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## 3. HYPERSPACES AND DUAL TREES

In this section, we assume that  $X$  is an  $n$ -dimensional simply connected cubical chamber complex of nonpositive curvature, endowed with the cubical metric.

## HYPERSPACES

Let  $P$  be a  $k$ -cell in  $X$ ,  $1 \leq k \leq n$ . Any subset of  $P$  of the form  $\{\frac{1}{2}\} \times [0, 1]^{k-1}$ , for any isometric identification of  $P$  with  $[0, 1]^k$ , is called a *wall* in  $P$ . If  $Q$  is a  $j$ -cell of  $X$  contained in  $P$ ,  $1 \leq j < k$ , and  $W$  is a wall in  $Q$ , then there is precisely one wall  $V$  in  $P$  such that  $V \cap P = W$ . Such a wall  $V$  is perpendicular to  $Q$  in  $P$ . In particular, if  $Q$  is an edge, there is precisely one wall  $V$  in  $P$  such that  $V \cap P$  is the midpoint of  $Q$  and  $V$  is perpendicular to  $Q$ .

LEMMA 3.1. *Let  $P$  be a  $k$ -cell in  $X$  and  $W$  a wall in  $P$ . Then  $\text{res } P$  is isometric to  $\text{res } W \times [0, 1]$ , where  $\text{res } W := \bigcup V$  and the union is over the walls  $V$  in cells  $Q \in \text{res } P$  such that  $V \cap P = W$ .  $\square$*

LEMMA 3.2. *A wall  $W$  in a cell  $P$  extends uniquely to a minimal connected subspace  $\Sigma = \Sigma_W \subset X$  such that*

- (1)  $\Sigma$  is a union of walls;
- (2)  $\text{res } V \subset \Sigma$  for any wall  $V \subset \Sigma$ .

Moreover,

- (3) if  $\Sigma$  intersects a cell  $P$  then  $\Sigma \cap \text{res } P = \text{res } W$  for some wall  $W$  of  $P$ ;
- (4)  $\Sigma$  is locally (and hence globally) convex; and
- (5)  $X \setminus \Sigma$  consists of two convex connected components.

*Proof.* Existence and uniqueness of a connected subspace satisfying Properties (1) and (2) is clear from what was said before. Property (3) follows from the observation that otherwise it would be possible to find in  $X$  a nontrivial geodesic (contained in  $\Sigma$ ) with the same initial and final point (belonging to the “selfintersection locus” of  $\Sigma$ ). Property (4) is then an immediate consequence of (3), Lemma 3.1 and Theorem 1.4(2). Property (5) follows from the contractibility of  $X$ : we have to exclude the existence of a closed curve in  $X$  that crosses  $\Sigma$  once. Now such a closed curve can be contracted to a constant curve and a contraction can be put into general position with respect to  $\Sigma$ . Then the number of transversal intersections with

$\Sigma$  does not change mod 2. Since this number is 0 for the final constant curve, it cannot be 1 for the initial curve. The two resulting components of  $X \setminus \Sigma$  are (globally) convex since, by (3) and Lemma 3.1 they are clearly locally convex.  $\square$

We call the subspaces  $\Sigma$  as above *hyperspaces* in  $X$ .

## DUAL TREES

From now on we assume that  $X$  is a simply connected foldable cubical chamber complex of nonpositive curvature. Fix a folding  $F: X \rightarrow C$  of  $X$  onto an  $n$ -dimensional cube  $C$ ,  $n = \dim X$ . Label the walls in  $C$  by the numbers  $1, \dots, n$  and the panels of  $C$  by the label of the corresponding parallel wall. Lift these labellings by  $F$  to the walls and panels in the chambers of  $X$ . Each hyperspace  $\Sigma$  in  $X$  is a union of walls of chambers of  $X$ , and the labels of the walls in  $\Sigma$  are the same. Thus we also obtain a labelling of the hyperspaces. Two different hyperspaces with the same label are disjoint.

