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EXTENSIONS OF Q(t) (A DIRECT, CONSTRUCTIVE AND

QUANTITATIVE APPROACH)

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some intersection  $W_{\sigma} \cap W_{\tau}$  of distinct conjugates. This has smaller dimension and induction applies.

In conclusion, for large p and B as above we have that the following are equivalent: (i) f is norm from  $\mathbf{Q}_p L$ ; (ii)  $V_B$  has a  $\mathbf{Q}_p$ -point; (iii)  $V_B$  has an  $\mathbf{F}_p$ -point; (iv) f is a norm from L(p).

We finally observe that the varieties  $V_B$  so defined satisfy the usual localglobal principle, in view of the above Corollary 2 (with  $\Sigma = \emptyset$ ) and in view of the Corollary to the Proposition (applied with  $\mathbf{k} = \mathbf{Q}$  and  $\mathbf{k} = \mathbf{Q}_v$ ).

REMARK 2. A proof of the equivalence of (i) and (iv) may also be given by arguments partially analogous to the proof of the Theorem, without invoking the Proposition or the varieties  $V_B$ . We start by finding a solution over a finite normal extension k of Q. We embed k in a finite extension  $k_v$  of  $\mathbf{Q}_p$ and we consider the functions  $\psi_{\sigma}$ ,  $L_{\sigma}$ ,  $Q_{\sigma,\tau}$  for  $\sigma,\tau\in G':=\mathrm{Gal}(k_v/\mathbb{Q}_p)$ ; for large p we may reduce everything modulo v, denoting it with a tilde, finding a similar situation over the residue field  $\mathbf{F}_v$  of  $k_v$ . Also, we may assume that  $Gal(\mathbf{F}_v/\mathbf{F}_p) \cong G'$ . By assumption, there exists  $\xi \in L(p)$  with norm  $\tilde{f}$ . Then  $\tilde{\varphi}$  and  $\xi$  have the same norm, whence  $\tilde{\varphi} = \xi(A/\gamma A)$  for some  $A \in \mathbf{F}_v L(p)$ . This easily leads to  $\widetilde{L}_{\sigma} = (A/\sigma A)\widetilde{B}_{\sigma}(t)$ , where  $\widetilde{B}_{\sigma} \in \mathbf{F}_v(t)$ . In turn we find that  $\widetilde{Q}_{\sigma,\tau} = \partial(\widetilde{B}_{\sigma})$ . If p is so large that no two zeros or poles of  $Q_{\sigma,\tau}$  may collapse after reduction, then is is easily seen that we may find rational functions  $B_{\sigma} \in k_v(t)$  such that  $Q_{\sigma,\tau}/\partial(B_{\sigma}) \in k_v$ , reducing to the case when the  $Q_{\sigma,\tau}$  are constant. Actually, by using equations (5), we reduce to the case when they are roots of unity in  $k_v$ , in which case the proof is easily completed.

## 6. EFFECTIVENESS

The problem is the following. How can we decide whether a given f admits a nontrivial representation in the form (13), with  $x_i \in \mathbf{Q}[t]$ ? An answer can be given with the methods at the end of the last section. In fact, we have proved that if some representation exists, then a certain projective variety V (whose equations can be found) has a  $\mathbf{Q}$ -point and conversely. We have observed that V satisfies the local-global principle. Known methods allow one to decide whether V has points over all  $\mathbf{Q}_v$  and this gives an answer to the original question.

Another, more direct, procedure is furnished by the method of proof of the Theorem. This has the advantage of yielding a representation when it exists. We start by finding a solution over  $\overline{\mathbf{Q}}$ . This can be done by e.g. Remark 1. We may then construct the number field k and the functions  $\psi_{\sigma}$ , as in (2) above. Now we can construct, as in the proof, the rational functions  $R_{\sigma}$ . Reversing the arguments in the proof of the Theorem, we see that the main problem may be solved if and only if

- (i) the conclusion of the Lemma holds for the  $R_{\sigma}$  and
- (ii) if (i) is in fact true, the function  $\zeta_{\sigma,\tau}$  given by (12) is of the form  $\partial \xi_{\sigma}$  for some  $\xi \colon G \to k^*$ .

Question (i), as in the proof of the Lemma, amounts to the fact that definition (9) is a good one and that (11) holds. Plainly this can be decided with a finite amount of computation.

As to the second question, it can be decided e.g. by the usual local-global principle for 2-cocycles over number fields or by the following method, which allows even to find a suitable function  $\xi$ , when it exists.

Suppose that such a function  $\xi$  exists. First, since the  $\zeta_{\sigma,\tau}$  are roots of unity, the divisor  $D_{\sigma}$  of  $\xi_{\sigma}$  satisfies  $\partial(D_{\sigma})=0$ . The group of divisors of k is however a permutation module for the action of  $G = Gal(k/\mathbb{Q})$ , so, as we have seen in §2, we may write  $D_{\sigma} = D - \sigma(D)$  for some divisor D. Since the class number of k is finite, we may write D = (y) + R, where (y) is the principal divisor of  $y \in k^*$  and R is in a finite set which can be computed. Replacing  $\xi_{\sigma}$  with  $\xi_{\sigma}\sigma(y)/y$  we may thus assume that the divisor of  $\xi_{\sigma}$  belongs to a finite set. Hence we may write  $\xi_{\sigma} = z_{\sigma}u_{\sigma}$ , where the  $z_{\sigma} \in k^*$  lie in a finite set and  $u_{\sigma} \in k^*$  are units. In particular we may suppose the  $z_{\sigma}$  to be fixed. Now, the unit group of k is of the form  $\mathbb{Z}/(m) \times \mathbb{Z}^{s}$ , for some integers m, s (and we may effectively find corresponding generators). The action of G corresponds to a certain linear action on this product. Our problem is thus easily reduced to a finite system of linear equations and congruences modulo m, to be solved in integers. It is an easy and well-known matter how to decide about the existence of integral solutions. This completes the argument.

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