequence of (x_m) converges to a point $\xi \in X(\infty)$.

Let $\phi \in \Gamma$ and choose $c = c_\phi$ as in Lemma 4.8. Let $t_0 > 0$ be given. By Lemma 4.8 we have $r \circ \gamma_m(t_0) \in F$ for all $m \geq m_0$. By Proposition 3.4 and Lemma 4.8, we have $d(\phi(\gamma_m(t_0)), \gamma_m(t_0)) \leq \sqrt{n}c_\phi$ for all such m . Now c_ϕ is independent of t_0 , hence $\phi(\xi) = \xi$. \square

We now complete the proof of Theorem 2 of the introduction. By Proposition 4.1, $\Delta \cong \ker h$ consists precisely of the elliptic elements of Γ . If indices of type 1 do not occur, then Proposition 4.3 applies: If $k = 0$, then $\Gamma \cong \Delta$ fixes a point of X and possibility (1) holds. If $k > 0$, then possibility (2) holds. If indices of type 1 occur, then possibility (3) holds by Proposition 4.9 and Corollary 4.5. Note that $\text{Stab}(x) \neq \Delta$ for any $x \in X$ in this case since Δ would have a fixed point otherwise.

5. PARALLEL TRANSPORT IN A CUBICAL MANIFOLD AND THE PROOF OF THEOREM 3

Let X be a cubical manifold of dimension n . Given two chambers P and Q in X with a common face of dimension $n - 1$, we define $t_{PQ}: P \rightarrow Q$ to be the *translation* which moves each point p of P along the unit geodesic segment starting at p and orthogonal to the common $(n - 1)$ -face of P to the end point in Q . The map t_{PQ} is an isomorphism and isometry of P with Q . Given a gallery $\pi = (P_1, \dots, P_n)$ in X , the *parallel transport* along π is the isomorphism $t_\pi: P_1 \rightarrow P_n$ given by

$$t_\pi := t_{P_{n-1}P_n} \circ \dots \circ t_{P_2P_3} \circ t_{P_1P_2}.$$

LEMMA 5.1. *Let X be a simply connected cubical manifold and assume that the number of chambers adjacent to each face of codimension 2 in X is divisible by 4. Then for any two chambers P and Q in X , the parallel transport t_π along a gallery π connecting P and Q is independent of π .*

Proof. It is enough to show that the parallel transport along any closed gallery is the identity. Let π be such a gallery with initial and final chamber P .

Represent π by a closed curve c which starts and ends in some interior point p of P , such that c misses the $(n-2)$ -skeleton of X and crosses $(n-1)$ -faces transversally and according to the pattern provided by π . Since X is simply connected, the curve c can be contracted in X to the point p . Since X is a manifold, the links of the vertices in X are $(n-2)$ -connected. Hence the contraction of c can be chosen to be generic in the sense that it misses the $(n-3)$ -skeleton of X and crosses the $(n-2)$ -skeleton transversally. Following the curve c along this contraction, we get a sequence of modifications of the gallery π . These modifications occur when c crosses an $(n-2)$ -face of X . The condition that the number of chambers adjacent to such faces is divisible by 4 implies that the parallel transport t_π does not change under these modifications. Since the parallel transport along the trivial gallery is the identity, $t_\pi = \text{id}_P$. \square

From now on we assume that X is a simply connected cubical manifold such that the number of chambers adjacent to each face of codimension 2 in X is divisible by 4. For chambers P and Q in X define $t_{PQ} = t_\pi$, where π is any gallery connecting P with Q . The above lemma shows that t_{PQ} is well defined.

We fix a chamber P_0 of X and define a homomorphism $\phi: \Gamma \rightarrow \text{Aut } P_0$ by

$$\phi(g) := t_{g(P_0)P_0} \circ g|_{P_0}.$$

The kernel $\Gamma' := \ker \phi$ is a finite index subgroup of Γ and consists precisely of those automorphisms of Γ whose restriction to any chamber commutes with the corresponding parallel transport.

COORIENTATIONS

A *coorientation of a wall* in a chamber is a choice of one of the two half-chambers determined by the wall. Once and for all, we choose coorientations of the walls in the above chamber P_0 . Now by Lemma 5.1, parallel transport gives rise to a consistent choice of coorientations for all walls in X .

By Corollary 1.3, X is foldable. We fix a folding and denote by Λ_i the set of hyperspaces of X with label i . Note that Λ_i is invariant under parallel transport. Along a hyperspace with label i , the half-chambers distinguished by the coorientation are all contained in the same halfspace with respect to the hyperspace. The above group Γ' preserves the families Λ_i together with the coorientations.

The index of intersection of an oriented curve c at a transversal crossing of a hyperspace $H \in \Lambda_i$ is defined to be equal to $+1$ or -1 respectively, according to whether the orientation of c coincides with the coorientation of H or not. Fix a point p_0 in the interior of P_0 which does not belong to any wall and any of the chosen coorientations. For $p \in X$ define $f_i(p)$ to be the sum of the indices of intersection of an oriented curve c connecting p_0 and p with the hyperspaces from Λ_i . Here we assume that c is generic, i.e. c does not meet the $(n-2)$ -skeleton and crosses hyperspaces transversally. The integer $f_i(p)$ does not depend on c since X is simply connected and any two such curves can be deformed into each other by a homotopy which misses the $(n-3)$ -skeleton of X and crosses the $(n-2)$ -skeleton of X transversally.

For $g \in \Gamma$ set $h_i(g) = f_i(g(p_0))$. Since the chosen system of coorientations is invariant under the action of Γ' , the maps $h_i: \Gamma' \rightarrow \mathbf{Z}$ are homomorphisms. We finish the proof of Theorem 3 by showing that the image of $h = (h_1, \dots, h_n)$ is of finite index in \mathbf{Z}^n .

We need to show that the image of h contains n linearly independent vectors. To that end, we show that the image contains non-zero vectors which span arbitrarily small angles with the unit vectors e_i in \mathbf{R}^n , $1 \leq i \leq n$. Let σ be a unit speed geodesic ray with $\sigma(0) = p_0$ which is perpendicular in P_0 to the wall with label i . By the choice of p_0 , the ray σ does not meet the $(n-2)$ -skeleton of X and is perpendicular to all $(n-1)$ -faces and walls with label i which it intersects. We have

$$f_j(\sigma(m)) = \delta_{ij} \cdot m, \quad m \geq 1.$$

By the cocompactness of the action of Γ' , there is an integer $k \geq 1$ such that for any $m \geq 1$ there is a $g_m \in \Gamma'$ with $d(\sigma(m), g_m(p_0)) \leq k$. By the definition of h_j this implies $|h_j(g_m) - f_j(\sigma(m))| \leq k$. Theorem 3 follows.