Denote by  $\Lambda_i$  the union of the walls with label  $i$  in the chambers of  $X$ . Then  $\Lambda_i$  is the union of the hyperspaces labelled  $i$ . Moreover, the intersection of the boundaries of two different connected components of  $X \setminus \Lambda_i$  is either empty or a hyperspace with label  $i$ . Therefore we can define a graph  $\Lambda_i^*$  as follows: the vertices of  $\Lambda_i^*$  correspond to the connected components of  $X \setminus \Lambda_i$ ; two vertices are connected by an edge if the corresponding components are adjacent along a hyperspace with label  $i$ . Observe that  $\Lambda_i^*$  is a tree since the complement of any of its edges is disconnected by the separating property of hyperspaces, see Lemma 3.2(5). We call  $\Lambda_i^*$  the *dual tree* to the system of hyperspaces with label  $i$ . Note that in general  $\Lambda_i^*$  may not be locally finite, even if the initial complex  $X$  is. We endow  $\Lambda_i^*$  with the length metric  $d_i^*$  such that each edge has length 1.

Note that the panels of  $X$  with label  $i$  do not belong to the set  $\Lambda_i$ ,  $1 \leq i \leq n$ . Thus we can define maps  $r_i: X \rightarrow \Lambda_i^*$  as follows: a panel of  $X$  is mapped by  $r_i$  to the vertex of  $\Lambda_i^*$  representing the component in  $X \setminus \Lambda_i$  to which it belongs. This extends uniquely to all chambers of  $X$  so that a chamber  $P$  is mapped by  $r_i$  onto the edge in  $\Lambda_i^*$  representing the hyperspace in  $X$  containing the wall of  $P$  labelled  $i$  and such that  $r_i$  is isometric in the direction perpendicular to the wall with label  $i$ .

The same argument as in the proof of Lemma 3.2(4) shows that the preimage  $r_i^{-1}(p)$  of any point  $p \in \Lambda_i^*$  distinct from a vertex is a convex subset of  $X$ . Moreover, if  $p$  is a vertex of  $\Lambda_i^*$ , then the convexity of the

subcomplex  $r_i^{-1}(p) \subset X$  follows from foldability of links of  $X$  at vertices in view of the following characterisation (see e.g. Lemma 1.7.1 in [DJS]): a connected subcomplex  $K$  in a simply connected nonpositively curved cubical complex  $L$  is convex if and only if for each vertex  $v$  of  $K$  the link  $K_v$  is a full subcomplex of the link  $L_v$  (which means that a simplex of  $L_v$  belongs to  $K_v$  whenever its vertices belong to  $K_v$ ). The above properties imply that if  $\sigma: I \rightarrow X$  is a geodesic, then  $r_i \circ \sigma$  is (weakly) monotonic:  $r_i \circ \sigma$  never turns. Furthermore, if  $\sigma$  is not constant, then for each  $t \in I$  there are  $i, j \in \{1, \dots, n\}$  such that  $r_i \circ \sigma$  is injective on  $(t - \varepsilon, t] \cap I$  and  $r_j \circ \sigma$  is injective on  $[t, t + \varepsilon) \cap I$ .

#### EMBEDDING INTO A PRODUCT OF TREES

Consider the map  $r: X \rightarrow \prod_{i=1}^n \Lambda_i^*$  defined by  $r(x) = (r_1(x), \dots, r_n(x))$ . Clearly  $r$  is a nondegenerate combinatorial map of cubical complexes, that is, it is isometric on each cell of  $X$ . By what we just said about the image of geodesics under the maps  $r_i$ , it follows immediately that  $r$  is injective. We call  $r$  the *canonical embedding* of  $X$  into the product of trees  $\prod_{i=1}^n \Lambda_i^*$ .

