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Artikel: LOCAL-GLOBAL PRINCIPLE FOR NORMS FROM CYCLIC EXTENSIONS OF Q(t) (A DIRECT, CONSTRUCTIVE AND QUANTITATIVE APPROACH)

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some intersection $W_\sigma \cap W_\tau$ of distinct conjugates. This has smaller dimension and induction applies.

In conclusion, for large p and B as above we have that the following are equivalent: *(i) f is norm from $\mathbf{Q}_p L$; (ii) V_B has a $\mathbf{Q}_p$-point; (iii) V_B has an $\mathbf{F}_p$-point; (iv) f is a norm from $L(p)$.*

We finally observe that the varieties V_B so defined satisfy the usual local-global principle, in view of the above Corollary 2 (with $\Sigma = \varnothing$) and in view of the Corollary to the Proposition (applied with $\mathbf{k} = \mathbf{Q}$ and $\mathbf{k} = \mathbf{Q}_v$).

REMARK 2. A proof of the equivalence of *(i)* and *(iv)* may also be given by arguments partially analogous to the proof of the Theorem, without invoking the Proposition or the varieties V_B. We start by finding a solution over a finite normal extension k of $\mathbf{Q}$. We embed k in a finite extension k_v of $\mathbf{Q}_p$ and we consider the functions ψ_σ, L_σ, $Q_{\sigma,\tau}$ for $\sigma, \tau \in G' := \mathrm{Gal}(k_v/\mathbf{Q}_p)$; for large p we may reduce everything modulo v, denoting it with a tilde, finding a similar situation over the residue field $\mathbf{F}_v$ of k_v. Also, we may assume that $\mathrm{Gal}(\mathbf{F}_v/\mathbf{F}_p) \cong G'$. By assumption, there exists $\xi \in L(p)$ with norm $\widetilde{f}$. Then $\widetilde{\varphi}$ and ξ have the same norm, whence $\widetilde{\varphi} = \xi(A/\gamma A)$ for some $A \in \mathbf{F}_v L(p)$. This easily leads to $\widetilde{L_\sigma} = (A/\sigma A)\widetilde{B_\sigma}(t)$, where $\widetilde{B_\sigma} \in \mathbf{F}_v(t)$. In turn we find that $\widetilde{Q_{\sigma,\tau}} = \partial(\widetilde{B_\sigma})$. If p is so large that no two zeros or poles of $Q_{\sigma,\tau}$ may collapse after reduction, then is is easily seen that we may find rational functions $B_\sigma \in k_v(t)$ such that $Q_{\sigma,\tau}/\partial(B_\sigma) \in k_v$, reducing to the case when the $Q_{\sigma,\tau}$ are constant. Actually, by using equations (5), we reduce to the case when they are roots of unity in k_v, in which case the proof is easily completed.

6. EFFECTIVENESS

The problem is the following. How can we decide whether a given f admits a nontrivial representation in the form (13), with $x_i \in \mathbf{Q}[t]$? An answer can be given with the methods at the end of the last section. In fact, we have proved that if some representation exists, then a certain projective variety V (whose equations can be found) has a $\mathbf{Q}$-point and conversely. We have observed that V satisfies the local-global principle. Known methods allow one to decide whether V has points over all $\mathbf{Q}_v$ and this gives an answer to the original question.

Another, more direct, procedure is furnished by the method of proof of the Theorem. This has the advantage of yielding a representation when it exists. We start by finding a solution over $\overline{\mathbf{Q}}$. This can be done by e.g. Remark 1. We may then construct the number field k and the functions ψ_σ, as in (2) above. Now we can construct, as in the proof, the rational functions R_σ. Reversing the arguments in the proof of the Theorem, we see that the main problem may be solved if and only if

(i) the conclusion of the Lemma holds for the R_σ and

(ii) if (i) is in fact true, the function $\zeta_{\sigma,\tau}$ given by (12) is of the form $\partial\xi_\sigma$ for some $\xi\colon G \to k^*$.

Question (i), as in the proof of the Lemma, amounts to the fact that definition (9) is a good one and that (11) holds. Plainly this can be decided with a finite amount of computation.

As to the second question, it can be decided e.g. by the usual local-global principle for 2-cocycles over number fields or by the following method, which allows even to find a suitable function ξ, when it exists.

Suppose that such a function ξ exists. First, since the $\zeta_{\sigma,\tau}$ are roots of unity, the divisor D_σ of ξ_σ satisfies $\partial(D_\sigma) = 0$. The group of divisors of k is however a permutation module for the action of $G = \mathrm{Gal}(k/\mathbf{Q})$, so, as we have seen in §2, we may write $D_\sigma = D - \sigma(D)$ for some divisor D. Since the class number of k is finite, we may write $D = (y) + R$, where (y) is the principal divisor of $y \in k^*$ and R is in a finite set which can be computed. Replacing ξ_σ with $\xi_\sigma\sigma(y)/y$ we may thus assume that the divisor of ξ_σ belongs to a finite set. Hence we may write $\xi_\sigma = z_\sigma u_\sigma$, where the $z_\sigma \in k^*$ lie in a finite set and $u_\sigma \in k^*$ are units. In particular we may suppose the z_σ to be fixed. Now, the unit group of k is of the form $\mathbf{Z}/(m) \times \mathbf{Z}^s$, for some integers m, s (and we may effectively find corresponding generators). The action of G corresponds to a certain linear action on this product. Our problem is thus easily reduced to a finite system of linear equations and congruences modulo m, to be solved in integers. It is an easy and well-known matter how to decide about the existence of integral solutions. This completes the argument.

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