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some intersection $W_\sigma \cap W_\tau$ of distinct conjugates. This has smaller dimension and induction applies.

In conclusion, for large p and B as above we have that the following are equivalent: (i) f is norm from $\mathbf{Q}_p L$; (ii) V_B has a \mathbf{Q}_p -point; (iii) V_B has an \mathbf{F}_p -point; (iv) f is a norm from $L(p)$.

We finally observe that the varieties V_B so defined satisfy the usual local-global principle, in view of the above Corollary 2 (with $\Sigma = \emptyset$) and in view of the Corollary to the Proposition (applied with $\mathbf{k} = \mathbf{Q}$ and $\mathbf{k} = \mathbf{Q}_v$).

REMARK 2. A proof of the equivalence of (i) and (iv) may also be given by arguments partially analogous to the proof of the Theorem, without invoking the Proposition or the varieties V_B . We start by finding a solution over a finite normal extension k of \mathbf{Q} . We embed k in a finite extension k_v of \mathbf{Q}_p and we consider the functions ψ_σ , L_σ , $Q_{\sigma,\tau}$ for $\sigma, \tau \in G' := \text{Gal}(k_v/\mathbf{Q}_p)$; for large p we may reduce everything modulo v , denoting it with a tilde, finding a similar situation over the residue field \mathbf{F}_v of k_v . Also, we may assume that $\text{Gal}(\mathbf{F}_v/\mathbf{F}_p) \cong G'$. By assumption, there exists $\xi \in L(p)$ with norm \tilde{f} . Then $\tilde{\varphi}$ and ξ have the same norm, whence $\tilde{\varphi} = \xi(A/\gamma A)$ for some $A \in \mathbf{F}_v L(p)$. This easily leads to $\tilde{L}_\sigma = (A/\sigma A)\tilde{B}_\sigma(t)$, where $\tilde{B}_\sigma \in \mathbf{F}_v(t)$. In turn we find that $\tilde{Q}_{\sigma,\tau} = \partial(\tilde{B}_\sigma)$. If p is so large that no two zeros or poles of $Q_{\sigma,\tau}$ may collapse after reduction, then it is easily seen that we may find rational functions $B_\sigma \in k_v(t)$ such that $Q_{\sigma,\tau}/\partial(B_\sigma) \in k_v$, reducing to the case when the $Q_{\sigma,\tau}$ are constant. Actually, by using equations (5), we reduce to the case when they are roots of unity in k_v , in which case the proof is easily completed.

6. EFFECTIVENESS

The problem is the following. How can we decide whether a given f admits a nontrivial representation in the form (13), with $x_i \in \mathbf{Q}[t]$? An answer can be given with the methods at the end of the last section. In fact, we have proved that if some representation exists, then a certain projective variety V (whose equations can be found) has a \mathbf{Q} -point and conversely. We have observed that V satisfies the local-global principle. Known methods allow one to decide whether V has points over all \mathbf{Q}_v and this gives an answer to the original question.

Another, more direct, procedure is furnished by the method of proof of the Theorem. This has the advantage of yielding a representation when it exists. We start by finding a solution over $\overline{\mathbf{Q}}$. This can be done by e.g. Remark 1. We may then construct the number field k and the functions ψ_σ , as in (2) above. Now we can construct, as in the proof, the rational functions R_σ . Reversing the arguments in the proof of the Theorem, we see that the main problem may be solved if and only if

- (i) the conclusion of the Lemma holds for the R_σ and
- (ii) if (i) is in fact true, the function $\zeta_{\sigma,\tau}$ given by (12) is of the form $\partial\xi_\sigma$ for some $\xi: G \rightarrow k^*$.

Question (i), as in the proof of the Lemma, amounts to the fact that definition (9) is a good one and that (11) holds. Plainly this can be decided with a finite amount of computation.

As to the second question, it can be decided e.g. by the usual local-global principle for 2-cocycles over number fields or by the following method, which allows even to find a suitable function ξ , when it exists.

Suppose that such a function ξ exists. First, since the $\zeta_{\sigma,\tau}$ are roots of unity, the divisor D_σ of ξ_σ satisfies $\partial(D_\sigma) = 0$. The group of divisors of k is however a permutation module for the action of $G = \text{Gal}(k/\mathbf{Q})$, so, as we have seen in §2, we may write $D_\sigma = D - \sigma(D)$ for some divisor D . Since the class number of k is finite, we may write $D = (y) + R$, where (y) is the principal divisor of $y \in k^*$ and R is in a finite set which can be computed. Replacing ξ_σ with $\xi_\sigma \sigma(y)/y$ we may thus assume that the divisor of ξ_σ belongs to a finite set. Hence we may write $\xi_\sigma = z_\sigma u_\sigma$, where the $z_\sigma \in k^*$ lie in a finite set and $u_\sigma \in k^*$ are units. In particular we may suppose the z_σ to be fixed. Now, the unit group of k is of the form $\mathbf{Z}/(m) \times \mathbf{Z}^s$, for some integers m, s (and we may effectively find corresponding generators). The action of G corresponds to a certain linear action on this product. Our problem is thus easily reduced to a finite system of linear equations and congruences modulo m , to be solved in integers. It is an easy and well-known matter how to decide about the existence of integral solutions. This completes the argument.

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