Recall that  $d_i^*$  is the natural metric in  $\Lambda_i^*$ . Define two metrics  $d_{(1)}$  and  $d_{(2)}$  on the product  $\prod_{i=1}^n \Lambda_i^*$  by

$$(3.3) \quad d_{(1)} = \sum_{i=1}^n d_i^* \quad \text{and} \quad d_{(2)} = \left( \sum_{i=1}^n (d_i^*)^2 \right)^{\frac{1}{2}}.$$

It is easy to see that  $d_{(2)} \leq d_{(1)} \leq \sqrt{n} \cdot d_{(2)}$ , and hence the two metrics are Lipschitz equivalent. Moreover, we have

**PROPOSITION 3.4.** *The map  $r$  is a biLipschitz embedding. More precisely, if  $x$  and  $y$  are points in  $X$ , then*

$$d_{(2)}(r(x), r(y)) \leq d(x, y) \leq d_{(1)}(r(x), r(y)).$$

where  $d$  denotes the cubical metric on  $X$ .

*Proof.* The first inequality follows from the fact that  $r$  restricted to any chamber of  $X$  is an isometry. The second inequality is obviously true for  $x$  and  $y$  belonging to the same chamber of  $X$ . It extends to arbitrary  $x$  and  $y$  since for each geodesic  $\sigma$  in  $X$ ,  $r_i \circ \sigma$  is monotonic and hence, up to parameter, a geodesic in  $\Lambda_i^*$ .  $\square$

## EQUIVARIANCE PROPERTIES OF THE CANONICAL EMBEDDING

It follows from gallery connectedness of  $X$  that the folding map  $F: X \rightarrow C$  is unique up to an automorphism of  $C$ , so that a group  $\Gamma$  acting by automorphisms on  $X$  has a well defined homomorphism into the group  $\text{Aut}(C)$  of all automorphisms of  $C$ . The kernel  $\Gamma'$  of this homomorphism is a finite index subgroup in  $\Gamma$ , it preserves all the sets  $\Lambda_i$  and hence acts by automorphisms on the dual trees  $\Lambda_i^*$ .

From now on, we assume that  $\Gamma$  preserves the folding of  $X$  and hence the labelling of the walls. Then  $\Gamma$  acts on the dual trees  $\Lambda_i^*$  and the maps  $r_i$  are equivariant with respect to these actions. Therefore the canonical embedding  $r$  is equivariant with respect to the diagonal action of  $\Gamma$  on the product  $\prod_{i=1}^n \Lambda_i^*$ . This completes the proof of the first assertion of Theorem 1 in the introduction.

Since  $r$  is equivariant, it follows that  $\text{Stab}(\Gamma, x) \subset \text{Stab}(\Gamma, r(x))$  for each  $x \in X$ , where  $\text{Stab}(G, p)$  denotes the stabilizer of a point  $p$  with respect to a transformation group  $G$ .

**PROPOSITION 3.5.** *For each  $p \in \prod_{i=1}^n \Lambda_i^*$ , there is a point  $x_p \in X$  such that  $\text{Stab}(\Gamma, p) \subset \text{Stab}(\Gamma, x_p)$ . In particular, if  $\Gamma$  does not have a fixed point in  $X$ , then  $\Gamma$  acts without a fixed point on at least one of the trees  $\Lambda_i^*$ .*

*Proof.* If  $p$  is in the image of  $r$ , then the assertion follows from the injectivity of  $r$ . If not, let  $\delta$  be the distance of  $p$  to the image of  $r$  with respect to the metric  $d_{(2)}$ . Take the ball  $B(p, 2\delta)$  of radius  $2\delta$  about  $p$  in  $(\prod_{i=1}^n \Lambda_i^*, d_{(2)})$ . The preimage  $r^{-1}(B(p, 2\delta))$  is then a bounded nonempty subset of  $X$  by Proposition 3.4. Let  $x_p$  be its circumcenter, i.e. the center of the unique ball with smallest radius containig this subset, see [Ba, p. 26]. Since  $\Gamma$  acts by isometries with respect to  $d_{(2)}$ ,  $B(p, 2\delta)$  is fixed by each automorphism in  $\text{Stab}(\Gamma, p)$ . Since  $r$  is equivariant and  $\Gamma$  acts by isometries on  $X$ , each such automorphism fixes  $r^{-1}(B(p, 2\delta))$  and hence  $x_p$ .  $\square$

Our next proposition is a special case of a more general result of M. Bridson [B2]. Together with Proposition 3.5, it completes the proof of Theorem 1 of the introduction. For the convenience of the reader we include a short proof adapted to our case of folded cubical complexes.

**PROPOSITION 3.6.** *Let  $X$  be a simply connected, folded cubical chamber complex of nonpositive curvature. Then any automorphism of  $X$  is semisimple, i.e. elliptic or axial.*

*Proof.* Let  $\varphi$  be an automorphism of  $X$ . If  $\varphi$  fixes a point  $p$  of  $\Lambda_i^*$ , then  $p$  can be chosen as a vertex or a midpoint of an edge. If  $p$  is a vertex, then the preimage  $X'$  of  $p$  under  $r_i$  is a closed and convex subcomplex of  $X$ . If  $p$  is the midpoint of an edge,  $X'$  is a hyperspace and as a union of walls, carries a natural cubical structure. In either case,  $X'$  is a closed, convex and  $\varphi$ -invariant subset of  $X$ , and therefore  $\varphi$  is semisimple if and only if the restriction  $\varphi|_{X'}$  is semisimple. Since moreover  $X'$  is a simply connected folded cubical chamber complex of nonpositive curvature and of dimension lower than  $X$ , we can assume by induction on  $\dim X$  that the action of  $\varphi$  on all the trees  $\Lambda_i^*$  is axial.

Let  $a_i$  be an axis of  $\varphi$  in  $\Lambda_i^*$  (unique up to parameter). Let  $X_i = r_i^{-1}(a_i)$ . Since  $r_i$  is surjective,  $X_i$  is non-empty. Furthermore,  $X_i$  is a closed, convex and  $\varphi$ -invariant subcomplex of  $X$ .

Set  $Y_1 := X_1$ . The image of  $Y_1$  under  $r_2$  is path connected and  $\varphi$ -invariant, hence contains  $a_2$ . Let  $Y_2 = Y_1 \cap X_2$ . Then  $Y_2$  is non-empty, closed, convex and  $\varphi$ -invariant. By induction we get that  $Y = X_1 \cap \dots \cap X_n$  is a non-empty, closed, convex and  $\varphi$ -invariant subcomplex of  $X$ . It is then sufficient to prove semisimplicity for the restriction  $\varphi|_Y$ . Note that  $Y = r^{-1}(F)$ , where  $F \cong \mathbf{R}^n$  is the flat

$$F = \{(a_1(t_1), \dots, a_n(t_n)) \mid t_i \in \mathbf{R}\}$$

in the product of trees. Now  $\varphi$  operates as a translation on  $F$ , hence the displacement of  $\varphi$  on  $F$  is constant, say  $= \delta$ . Since  $r$  is injective, we can consider  $Y$  as a closed subcomplex of  $F$ , namely a union of chambers. The metric on  $Y$  is the induced path metric. It follows easily that there are only finitely many possible values for the distance in  $Y$  from a point  $x$  to its image  $\varphi x$ , if the location of  $x$  in its chamber is given.  $\square$

#### 4. NONEXISTENCE OF FREE SUBGROUPS

In this section we discuss the proof of Theorem 2 of the introduction. We assume throughout this section that  $X$  is a simply connected folded cubical chamber complex of nonpositive curvature and that  $\Gamma \subset \text{Aut}(X)$  is a group that preserves the folding of  $X$  (this can be always assumed by passing to a finite index normal subgroup if necessary) and does not contain a free nonabelian subgroup acting freely on  $X$ . By equivariance of the maps  $r_i$ , the same holds for the actions of  $\Gamma$  on the trees  $\Lambda_i^*$ . Up to a subgroup of index two, there are three possibilities for each particular  $i$  [PV]: