L'Enseignement Mathématique, t. 45 (1999), p. 51–81

ON GROUPS ACTING ON NONPOSITIVELY CURVED CUBICAL COMPLEXES

by Werner BALLMANN [1]) and Jacek ŚWIĄTKOWSKI [2])

ABSTRACT. We study groups acting on simply connected cubical complexes of nonpositive curvature. Our main objectives are related actions on trees, the existence of free subgroups and the existence of homomorphisms onto free abelian groups.

INTRODUCTION

We study groups acting on simply connected cubical complexes of nonpositive curvature. Examples of such groups and spaces arise naturally from many constructions. Among them are graph products of groups and other groups acting on right-angled buildings, fundamental groups of hyperbolizations of polyhedra, of toric manifolds and of blow-ups of arrangements of hyperplanes, and many others (see [Da], [DJ1], [DJ2], [DJS] and Section 2 below). Roughly speaking, a *cubical complex* is a cell complex whose cells are cubes. As a definition of nonpositive curvature we use the comparison triangle condition $CAT(0)$ with respect to the natural *cubical metric* of a cubical complex (see Section 1 below for more details).

It turns out that groups acting on nonpositively curved cubical complexes share many properties with groups acting on trees and with infinite Coxeter groups. For example, if Γ is a group satisfying Property (T), then any automorphic action of Γ on a tree, a Coxeter complex, a Euclidean space or a hyperbolic space has a fixed point, see [HV], Chapter 6. The same result holds for actions of Γ on cubical complexes, a result recently proved by Niblo and Reeves, see [NR]. This result and our related results in [BS] are the source of our interest in cubical complexes.

---

[1]) Partially supported by Sonderforschungsbereich 256, Universität Bonn

[2]) Partially supported by Max-Planck-Institut für Mathematik (Bonn), SFB256 (Bonn) and Polish Committee of Scientific Research, grant no. 1262/P03/95/08

For the results in this paper we require stronger assumptions on cubical complexes, related to finer results in [BS]. These additional requirements are still natural and are satisfied by many examples. First of all, we only consider *chamber complexes*. Our second and main requirement is that the complex X is *foldable*, that is, it admits a combinatorial map onto an n-cube, where $n = \dim X$. Since a folding of an n-dimensional cubical chamber complex X is unique up to an automorphism of the n-cube, any group Γ of automorphisms of X contains a finite index subgroup Γ' preserving a given (and hence any) folding of X. We refer the reader to Section 1 for definitions and basic facts.

We recall that a *Hadamard space* is a simply connected complete space of nonpositive curvature. The theory of Hadamard spaces is fundamental for the arguments in this paper. Isometries of Hadamard spaces fall into three classes according to the behaviour of their corresponding displacement function. If this function assumes its infimum, then the corresponding isometry is called *semisimple*, otherwise *parabolic*. The semisimple isometries fall into two subclasses, the *elliptic* ones which fix a point and the *axial* ones which translate a geodesic of the space.

Associated to a Hadamard space X is the *ideal boundary* $X(\infty)$ at infinity and the *closure* $\bar{X} = X \cup X(\infty)$. These objects generalize the corresponding objects for trees and the hyperbolic plane in an appropriate way. For details we refer to [Ba].

As we mentioned above, a group does not satisfy Property (T) if it acts without fixed points on a tree. In this sense, the result below gives a strengthening of the result of Niblo and Reeves.

THEOREM 1. *Let X be a simply connected foldable cubical chamber complex of nonpositive curvature, and let $\mathrm{Aut}_f(X)$ be the group of automorphisms of X preserving the foldings. Then we have:*

(1) *there are simplicial trees $\Lambda_1^*, \dots, \Lambda_n^*$, $n = \dim X$, actions of $\mathrm{Aut}_f(X)$ on $\Lambda_1^*, \dots, \Lambda_n^*$ and a biLipschitz embedding $r\colon X \to \Lambda_1^* \times \cdots \times \Lambda_n^*$ such that r is equivariant with respect to the diagonal action of $\mathrm{Aut}_f(X)$ on the product of the trees Λ_i^*;*

(2) *an automorphism $\phi \in \mathrm{Aut}_f(X)$ is elliptic if and only if the action of ϕ on each of the trees Λ_i^* is elliptic and axial if and only if the action of ϕ on at least one of the trees Λ_i^* is axial;*

(3) *if $\Gamma \subset \mathrm{Aut}_f(X)$ is a subgroup that does not have a fixed point in X, then Γ acts without fixed point on at least one of the trees Λ_i^*.*

The next result is a version of the Tits Alternative on the existence of free subgroups. Our result extends and our proof relies on a corresponding result for the action of a group Γ on a tree T: if Γ does not fix a point or an end or a pair of ends of T, then Γ contains a free nonabelian subgroup acting freely on T, see [PV].

THEOREM 2. *Let X be an n-dimensional simply connected foldable cubical chamber complex of nonpositive curvature and $\Gamma \subset \mathrm{Aut}(X)$ a subgroup. Suppose that Γ does not contain a free nonabelian subgroup acting freely on X. Then up to passing to a subgroup of finite index, there is a surjective homomorphism $h\colon \Gamma \to \mathbf{Z}^k$ for some $k \in \{0, \dots, n\}$ such that the kernel Δ of h consists precisely of the elliptic elements of Γ and, furthermore, precisely one of the following three possibilities occurs:*

(1) *Γ fixes a point in X (then $k = 0$).*

(2) *$k \geq 1$ and there is a Γ-invariant convex subset $E \subset X$ isometric to k-dimensional Euclidean space such that Δ fixes E pointwise and such that Γ/Δ acts on E as a cocompact lattice of translations. In particular, Γ fixes each point of $E(\infty) \subset X(\infty)$.*

(3) *Γ fixes a point of $X(\infty)$, but Δ does not fix a point in X. There is a sequence (x_m) in X with strictly increasing stabilizers, $\mathrm{Stab}_{x_m} \subsetneq \mathrm{Stab}_{x_{m+1}}$, with $\bigcup \mathrm{Stab}_{x_m} = \Delta$. Up to passing to a subsequence, any such sequence converges to a fixed point of Γ in $X(\infty)$.*

If the action of Γ is free or, more generally, if there is a universal upper bound on the order of the stabilizers of the action, then possibility (3) in Theorem 2 cannot occur.

Related to Property (T) there is the question of the existence of an epimorphism onto the group $\mathbf{Z}$ of integers for a finite index subgroup of a group. Recently, C. Gonciulea gave a positive answer to this question in the case of infinite Coxeter groups [Go]. We give a positive answer for groups acting on a class of cubical manifolds.

THEOREM 3. *Let X be a simply connected cubical manifold of nonpositive curvature and assume that the number of chambers adjacent to each face of codimension 2 in X is divisible by 4. Let Γ be a group acting on X cocompactly by automorphisms. Then a finite index subgroup of Γ admits a surjective homomorphism onto $\mathbf{Z}^n$, where $n = \dim X$.*

The paper is organized as follows. In Section 1 we recall basic definitions and facts related to cubical complexes, prove some criteria for foldability and discuss nonpositive curvature. In Section 2 we recall some constructions and examples of foldable cubical complexes. In Section 3, we introduce hyperplanes in cubical complexes as in [NR]. Foldability then leads to systems of disjoint hyperplanes and their "dual trees" which will accomplish the proof of Theorem 1. In Section 4 we investigate the induced actions on the dual trees and obtain the proof of Theorem 2. In Section 5 we develop the idea of parallel transport in cubical manifolds and use it to prove Theorem 3.

We are grateful to M. Bridson, T. Januszkiewicz, S. Mozes and the referee for helpful discussions and hints.

1. Cubical complexes

In this section we briefly recall basic notions and facts related to cubical complexes.

Cubical complexes and cubical metric

A *cell* P is the convex hull of a finite set of points in a real vector space. Faces of P are then well defined, and they are also cells (see e.g. [Br]). The set $\mathcal{P}$ of faces of P is partially ordered by inclusion and called the *poset* of P. Two cells are *combinatorially equivalent* if their posets are isomorphic. For example, every convex quadrilateral polygon is combinatorially equivalent to the unit square. An isomorphism of posets induces a bijection between sets of barycenters of faces and thus determines a piecewise linear homeomorphism between two cells. We call such a homeomorphism a *realization* of a combinatorial equivalence.

A *cell complex* is a collection X of cells which are glued by realizations of combinatorial equivalences along faces. We also assume that different faces of the same cell are not identified and that the intersection of different cells is either empty or consists of one cell. These latter assumptions are not essential, but they simplify the exposition considerably. However, we do not require that X is locally finite, so that, if not explicitely stated otherwise, a vertex in X may belong to infinitely many distinct cells.

We say that a cell complex X is *simplicial* if the cells of X are simplices. Because of our assumptions on the glueing of faces, this coincides with the standard terminology. We say that X is *cubical* if the cells of X are combinatorially equivalent to cubes.

Let X be a cubical complex. Any combinatorial equivalence of a Euclidean unit cube is an isometry, hence any cell P in X is endowed with a canonical metric d_P which makes it isometric to the Euclidean unit cube. This allows to measure the lengths of finite polygonal paths in X. Let d be the associated length pseudometric on X. Then d is actually a metric and turns X into a complete geodesic space, see [B1]. We call d the *cubical metric.*

Residues and links

For a cell complex X and a cell P in X, the *residue* of P, denoted $\operatorname{res} P$, consists of all cells of X containing P. The residue of a cell is a closed subcomplex of X.

Let X be a cell complex. If P and Q are cells in X with $Q \in \operatorname{res} P$, then the poset consisting of all faces R of Q with $P \neq R \supset P$ is a poset of a cell Q_P, well defined up to combinatorial equivalence and of dimension $\dim Q_P = \dim Q - \dim P$. We define the *link* X_P of a cell complex X at a cell P as the collection of the cells Q_P, one for each cell Q in $\operatorname{res} P$, with the natural identifications of faces induced from X.

We will need residues and links only in the case when X is simplicial or cubical. In both cases, the links are simplicial. In the simplicial case, the residue of a simplex P of X is naturally homeomorphic to the simplicial join of P and X_P, in the cubical case to the cubical cone over X_P times P. (See the subsection on right angled Coxeter complexes in Section 2 for the definition of the cubical cone.)

Galleries and chamber complexes

An n-dimensional cell complex X is called *dimensionally homogeneous* if each cell of X is contained in an n-dimensional cell. If X is dimensionally homogeneous, then the top-dimensional cells of X will be called *chambers*, the cells of codimension 1 *panels*.

A special case occurs when X is homeomorphic to a manifold. In this case we say that X is a *cellular manifold*, speaking also about simplicial or cubical manifolds if all the chambers are simplices or cubes respectively.

Let X be a dimensionally homogeneous cell complex. A *gallery* in X is a sequence of chambers where any two consecutive chambers have a panel in common. We say that X is *gallery connected* if any two chambers of X can be connected by a gallery. If X is gallery connected, then we say that X is a *chamber complex.* Tits buildings and connected cellular manifolds are chamber complexes.

We say that X is *locally gallery connected* if the link of each cell of X of codimension greater than 1 is gallery connected. If X is connected and locally gallery connected, then X is gallery connected and hence a chamber complex.

FOLDINGS

A *folding* of an n-dimensional simplicial (respectively cubical) complex X is a combinatorial map of X onto an n-simplex (respectively n-cube) which is injective on each cell of X. A *folded simplicial* (respectively *folded cubical*) *complex* is a simplicial (respectively cubical) complex together with a folding.

A simplicial (respectively cubical) complex X is *foldable* if it admits a folding, *locally foldable* if the link of each cell of X is foldable. The following lemma gives a criterion for foldability of a cubical complex in terms of local properties.

LEMMA 1.1. *Let X be a simply connected cubical chamber complex of dimension n. If X is locally gallery connected and locally foldable, then X is foldable and a folding of X is unique up to an automorphism of the n-cube.*

Proof. We observe that foldability (respectively gallery connectedness) holds for the residue of a cell P of X if and only if it holds for the link X_P. Therefore the assumptions of the lemma imply that all residues in X are foldable and gallery connected.

A curve $c\colon [0,1] \to X$ is called *generic* if it crosses the codimension one skeleton of X at finitely many points. We will call such points *singular*. Since X is dimensionally homogeneous, generic curves are dense in the space of all curves in X.

Let c be a generic curve connecting interior points p and q of chambers P and Q of X. Define an isomorphism $f_c\colon Q \to P$ as follows. If c has no singular point we set $f_c = \mathrm{id}_P$. If c has one singular point, let R be a cell of X containing this singular point in its interior. Then the whole curve c is contained in the residue of R. Since $\operatorname{res} R$ is foldable, there exists a folding map $f\colon \operatorname{res} R \to P$ which extends id_P; since $\operatorname{res} R$ is gallery connected, f is unique. We set $f_c := f|_Q$. Finally, if c has more than one singular point, we cut c into a sequence c_i of curves, each of which has exactly one singular point in its interior, and define f_c to be the composition of the isomorphisms f_{c_i}.

We show now that $f_c = \mathrm{id}$ for each closed generic curve at p. Let c be such a curve. Since X is simply connected, c can be contracted to p. Such a contraction can be chosen to be generic, that is, it consists of generic curves only and singular points appear or disappear only at a finite number of times during the contraction. At each such time, c can be cut into finitely many pieces such that each piece is contained in the residue of a cell and such that the appearance or disappearance of singular points occurs in (some of) the pieces. Since residues are gallery connected and foldable, we conclude that f_c remains unchanged during the contraction. Now $f_p = \mathrm{id}$ for the point curve p, hence $f_c = \mathrm{id}$.

Fix a chamber P in X and an interior point p of P. For each other chamber Q of X choose an interior point $q \in Q$ and a generic curve c connecting p with q. Define a map $F: X \to P$ by $F|_Q = f_c$. The above considerations show that F is well defined, hence F is a folding of X. This proves the first assertion of the lemma.

The remaining assertion that the folding is unique up to an automorphism of the n-cube follows immediately from gallery connectedness of X. □

LEMMA 1.2. *Let X be a simply connected cubical chamber complex of dimension n. Suppose that*

(1) *the links at cells of X of codimension > 2 are simply connected;*

(2) *the links at the cells of X of codimension $= 2$ are connected bipartite graphs.*

Then X is foldable, and a folding of X is unique up to an automorphism of the n-cube.

Proof. For the purpose of this proof a curve in X is called *generic* if it misses the skeleton of codimension 2 and crosses the cells of codimension 1 transversally (note that this notion here is slightly different from the one in the proof of the previous lemma). It is clear that any two points in the interior of some chambers of X can be connected by a generic curve. If such a curve is closed, it can be contracted to a point by a contraction that misses the skeleton of codimension 3 and crosses the higher dimensional skeleta transversally.

Now we repeat the arguments of the proof of Lemma 1.1 taking only the residues of cells of codimension 2 into account. These residues consist of chambers arranged according to the corresponding links. Because the links are bipartite graphs, the residues are foldable. □

COROLLARY 1.3. *Let X be a simply connected cubical manifold of dimension n with the property that the number of chambers adjacent to each face of codimension 2 in X is even. Then X is foldable, and a folding of X is unique up to an automorphism of the n-cube.* □

NONPOSITIVE CURVATURE

We will need some elementary facts from the theory of spaces with upper curvature bounds. The main reference is [Ba].

Let (X, d) be a metric space. A curve in X is called a *geodesic* if it has constant speed and realizes the distance locally. We say that X is *geodesic* if any two points of X can be connected by a minimal geodesic. From now on we assume that X is a complete geodesic space.

Let $\kappa \in \mathbf{R}$, and let M^2_κ be the model surface of constant Gauss curvature κ. Denote by $D(\kappa)$ the diameter of M^2_κ. We say that our geodesic space X is a *CAT*(κ)-space if any geodesic triangle in X with minimal sides and of perimeter $< D(\kappa)$ is not thicker than its comparison triangle in M^2_κ. We say that X has curvature $\le \kappa$ if any point of X has a neighborhood that is *CAT*(κ) with respect to the induced metric.

For nonpositively curved spaces, that is, spaces with upper curvature bound 0, there is the following extension of the Hadamard-Cartan Theorem.

THEOREM 1.4 (Gromov [Gr], Alexander-Bishop [AB]). *Let X be a simply connected, complete geodesic space of nonpositive curvature. Then geodesic triangles in X are not thicker than their corresponding comparison triangles in the Euclidean plane. In particular,*

(1) *for any two points $x, y \in X$, there is a unique geodesic $\sigma_{xy}\colon [0,1] \to X$ from x to y and σ_{xy} depends continuously on x and y;*

(2) *locally convex subsets of X are globally convex;*

(3) *X is contractible.*

We say that a cubical complex is *nonpositively curved* if it is nonpositively curved with respect to the cubical metric. The lemma below presents a necessary and sufficient condition for a cubical complex to be nonpositively curved in terms of its combinatorics.

A simplicial complex X is a *flag complex* if each set of vertices of X, in which any two vertices are connected by an edge, spans a simplex of X.

LEMMA 1.5 (Gromov [Gr]). *A cubical complex is nonpositively curved if and only if the link X_v at each vertex v of X is a flag complex.*

REMARK 1.6. If X is a simply connected nonpositively curved cubical complex, then the restriction of the cubical metric to any of its cells coincides with the standard Euclidean metric on the cell.

2. Examples

In this section we show how examples of cubical chamber complexes of nonpositive curvature arise naturally in many constructions. We indicate the features of the constructions which lead to foldability of the universal covers of the resulting complexes.

Barycentric subdivision

Recall that if P is a cell then the barycentric subdivision P' of P is the following simplicial complex: k-simplices of P' correspond to sequences $F_0 \subset F_1 \subset \cdots \subset F_k$ of faces of P, where $F_{i-1} \neq F_i$ for $i = 1, \dots, k$, and the relation of being a face corresponds to being a subsequence. One can realize the subdivision P' geometrically inside P as follows: For each face F of P choose a point p_F in the interior of F (if F is a vertex then $p_F = F$). Then the simplex corresponding to a sequence $F_0 \subset F_1 \subset \cdots \subset F_k$ is identified with the convex hull of the set $\{p_{F_0}, p_{F_1}, \dots, p_{F_k}\}$ in P.

For a cell complex X the barycentric subdivision X' of X is the simplicial complex whose k-simplices correspond to sequences $C_0 \subset C_1 \subset \cdots \subset C_k$ of cells of X, with $C_{i-1} \neq C_i$ for $i = 1, \dots, k$. This corresponds to subdividing barycentrically all the cells of X in a consistent way.

We recall some well known facts related to barycentric subdivision.

Lemma 2.1. *The barycentric subdivision X' of a cell complex X is a foldable flag complex.*

Proof. We note that a folding of X' onto the simplex spanned by the set $\{0, 1, \dots, \dim X\}$ is well defined by assigning to each vertex in X' the dimension of the corresponding cell in X.

If A is a set of vertices of X' pairwise connected by edges, then the set of corresponding cells of X can be ordered by inclusion. But this means that A spans the simplex in X' corresponding to this ordered sequence. Thus X' is a flag complex. □

Lemma 2.2. *Let v be a vertex of a cell complex X. Then the complexes $(X')_v$ and $(X_v)'$ are isomorphic.*

Proof. Simplices in both $(X')_v$ and $(X_v)'$ correspond to sequences $C_0 \subset C_1 \subset \cdots \subset C_k$ of cells of X containing v and distinct from v. □

LEMMA 2.3. *Let v be a vertex of a cell P. Then the complex $(P')_v$ is isomorphic to the complex $[(\partial P)' \times I]_{v\times 1}$, where I is a 1-cell with a vertex 1, and $\times$ denotes the product of cell complexes.*

Proof. Clearly, both complexes are isomorphic to the simplicial cone over the complex $(\partial P)'_v$. □

HYPERBOLIZATIONS

We briefly describe two procedures which turn cell complexes into nonpositively curved cubical complexes (for more details, see [DJ1], [CD]). We also discuss when the resulting complexes are foldable chamber complexes.

THE PRODUCT WITH INTERVAL PROCEDURE. Define a functor h_1 from the category of cell complexes to the category of cubical complexes, inductively with respect to the dimension of initial cell complexes. Let K be a cell complex. If $\dim K \leq 1$, set $h_1(K) = K$. Now consider a cell complex K with $\dim K = i+1$. Assuming inductively that h_1 has been already defined for all cell complexes of dimension $\leq i$, define a hyperbolized complex $h_1(K)$ as follows. Glue "hyperbolized $(i+1)$-cell" $h_1(C) := h_1(\partial C) \times [-1, 1]$ (corresponding to $(i+1)$-cells C of K) to the complex $h_1(K^{(i)}) \times \{-1, 1\}$ according to the identifications of the two copies of the sets $h_1(\partial C) \times \{-1, 1\}$, one in $h_1(C)$ and second in $h_1(K^{(i)}) \times \{-1, 1\}$, by the identity maps. Note that $h_1(\partial C)$ is identified with a subset of $h_1(K^{(i)})$ by the (inductively verified) functoriality of h_1 for cell complexes of dimension $\leq i$.

THE MÖBIUS BAND PROCEDURE. Define a functor h_2 on the category of cubical complexes, inductively with respect to the dimension of initial complexes. Let K be a cubical complex. If $\dim K \leq 1$, set $h_2(K) = K$. Now consider a cubical complex K with $\dim K = i+1$. Assuming inductively that h_2 has been already defined for all cubical complexes of dimension $\leq i$, define a hyperbolized complex $h_2(K)$ as follows. For each $(i+1)$-cell $C \equiv [-1,1]^{i+1}$ in K put $h_2(C) = (h_2(\partial C) \times [-1, 1])/\tau$. Here τ is the involution $\tau(x, t) = (a(x), -t)$ on $h_2(\partial C) \times [-1, 1]$, where a is the combinatorial automorphism of $h_2(\partial C)$ induced from the antipodal automorphism of ∂C by the (inductively verified) functoriality of h_2 for cubical complexes of dimension $\leq i$. Identify $h_2(\partial C)$ with the image of $h_2(\partial C) \times 1$ in $h_2(C)$, and then glue each $h_2(C)$ to $h_2(K^{(i)})$ along $h_2(\partial C)$ using the identity map.

Note that for both procedures above each vertex in the hyperbolized complex $h_j(K)$ corresponds to a unique vertex in the initial complex K. The next lemma shows how the links at such vertices in the corresponding complexes are related.

LEMMA 2.4. *Let v be a vertex of $h_j(K)$ and v_0 the corresponding vertex of K. Then the links $(h_j(K))_v$ and $(K')_{v_0}$ are isomorphic.*

Proof. We proceed by induction with respect to dimension of K. Clearly, if $\dim K = 1$ then $v_0 = v$, $h_j(K) = K$ and $K_v \cong (K')_v$. Let $\dim K = n \geq 2$. Denote by $\overline{v}$ the vertex in $h_j(K^{(n-1)})$ corresponding to v. Then by the inductive hypothesis, the links $[h_j(K^{(n-1)})]_{\overline{v}}$ and $[(K^{(n-1)})']_{v_0}$ are isomorphic. On the other hand, it follows from the descriptions of the procedures h_j that for each hyperbolized n-cell $h_j(C)$ containing v we have $[h_j(C)]_v \cong [h_j(\partial C) \times I]_{\overline{v}\times 1}$. The lemma follows then from Lemma 2.3. □

PROPOSITION 2.5. *Let K be a cell (respectively cubical) chamber complex which is locally gallery connected. Then the hyperbolized complex $h_1(K)$ (respectively $h_2(K)$) is a nonpositively curved cubical chamber complex. Moreover, its universal cover, with the induced cubical structure, is foldable.*

Proof. It is clear from the construction that $h_j(K)$ is a cubical chamber complex. By Lemmas 2.2 and 2.4, we have $[h_j(K)]_v \cong (K_{v_0})'$. It follows from Lemma 2.1 that the links of $h_j(K)$, at all vertices, are foldable flag complexes. Now Gromov's Lemma 1.5 implies that the complexes $h_j(K)$ are nonpositively curved. It is immediate from Lemma 2.4 that $h_j(K)$ is locally gallery connected if K is. The last part of Proposition 2.5 follows then from Lemma 1.1. □

Note that if X is the universal cover of a hyperbolized complex $h_j(K)$ as in the above Proposition, then the fundamental group $\pi_1(h_j(K))$ acts on X freely by automorphisms. The complex X and the group Γ are then examples of a complex and a group as in Theorems 1 and 2 of the introduction.

The universal cover of $h_j(K)$ is hyperbolic in many cases, but not always, see [Gr], [CD].

ZONOTOPAL COMPLEXES

In this subsection we briefly describe an extended class of cell complexes to which the Möbius band hyperbolization procedure can be applied, see [DJS] for more details. (Recall that the product with interval procedure applies to all cell complexes.)

An *arrangement* in a real vector space V is a finite collection $\mathcal{H}$ of linear subspaces of V with codimension one. Elements of $\mathcal{H}$ are called *hyperplanes*. An arrangement $\mathcal{H}$ is *essential* if the intersection $\bigcap \mathcal{H}$ of all hyperplanes in $\mathcal{H}$ is $\{0\}$.

Let $\mathcal{H}$ be an essential arrangement. For each hyperplane $H \in \mathcal{H}$ consider a linear functional $f_H \in V^*$ with $\ker f_H = H$. Denote by $Z_{\mathcal{H}}$ the convex polytope in V^* which is the convex hull of the set

$$\Big\{ \sum_{H \in \mathcal{H}} \varepsilon_H \cdot f_H \mid \varepsilon_H = \pm 1 \Big\}.$$

It turns out that the combinatorial structure of $Z_{\mathcal{H}}$ does not depend on the choice of the functionals f_H. In fact the polytope $Z_{\mathcal{H}}$ is dual to the arrangement $\mathcal{H}$ in the sense that its boundary is dual to the spherical cell complex determined by the intersection of $\mathcal{H}$ with the unit sphere in V.

Polytopes of the form $Z_{\mathcal{H}}$ as above are called *zonotopes* (see [B-Z]). A cell complex is *zonotopal* if all of its cells are zonotopes. The boundary of a zonotope is an example of a zonotopal complex, since each face of a zonotope is a zonotope.

The important feature of a zonotope $Z = Z_{\mathcal{H}}$ is that the central symmetry $f \mapsto -f$ of V^* induces a combinatorial antipodal automorphism of Z. This allows to apply the Möbius band hyperbolization h_2 to zonotopal complexes. By the same arguments as in the previous subsection we get the following result.

PROPOSITION 2.6. *Let K be a zonotopal chamber complex which is locally gallery connected. Then the hyperbolized complex $h_2(K)$ is a nonpositively curved cubical chamber complex. Moreover, the universal cover of $h_2(K)$, with the induced cubical structure, is foldable.*

BLOW-UPS OF ARRANGEMENTS

An arrangement $\mathcal{H}$ in a real vector space V determines an arrangement $P(\mathcal{H})$ of projective hyperplanes in the projective space $P(V)$. If $\mathcal{H}$ is essential then $P(\mathcal{H})$ divides the space $P(V)$ into convex spherical polytopes, so that it becomes a chamber complex. It is proved in [DJS] that the cell structure dual to the above converts the space $P(V)$ into a zonotopal chamber complex.

It is possible to interpret the hyperbolization procedure h_2, applied to a zonotopal complex as above, as a sort of blow-up with respect to the divisor in $P(V)$ consisting of all subspaces of codimension greater than one which are intersections of hyperplanes in $P(\mathcal{H})$, see [DJS]. By Proposition 2.6, this blow-up produces a nonpositively curved cubical chamber complex whose universal cover is foldable.

In [DJS], the procedure described above is called the maximal blow-up. In the same paper some refinements of this procedure, called partial blow-ups, are discussed. In many natural cases these partial blow-ups result in cubical chamber complexes of nonpositive curvature.

SIMPLE POLYTOPES

A convex polytope P is *simple* if the link of P at any vertex is a simplex. Equivalently, P is simple if the boundary complex $\partial\widetilde{P}$ of the dual polytope $\widetilde{P}$ is a simplicial complex.

Any n-dimensional simple polytope P can be subdivided canonically into a cubical complex $P_\square$ in such a way that vertices of $P_\square$ correspond to cells of P and each cubical n-cell of $P_\square$ is spanned by the set of vertices corresponding to cells of P containing a fixed vertex of P. See section 1.2 of [DJS] for a more detailed description of this subdivision and for the proof of the following lemma.

LEMMA 2.7. *Let v be the vertex of $P_\square$ corresponding to P. Then the link $(P_\square)_v$ is isomorphic to $\partial\widetilde{P}$.*

COROLLARY 2.8. *The following conditions are equivalent:*

(1) *$P_\square$ is foldable;*

(2) *$\partial\widetilde{P}$ is foldable;*

(3) *for each codimension 2 simplex C in $\partial\widetilde{P}$ the link $(\partial\widetilde{P})_C$ is even-gonal;*

(4) *each 2-dimensional face F of P is even-gonal.*

Proof. Conditions (1) and (2) are equivalent by Lemma 2.7. The equivalence of (2) and (3) follows from Lemma 1.2. And (3) and (4) are just the dual expressions of the same condition. □

REMARK 2.9. A polytope P satisfies Condition (4) of Corollary 2.8 if and only if P is a zonotope, see Proposition 2.2.14, p. 64, in [B-Z]. Therefore, the cubical subdivision $P_\square$ of a simple polytope P is foldable if and only if P is a (simple) zonotope.

Any face F of a simple polytope P is also simple and the cubical subdivision $P_\square$ restricted to F agrees with the subdivision $F_\square$. Let X be a *simple cell complex*, i.e. a complex all cells of which are simple. Then the canonical cubical subdivisions of the cells of X are consistent and determine a subdivision $X_\square$ of X.

From [DJS] we recall the following

LEMMA 2.10. *The canonical cubical subdivision of a simple chamber complex X is nonpositively curved if and only if the following two conditions are satisfied:*

(1) *for each chamber P of X the boundary $\partial\widetilde{P}$ of the dual simplicial polytope is a flag complex;*

(2) *for each vertex v of X the link X_v is a flag complex.*

In view of Lemma 1.1, we can summarize the considerations of this subsection in the following

PROPOSITION 2.11. *Let K be a chamber complex satisfying the following conditions:*

(1) *all cells K are simple zonotopes;*

(2) *the links of K at all vertices are gallery connected and foldable.*

Then the cubical subdivision $K_\square$ is nonpositively curved and its universal cover with the induced cubical structure is a foldable chamber complex.

POLYGONAL COMPLEXES

Recall that a 2-dimensional cell complex is called a *polygonal complex*. Polygonal complexes arise naturally in combinatorial group theory. The class of polygonal complexes is very rich, see [Bar], [BB], [BS], [Be], [Sw].

Since polygonal complexes are simple, Remark 2.9 and Proposition 2.11 imply the following assertion.

PROPOSITION 2.12. *Let K be a polygonal complex satisfying the following conditions:*

(1) *all 2-cells of K have an even number of sides;*

(2) *the links of K at all vertices are connected bipartite graphs.*

Then the cubical subdivision $K_\square$ is nonpositively curved and its universal cover with the induced cubical structure is a foldable chamber complex.

A big class of polygonal complexes is constituted by Cayley complexes of presentations of groups, on which the corresponding groups act freely by combinatorial automorphisms. Using Proposition 2.12, it is then easy to decide in terms of the presentation whether a group acting on its Cayley complex satisfies the assumptions of Theorems 1–3 of the introduction. Many other examples satisfying these assumptions can be constructed using various methods, see [BB], [BS], [Sw].

TORIC MANIFOLDS

In this subsection, we recall the construction of toric manifolds from [DJ2]. Let $\mathcal{F}$ be the set of codimension 1 faces of a simple polytope P of dimension n. A map $\lambda\colon \mathcal{F} \to (Z_2)^n$ is a *characteristic function* for P, if for every vertex v of P the set $\{\lambda(F) \mid F \in \mathcal{F}, v \in F\}$ is a basis for $(Z_2)^n$. Let $\sim$ be the equivalence relation on the set $P \times (Z_2)^n$ defined by $(x,s) \sim (x,t)$ if $x \in F$ and $s \equiv t \bmod \lambda(F)$. Put $M(P,\lambda) := P \times (Z_2)^n/\sim$ and note that $M(P,\lambda)$ is a simple chamber complex with chambers the images of the sets $P \times \{s\}$ in the quotient. The projection $P \times (Z_2)^n \to P$ induces a combinatorial map $\pi\colon M(P,\lambda) \to P$ which is injective on cells of $M(P,\lambda)$. By Proposition 1.7 of [DJ2], $M(P,\lambda)$ is a closed manifold and it is called a *toric manifold.*

PROPOSITION 2.13. *Let P be a simple polytope with even-gonal 2-dimensional faces and λ be a characteristic function for P. Then the standard cubical subdivision of the toric manifold $M(P,\lambda)$ is foldable and nonpositively curved.*

Proof. According to Corollary 2.8, there is a folding ϕ of $P_\square$. Furthermore, we can view the map π as a nondegenerate combinatorial map $M(P,\lambda)_\square \to P_\square$. Then the composition $\phi \circ \pi$ is a folding of $M(P,\lambda)_\square$.

Nonpositive curvature of $M(P,\lambda)_\square$ follows from Lemma 2.10 since the links of $M(P,\lambda)$ are isomorphic to the boundaries of hyperoctahedra (simplicial polytopes dual to cubes). $\square$

Let X be the universal cover of $M(P,\lambda)_\square$ with the induced cubical structure. Then the fundamental group Γ of $M(P,\lambda)$ acts on X freely by combinatorial automorphisms and the pair X and Γ satisfies the assumptions of Theorems 1 and 2 of the introduction.

RIGHT ANGLED COXETER COMPLEXES

Given a simplicial complex K define the *cubical cone* C_cK to be the unique cubical complex with distinguished vertex v_0 satisfying the following properties:

(1) C_cK is the union of those cells which contain v_0;

(2) the link $(C_cK)_{v_0}$ is isomorphic to K.

Define the *base* B_K of this cone to be the subcomplex consisting of all cells not containing v_0. Then B_K is canonically isomorphic to the standard cubical subdivision $K_\square$ of K and hence the vertices of B_K naturally correspond to the simplices of K.

For each vertex v of K define a *coface* F_v in B_K as follows. Let v' be the vertex in B_K corresponding to v. Then F_v is a subcomplex consisting of all cells of B_K which contain v'.

Let I be a finite set and $M = [m_{ij}]$ a symmetric matrix indexed by $I \times I$. Assume that $m_{ii} = 1$ and $m_{ij} \in \{2, +\infty\}$ for all $i, j \in I$, $i \neq j$. A *right angled Coxeter group* is a group W_M given by a presentation

$$W_M = \langle s_i \mid (s_i s_j)^{m_{ij}} \rangle$$

for some matrix M as above. Any such matrix will be called a *right angled matrix.*

For a right angled matrix M define the graph Γ_M as follows. The set $\{v_i \mid i \in I\}$ of vertices of Γ_M is in 1-1 correspondence with I, and vertices v_i, v_j are connected by an edge if and only if $m_{ij} = 2$. The graph Γ_M determines uniquely a flag complex K_M with the same vertex and edge set: a set of vertices spans a simplex in K_M if and only if any two vertices in this set are connected in Γ_M by an edge.

The *Coxeter complex* of the right angled Coxeter group W_M is the quotient $\Sigma_M = W_M \times C_cK_M/\sim$ modulo the equivalence relation determined by all the equivalences $(w_1, x) \sim (w_2, x)$ with $x \in F_{v_i}$ and $w_1^{-1}w_2 = s_i$ for all $i \in I$. G. Moussong proved the following [Mo].

PROPOSITION 2.14. *The Coxeter complex of a right angled Coxeter group is a locally compact simply connected nonpositively curved cubical complex on which the group acts properly and cocompactly by combinatorial automorphisms.*

In addition to Proposition 2.14 we have the following

LEMMA 2.15. *Assume that for some right angled matrix M the complex K_M is a foldable chamber complex. Then the Coxeter complex Σ_M is also a foldable chamber complex.*

Proof. Note that $C_c K_M$ is foldable since K_M is foldable. Let ϕ be a folding of $C_c K_M$ and $p \colon \Sigma_M \to C_c K_M$ be the nondegenerate combinatorial map induced by the projection $W_M \times C_c K_M \to C_c K_M$. Then clearly the composition $\phi \circ p$ is a folding of Σ_M.

Observe that $C_c K_M$ and hence Σ_M is dimensionally homogeneous since K_M is. Hence to show that Σ_M is a chamber complex, it remains to prove gallery connectedness. To that end let $[(w_1, C_1)]$ and $[(w_2, C_2)]$ be two chambers of Σ_M. By the gallery connectedness of $C_c K_M$ — which is immediate from the gallery connectedness of K_M — it is clear that there is a gallery connecting the above chambers if $w_1 w_2^{-1} = s_i$. The existence of a connecting gallery in the general case follows by induction on the word length of $w_1 w_2^{-1}$ in W_M. □

RIGHT ANGLED BUILDINGS AND GRAPH PRODUCTS OF GROUPS

In [Da] M. Davis defines buildings of type M for a class of matrices which contains right angled matrices. If M is a right angled matrix then any building of type M is a cubical complex and its apartments are isomorphic to the Coxeter complex Σ_M. It is proved in [Da] that any such building is nonpositively curved and simply connected. Moreover, since any two cells of a building lie in a common apartment and since there is a nondegenerate combinatorial map of a building onto any of its apartments, we have the following

PROPOSITION 2.16. *Let M be a right angled matrix for which the complex K_M is a foldable chamber complex. Then any building of type M is a simply connected foldable cubical chamber complex of nonpositive curvature.*

Let M be a right angled matrix over I, and for each $i \in I$ let G_i be a group. Define the *graph product* of the groups G_i (with respect to M) as the quotient of the free product of the groups G_i, $i \in I$, by the normal subgroup generated by all commutators of the form $[g_i, g_j]$, where $g_i \in G_i$, $g_j \in G_j$ and $m_{ij} = 2$. Davis proved [Da] that the graph product of groups (with respect to M) acts cocompactly by automorphisms on a building of type M. He also showed that the building is locally compact and the action is proper if the groups G_i in the product are finite. Moreover, if the assumptions of Corollary 2.16 for M are satisfied, then the action preserves the folding of the building.

3. Hyperspaces and dual trees

In this section, we assume that X is an n-dimensional simply connected cubical chamber complex of nonpositive curvature, endowed with the cubical metric.

Hyperspaces

Let P be a k-cell in X, $1 \leq k \leq n$. Any subset of P of the form $\{\frac{1}{2}\} \times [0,1]^{k-1}$, for any isometric identification of P with $[0,1]^k$, is called a *wall* in P. If Q is a j-cell of X contained in P, $1 \leq j < k$, and W is a wall in Q, then there is precisely one wall V in P such that $V \cap P = W$. Such a wall V is perpendicular to Q in P. In particular, if Q is an edge, there is precisely one wall V in P such that $V \cap P$ is the midpoint of Q and V is perpendicular to Q.

LEMMA 3.1. *Let P be a k-cell in X and W a wall in P. Then* $\operatorname{res} P$ *is isometric to* $\operatorname{res} W \times [0,1]$, *where* $\operatorname{res} W := \bigcup V$ *and the union is over the walls V in cells $Q \in \operatorname{res} P$ such that $V \cap P = W$.* $\square$

LEMMA 3.2. *A wall W in a cell P extends uniquely to a minimal connected subspace $\Sigma = \Sigma_W \subset X$ such that*

(1) *Σ is a union of walls;*

(2) $\operatorname{res} V \subset \Sigma$ *for any wall $V \subset \Sigma$.*

Moreover,

(3) *if Σ intersects a cell P then* $\Sigma \cap \operatorname{res} P = \operatorname{res} W$ *for some wall W of P;*

(4) *Σ is locally (and hence globally) convex; and*

(5) *$X \setminus \Sigma$ consists of two convex connected components.*

Proof. Existence and uniqueness of a connected subspace satisfying Properties (1) and (2) is clear from what was said before. Property (3) follows from the observation that otherwise it would be possible to find in X a nontrivial geodesic (contained in Σ) with the same initial and final point (belonging to the "selfintersection locus" of Σ). Property (4) is then an immediate consequence of (3), Lemma 3.1 and Theorem 1.4(2). Property (5) follows from the contractibility of X: we have to exclude the existence of a closed curve in X that crosses Σ once. Now such a closed curve can be contracted to a constant curve and a contraction can be put into general position with respect to Σ. Then the number of transversal intersections with

Σ does not change mod 2. Since this number is 0 for the final constant curve, it cannot be 1 for the initial curve. The two resulting components of $X \setminus \Sigma$ are (globally) convex since, by (3) and Lemma 3.1 they are clearly locally convex. □

We call the subspaces Σ as above *hyperspaces* in X.

DUAL TREES

From now on we assume that X is a simply connected foldable cubical chamber complex of nonpositive curvature. Fix a folding $F: X \to C$ of X onto an n-dimensional cube C, $n = \dim X$. Label the walls in C by the numbers $1, \ldots, n$ and the panels of C by the label of the corresponding parallel wall. Lift these labellings by F to the walls and panels in the chambers of X. Each hyperspace Σ in X is a union of walls of chambers of X, and the labels of the walls in Σ are the same. Thus we also obtain a labelling of the hyperspaces. Two different hyperspaces with the same label are disjoint.

Denote by Λ_i the union of the walls with label i in the chambers of X. Then Λ_i is the union of the hyperspaces labelled i. Moreover, the intersection of the boundaries of two different connected components of $X \setminus \Lambda_i$ is either empty or a hyperspace with label i. Therefore we can define a graph Λ_i^* as follows: the vertices of Λ_i^* correspond to the connected components of $X \setminus \Lambda_i$; two vertices are connected by an edge if the corresponding components are adjacent along a hyperspace with label i. Observe that Λ_i^* is a tree since the complement of any of its edges is disconnected by the separating property of hyperspaces, see Lemma 3.2(5). We call Λ_i^* the *dual tree* to the system of hyperspaces with label i. Note that in general Λ_i^* may not be locally finite, even if the initial complex X is. We endow Λ_i^* with the length metric d_i^* such that each edge has length 1.

Note that the panels of X with label i do not belong to the set Λ_i, $1 \leq i \leq n$. Thus we can define maps $r_i: X \to \Lambda_i^*$ as follows: a panel of X is mapped by r_i to the vertex of Λ_i^* representing the component in $X \setminus \Lambda_i$ to which it belongs. This extends uniquely to all chambers of X so that a chamber P is mapped by r_i onto the edge in Λ_i^* representing the hyperspace in X containing the wall of P labelled i and such that r_i is isometric in the direction perpendicular to the wall with label i.

The same argument as in the proof of Lemma 3.2(4) shows that the preimage $r_i^{-1}(p)$ of any point $p \in \Lambda_i^*$ distinct from a vertex is a convex subset of X. Moreover, if p is a vertex of Λ_i^*, then the convexity of the subcomplex $r_i^{-1}(p) \subset X$ follows from foldability of links of X at vertices in view of the following characterisation (see e.g. Lemma 1.7.1 in [DJS]): a connected subcomplex K in a simply connected nonpositively curved cubical complex L is convex if and only if for each vertex v of K the link K_v is a full subcomplex of the link L_v (which means that a simplex of L_v belongs to K_v whenever its vertices belong to K_v). The above properties imply that if $\sigma: I \to X$ is a geodesic, then $r_i \circ \sigma$ is (weakly) monotonic: $r_i \circ \sigma$ never turns. Furthermore, if σ is not constant, then for each $t \in I$ there are $i, j \in \{1, \dots, n\}$ such that $r_i \circ \sigma$ is injective on $(t-\varepsilon, t] \cap I$ and $r_j \circ \sigma$ is injective on $[t, t+\varepsilon) \cap I$.

EMBEDDING INTO A PRODUCT OF TREES

Consider the map $r: X \to \prod_{i=1}^n \Lambda_i^*$ defined by $r(x) = (r_1(x), \dots, r_n(x))$. Clearly r is a nondegenerate combinatorial map of cubical complexes, that is, it is isometric on each cell of X. By what we just said about the image of geodesics under the maps r_i, it follows immediately that r is injective. We call r the *canonical embedding* of X into the product of trees $\prod_{i=1}^n \Lambda_i^*$.

Recall that d_i^* is the natural metric in Λ_i^*. Define two metrics $d_{(1)}$ and $d_{(2)}$ on the product $\prod_{i=1}^n \Lambda_i^*$ by

$$d_{(1)} = \sum_{i=1}^n d_i^* \quad \text{and} \quad d_{(2)} = \Big(\sum_{i=1}^n (d_i^*)^2\Big)^{\frac{1}{2}}. \tag{3.3}$$

It is easy to see that $d_{(2)} \leq d_{(1)} \leq \sqrt{n} \cdot d_{(2)}$, and hence the two metrics are Lipschitz equivalent. Moreover, we have

PROPOSITION 3.4. *The map r is a biLipschitz embedding. More precisely, if x and y are points in X, then*

$$d_{(2)}(r(x), r(y)) \leq d(x, y) \leq d_{(1)}(r(x), r(y)).$$

where d denotes the cubical metric on X.

Proof. The first inequality follows from the fact that r restricted to any chamber of X is an isometry. The second inequality is obviously true for x and y belonging to the same chamber of X. It extends to arbitrary x and y since for each geodesic σ in X, $r_i \circ \sigma$ is monotonic and hence, up to parameter, a geodesic in Λ_i^*. $\square$

EQUIVARIANCE PROPERTIES OF THE CANONICAL EMBEDDING

It follows from gallery connectedness of X that the folding map $F\colon X \to C$ is unique up to an automorphism of C, so that a group Γ acting by automorphisms on X has a well defined homomorphism into the group $\mathrm{Aut}(C)$ of all automorphisms of C. The kernel Γ' of this homomorphism is a finite index subgroup in Γ, it preserves all the sets Λ_i and hence acts by automorphisms on the dual trees Λ_i^*.

From now on, we assume that Γ preserves the folding of X and hence the labelling of the walls. Then Γ acts on the dual trees Λ_i^* and the maps r_i are equivariant with respect to these actions. Therefore the canonical embedding r is equivariant with respect to the diagonal action of Γ on the product $\prod_{i=1}^n \Lambda_i^*$. This completes the proof of the first assertion of Theorem 1 in the introduction.

Since r is equivariant, it follows that $\mathrm{Stab}(\Gamma, x) \subset \mathrm{Stab}(\Gamma, r(x))$ for each $x \in X$, where $\mathrm{Stab}(G, p)$ denotes the stabilizer of a point p with respect to a transformation group G.

PROPOSITION 3.5. *For each $p \in \prod_{i=1}^n \Lambda_i^*$, there is a point $x_p \in X$ such that $Stab(\Gamma, p) \subset Stab(\Gamma, x_p)$. In particular, if Γ does not have a fixed point in X, then Γ acts without a fixed point on at least one of the trees Λ_i^*.*

Proof. If p is in the image of r, then the assertion follows from the injectivity of r. If not, let δ be the distance of p to the image of r with respect to the metric $d_{(2)}$. Take the ball $B(p, 2\delta)$ of radius 2δ about p in $(\prod_{i=1}^n \Lambda_i^*, d_{(2)})$. The preimage $r^{-1}(B(p, 2\delta))$ is then a bounded nonempty subset of X by Proposition 3.4. Let x_p be its circumcenter, i.e. the center of the unique ball with smallest radius containig this subset, see [Ba, p. 26]. Since Γ acts by isometries with respect to $d_{(2)}$, $B(p, 2\delta)$ is fixed by each automorphism in $\mathrm{Stab}(\Gamma, p)$. Since r is equivariant and Γ acts by isometries on X, each such automorphism fixes $r^{-1}(B(p, 2\delta))$ and hence x_p. □

Our next proposition is a special case of a more general result of M. Bridson [B2]. Together with Proposition 3.5, it completes the proof of Theorem 1 of the introduction. For the convenience of the reader we include a short proof adapted to our case of folded cubical complexes.

PROPOSITION 3.6. *Let X be a simply connected, folded cubical chamber complex of nonpositive curvature. Then any automorphism of X is semisimple, i.e. elliptic or axial.*

Proof. Let φ be an automorphism of X. If φ fixes a point p of Λ_i^*, then p can be chosen as a vertex or a midpoint of an edge. If p is a vertex, then the preimage X' of p under r_i is a closed and convex subcomplex of X. If p is the midpoint of an edge, X' is a hyperspace and as a union of walls, carries a natural cubical structure. In either case, X' is a closed, convex and φ-invariant subset of X, and therefore φ is semisimple if and only if the restriction $\varphi|_{X'}$ is semisimple. Since moreover X' is a simply connected folded cubical chamber complex of nonpositive curvature and of dimension lower than X, we can assume by induction on $\dim X$ that the action of φ on all the trees Λ_i^* is axial.

Let a_i be an axis of φ in Λ_i^* (unique up to parameter). Let $X_i = r_i^{-1}(a_i)$. Since r_i is surjective, X_i is non-empty. Furthermore, X_i is a closed, convex and φ-invariant subcomplex of X.

Set $Y_1 := X_1$. The image of Y_1 under r_2 is path connected and φ-invariant, hence contains a_2. Let $Y_2 = Y_1 \cap X_2$. Then Y_2 is non-empty, closed, convex and φ-invariant. By induction we get that $Y = X_1 \cap \ldots \cap X_n$ is a non-empty, closed, convex and φ-invariant subcomplex of X. It is then sufficient to prove semisimplicity for the restriction $\varphi|_Y$. Note that $Y = r^{-1}(F)$, where $F \cong \mathbf{R}^n$ is the flat

$$F = \{(a_1(t_1), \ldots, a_n(t_n)) \mid t_i \in \mathbf{R}\}$$

in the product of trees. Now φ operates as a translation on F, hence the displacement of φ on F is constant, say $= \delta$. Since r is injective, we can consider Y as a closed subcomplex of F, namely a union of chambers. The metric on Y is the induced path metric. It follows easily that there are only finitely many possible values for the distance in Y from a point x to its image φx, if the location of x in its chamber is given. □

4. Nonexistence of free subgroups

In this section we discuss the proof of Theorem 2 of the introduction. We assume throughout this section that X is a simply connected folded cubical chamber complex of nonpositive curvature and that $\Gamma \subset \mathrm{Aut}(X)$ is a group that preserves the folding of X (this can be always assumed by passing to a finite index normal subgroup if necessary) and does not contain a free nonabelian subgroup acting freely on X. By equivariance of the maps r_i, the same holds for the actions of Γ on the trees Λ_i^*. Up to a subgroup of index two, there are three possibilities for each particular i [PV]:

(0) Γ fixes a point of Λ_i^* ;

(1) Γ fixes no point of Λ_i^*, but precisely one end of Λ_i^* ;

(2) Γ fixes no point of Λ_i^*, but precisely two ends of Λ_i^*.

Thus by passing to a subgroup of Γ of index at most 2^n, we can assume that the above three alternatives hold for all i. Corresponding to the alternative, we say that i is an *index of type* 0, 1 or 2 respectively.

We first construct a homomorphism $h = (h_1, \dots, h_n) \colon \Gamma \to \mathbf{Z}^n$ as claimed. If Γ fixes a point of Λ_i^*, we define h_i to be the trivial homomorphism. If Γ does not fix a point of Λ_i^*, we let ω_i be the end or one of the two ends of Λ_i^* fixed by Γ. The Busemann function $b_i \colon \Lambda_i^* \to \mathbf{R}$ at ω_i is well defined up to an additive constant (see [Ba], Section 1 of Chapter II). Since Γ fixes ω_i,

$$h_i(\phi) := b_i(\phi p) - b_i(p), \quad p \in \Lambda_i^* ,$$

is a well defined homomorphism $h_i \colon \Gamma \to \mathbf{Z}$, called the *Busemann homomorphism.* Note that h_i is integer valued since Λ_i^* is a simplicial tree and Γ acts by automorphisms. This completes the definition of $h = (h_1, \dots, h_n)$. We set

$$\Delta_i = \ker h_i \quad \text{and} \quad \Delta = \bigcap \Delta_i = \ker h .$$

PROPOSITION 4.1. *Δ consists precisely of the elliptic elements of Γ.*

Proof. If the action of Γ on Λ_i^* has a fixed point, then any $\phi \in \Gamma$ is elliptic on Λ_i^* and $\Delta_i = \Gamma$. If Γ does not have a fixed point in Λ_i^*, but fixes a point $\xi_i \in \Lambda_i^*(\infty)$ and $\phi \in \Gamma$ is axial on Λ_i^*, then ξ_i is an end point of the axis of ϕ. Then $h_i(\phi) \neq 0$. Hence by Proposition 3.5, any $\phi \in \Delta$ is elliptic on X. Conversely, if $\phi \in \Gamma$ is elliptic on X, then $\phi \in \Delta$. □

For the proof of the other assertions of Theorem 2 we need some more preparations.

LEMMA 4.2. *Let Λ be a simplicial tree on which Γ acts by automorphisms. Suppose Δ fixes a point of Λ. Then either Γ fixes a point of Λ or exactly two points in $\Lambda(\infty)$.*

Proof. Since Δ is a normal subgroup of Γ, the set Φ of fixed points of Δ is Γ-invariant. Now Φ is a subtree of Λ, hence we can assume $\Phi = \Lambda$. Then the quotient action by Γ/Δ on Λ is well defined.

Suppose that Γ/Δ contains an element ϕ which is axial on Λ. Since Γ/Δ is abelian, it leaves the unique axis of ϕ invariant and fixes the endpoints of the axis.

Suppose now that all elements of Γ/Δ are elliptic on Λ. Let $\phi_1, \dots, \phi_k$ be a system of generators. The set of fixed points of ϕ_1 is a Γ/Δ-invariant subtree. Replacing Λ by this subtree, we can assume that $\phi_1 = \mathrm{id}_\Lambda$. The quotient of Γ/Δ by the subgroup generated by ϕ_1 is abelian and has a system of $k-1$ generators. Induction on k shows that Γ has a fixed point. □

If i is an index of type 0 and $p \in \Lambda_i^*$ a fixed point, then $X' := r_i^{-1}(p) \subset X$ is closed, convex and Γ-invariant. In particular, $X'(\infty) \subset X(\infty)$ is Γ-invariant. Although X' is not a subcomplex if p is not a vertex, it is parallel to the walls with label i in the chambers it intersects. Hence we obtain a natural cubical structure on X' with a folding onto an $(n-1)$-cube, and Γ preserves this cubical structure and folding. Hence by passing to such subspaces if necessary, we can assume that no indices of type 0 occur.

Let i be an index of type 2. Let $\alpha_i, \omega_i \in \Lambda_i^*(\infty)$ be the fixed points of Γ and σ_i the unit speed geodesic from α_i to ω_i. Then σ_i is Γ-invariant and $\Delta_i = \mathrm{Stab}(\sigma_i(t))$ for all $t \in \mathbf{R}$. Hence $X' = r_i^{-1}(\mathrm{im}\ \sigma_i)$ is a closed, convex and Γ-invariant subcomplex of X. Hence by passing to such subspaces if necessary, we can assume that $\Lambda_i^* = \mathrm{im}\ \sigma_i \cong \mathbf{R}$ for all indices i of type 2.

PROPOSITION 4.3. *If there are no indices of type* 1, *then there is a* Γ-*invariant convex subset* $E \subset X$ *isometric to a Euclidean space of dimension* $k \in \{0, \dots, n\}$ *and an exact sequence*

$$0 \to \Delta \to \Gamma \to \mathbf{Z}^k \to 0$$

such that Δ *fixes* E *pointwise and such that the quotient* $\Gamma/\Delta \cong \mathbf{Z}^k$ *acts on* E *as a cocompact lattice of translations.*

Proof. After reductions as above we can assume that all indices are of type 2, that $\Lambda_i^* \cong \mathbf{R}$ for all i and that Δ fixes each point of $\prod \Lambda_i^*$. Since r is an injection, Δ fixes each point of X.

The image im h of the homomorphism h is a subgroup of the group $\mathbf{Z}^n$, hence it is isomorphic to $\mathbf{Z}^k$ for some $k \leq n$. Thus we may identify the quotient group Γ/Δ with $\mathbf{Z}^k$. Consider the quotient action of $\mathbf{Z}^k = \Gamma/\Delta$ on X, which is well defined since Δ acts trivially on X. This action is free and the elements are semisimple by Proposition 3.6. Applying the Flat Torus Theorem, see [CE] and [BH], we get that there exists a $\mathbf{Z}^k$-invariant convex subspace $E \subset X$, isometric to k-dimensional Euclidean space, such that $\mathbf{Z}^k$ acts on it as a cocompact lattice of translations. □

We now discuss the more difficult case that indices of type 1 occur. As explained above, we can assume that no indices of type 0 occur and that $\Lambda_i^* \cong \mathbf{R}$ for all indices of type 2.

Choose a vertex $x_0 \in X$ as an origin. For indices of type 2 choose the parameter on the above geodesics σ_i such that $\sigma_i(0) = r_i(x_0)$. For indices of type 1 we denote by $\omega_i \in \Lambda_i^*(\infty)$ the corresponding fixed point. For these indices, we let $\sigma_i \colon [0, \infty) \to \Lambda_i^*$ be a unit speed geodesic ray with $\sigma_i(0) = r_i(x_0)$ and $\sigma_i(\infty) = \omega_i$.

We set $F = \operatorname{im} \sigma_1 \times \cdots \times \operatorname{im} \sigma_n$. Note that F is a closed and convex subspace of $\prod \Lambda_i^*$. We also define a geodesic ray

$$\sigma \colon [0, \infty) \to F \quad \text{by} \quad \sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t)).$$

By construction, $\sigma(0) = r(x_0)$.

LEMMA 4.4. $\mathrm{Stab}(\sigma_i(t)) \to \Delta_i$ *and* $\mathrm{Stab}(\sigma(t)) \to \Delta$ *as* $t \to \infty$, *where the limit of groups is understood as the union of increasing family.*

Proof. Let $\phi \in \Delta_i$. Then ϕ fixes $\omega_i = \sigma_i(\infty)$. Therefore $\phi \circ \sigma_i$ is asymptotic to σ_i. Now Λ_i^* is a tree, hence $\phi \circ \sigma_i(t) = \sigma_i(t + c)$ for all t sufficiently large, where c is some constant independent of t. Since $\phi \in \Delta_i$, $c = 0$ and therefore $\phi \in \mathrm{Stab}(\sigma_i(t))$ for all t sufficiently large. □

COROLLARY 4.5. *There exists a sequence* (x_m) *in* X *such that* $\mathrm{Stab}(x_m) \to \Delta$.

Proof. We observe that $\mathrm{Stab}(x) \subset \Delta$ for all $x \in X$. Now the assertion follows immediately from Proposition 3.5 and Lemma 4.4. □

LEMMA 4.6. *If the group* Γ *fixes precisely one point* $\omega_i \in \Lambda_i^*(\infty)$, *then* $\Delta \cap \mathrm{Stab}(\sigma_i(t))$ *has infinitely many jumps as* $t \to \infty$.

Proof. Let $\phi \in \Delta \subset \Delta_i$. By Lemma 4.4 there is $t_\phi \geq 0$ such that $\phi \in \mathrm{Stab}(\sigma_i(t))$ for all $t \geq t_\phi$. Hence if $\Delta \cap \mathrm{Stab}(\sigma_i(t)) = \Delta \cap \mathrm{Stab}(\sigma_i(t'))$ for all t, t' sufficiently large, then $\Delta \subset \mathrm{Stab}(\sigma_i(t))$ for all t sufficiently large. By Lemma 4.2, Γ either fixes a point of Λ_i^*, which is excluded by our reductions above, or Γ fixes exactly two points of $\Lambda_i^*(\infty)$, which is in contradiction to the assumption. □

LEMMA 4.7. *Let* (x_m) *be a sequence in* X *such that* $\mathrm{Stab}(x_m) \to \Delta$ *and* $\gamma_m \colon [0, s_m] \to X$ *be the unit speed geodesic from* x_0 *to* x_m, *where* $s_m = d(x_0, x_m)$. *Then given a constant* $t_0 > 0$, *there exists* m_0 *such that* $s_m \geq t_0$ *and* $r \circ \gamma_m([0, t_0]) \in F$ *for all* $m \geq m_0$.

Proof. For those i for which Γ fixes exactly one point $\omega_i \in \Lambda_i^*(\infty)$ we choose $\phi_i \in \Delta$ such that $\phi_i \notin \mathrm{Stab}(\sigma_i(t))$ for $t \le t_0$, see Lemma 4.6. By assumption, there is m_0 such that $\phi_i \in \mathrm{Stab}(x_m)$ for all $m \ge m_0$ and all such i. Now $r_i \circ \gamma_m$ is a monotonic curve in Λ_i^* from $\sigma_i(0) = r_i(x_0)$ to $r_i(x_m)$. By equivariance of r_i, $\phi_i \in \mathrm{Stab}(r_i(x_m))$ for all $m \ge m_0$. On the other hand, $r_i \circ \sigma$ has speed ≤ 1, hence by the choice of t_0, $s_m \ge t_0$ and $r_i(\gamma_m(t)) \in \sigma_i([0, t_0])$ for $0 \le t \le t_0$.

The claim follows since the image of r_i is σ_i for those i for which Γ fixes exactly two ends of Λ_i^*. □

LEMMA 4.8. *Given $\phi \in \Gamma$, there is a constant $c = c_\phi$ such that $d(\phi(p), p) \le c$ for all $p \in F$.*

Proof. We show that $d_i(\phi(p), p) \le c_i$ for each point p in the image of σ_i. This is clear for those indices i for which Γ fixes exactly two ends of Λ_i^*. Consider some other index i. Then σ_i is defined on $[0, \infty)$.

If ϕ is elliptic on Λ_i^*, then $\phi \in \Delta_i$. By Lemma 4.4, there exists a constant t_ϕ such that ϕ fixes $\sigma_i(t)$ for all $t \ge t_\phi$. We conclude that $d_i(\phi(p), p) \le 2t_\phi$ for each point p in the image of σ_i.

We assume now that ϕ is axial on Λ_i^* and let ρ be an axis of ϕ in Λ_i^*. We parametrize ρ such that $\rho(\infty) = \omega_i$. Since Λ_i^* is a tree and $\sigma_i(\infty) = \rho(\infty)$, we can actually choose the parameter such that $\sigma_i(t) = \rho(t)$ for all $t \ge t_\phi$, where t_ϕ is an appropriate constant. Now $\phi(\rho(t)) = \rho(t+\tau)$ for some constant τ independent of t. We conclude that $d_i(\phi(p), p) \le 2t_\phi + \tau$ for each point p in the image of σ_i. □

PROPOSITION 4.9. *Suppose that indices of type* 1 *occur. Then*

(1) *Δ does not fix a point of X;*

(2) *Γ fixes a point in $X(\infty)$. More precisely, if (x_m) is a sequence in X such that* $\mathrm{Stab}(x_m) \to \Delta$, *then after passing to a subsequence if necessary, (x_m) converges to a fixed point $\xi \in X(\infty)$ of Γ.*

Proof. The first assertion is an immediate consequence of Lemma 4.7. As for the proof of the second assertion, let (x_m) be a sequence in X with $\mathrm{Stab}(x_m) \to \Delta$. Let $\gamma_m \colon [0, s_m] \to X$ be the unit speed geodesic from x_0 to x_m as in Lemma 4.7. Note that $r \circ \gamma_m$ is a sequence of unit speed curves (with respect to the metric $d_{(2)}$, for which r restricted to any chamber of X is an isometry) in $\prod \Lambda_i^*$. For each constant $t_0 > 0$, $r \circ \gamma_m([0, t_0])$ is contained in F for all m sufficiently large. Now F is locally compact, hence a subsequence of the sequence of curves $r \circ \gamma_m$ converges locally uniformly. By Proposition 3.4, the corresponding subsequence of the sequence of unit speed geodesics γ_m converges locally uniformly. By definition, this means that the corresponding subsequence of (x_m) converges to a point $\xi \in X(\infty)$.

Let $\phi \in \Gamma$ and choose $c = c_\phi$ as in Lemma 4.8. Let $t_0 > 0$ be given. By Lemma 4.8 we have $r \circ \gamma_m(t_0) \in F$ for all $m \geq m_0$. By Proposition 3.4 and Lemma 4.8, we have $d(\phi(\gamma_m(t_0)), \gamma_m(t_0)) \leq \sqrt{n}c_\phi$ for all such m. Now c_ϕ is independent of t_0, hence $\phi(\xi) = \xi$. □

We now complete the proof of Theorem 2 of the introduction. By Proposition 4.1, $\Delta \cong \ker h$ consists precisely of the elliptic elements of Γ. If indices of type 1 do not occur, then Proposition 4.3 applies: If $k = 0$, then $\Gamma \cong \Delta$ fixes a point of X and possibility (1) holds. If $k > 0$, then possibility (2) holds. If indices of type 1 occur, then possibility (3) holds by Proposition 4.9 and Corollary 4.5. Note that $\mathrm{Stab}(x) \neq \Delta$ for any $x \in X$ in this case since Δ would have a fixed point otherwise.

5. PARALLEL TRANSPORT IN A CUBICAL MANIFOLD AND THE PROOF OF THEOREM 3

Let X be a cubical manifold of dimension n. Given two chambers P and Q in X with a common face of dimension $n-1$, we define $t_{PQ}\colon P \to Q$ to be the *translation* which moves each point p of P along the unit geodesic segment starting at p and orthogonal to the common $(n-1)$-face of P to the end point in Q. The map t_{PQ} is an isomorphism and isometry of P with Q. Given a gallery $\pi = (P_1, \dots, P_n)$ in X, the *parallel transport* along π is the isomorphism $t_\pi\colon P_1 \to P_n$ given by

$$t_\pi := t_{P_{n-1}P_n} \circ \cdots \circ t_{P_2P_3} \circ t_{P_1P_2} .$$

LEMMA 5.1. *Let X be a simply connected cubical manifold and assume that the number of chambers adjacent to each face of codimension 2 in X is divisible by 4. Then for any two chambers P and Q in X, the parallel transport t_π along a gallery π connecting P and Q is independent of π.*

Proof. It is enough to show that the parallel transport along any closed gallery is the identity. Let π be such a gallery with initial and final chamber P.

Represent π by a closed curve c which starts and ends in some interior point p of P, such that c misses the $(n-2)$-skeleton of X and crosses $(n-1)$-faces transversally and according to the pattern provided by π. Since X is simply connected, the curve c can be contracted in X to the point p. Since X is a manifold, the links of the vertices in X are $(n-2)$-connected. Hence the contraction of c can be chosen to be generic in the sense that it misses the $(n-3)$-skeleton of X and crosses the $(n-2)$-skeleton transversally. Following the curve c along this contraction, we get a sequence of modifications of the gallery π. These modifications occur when c crosses an $(n-2)$-face of X. The condition that the number of chambers adjacent to such faces is divisible by 4 implies that the parallel transport t_π does not change under these modifications. Since the parallel transport along the trivial gallery is the identity, $t_\pi = \mathrm{id}_P$. □

From now on we assume that X is a simply connected cubical manifold such that the number of chambers adjacent to each face of codimension 2 in X is divisible by 4. For chambers P and Q in X define $t_{PQ} = t_\pi$, where π is any gallery connecting P with Q. The above lemma shows that t_{PQ} is well defined.

We fix a chamber P_0 of X and define a homomorphism $\phi\colon \Gamma \to \mathrm{Aut}\, P_0$ by

$$\phi(g) := t_{g(P_0)P_0} \circ g|_{P_0}\,.$$

The kernel $\Gamma' := \ker\phi$ is a finite index subgroup of Γ and consists precisely of those automorphisms of Γ whose restriction to any chamber commutes with the corresponding parallel transport.

COORIENTATIONS

A *coorientation of a wall* in a chamber is a choice of one of the two half-chambers determined by the wall. Once and for all, we choose coorientations of the walls in the above chamber P_0. Now by Lemma 5.1, parallel transport gives rise to a consistent choice of coorientations for all walls in X.

By Corollary 1.3, X is foldable. We fix a folding and denote by Λ_i the set of hyperspaces of X with label i. Note that Λ_i is invariant under parallel transport. Along a hyperspace with label i, the half-chambers distinguished by the coorientation are all contained in the same halfspace with respect to the hyperspace. The above group Γ' preserves the families Λ_i together with the coorientations.

The index of intersection of an oriented curve c at a transversal crossing of a hyperspace $H \in \Lambda_i$ is defined to be equal to $+1$ or -1 respectively, according to whether the orientation of c coincides with the coorientation of H or not. Fix a point p_0 in the interior of P_0 which does not belong to any wall and any of the chosen coorientations. For $p \in X$ define $f_i(p)$ to be the sum of the indices of intersection of an oriented curve c connecting p_0 and p with the hyperspaces from Λ_i. Here we assume that c is generic, i.e. c does not meet the $(n-2)$-skeleton and crosses hyperspaces transversally. The integer $f_i(p)$ does not depend on c since X is simply connected and any two such curves can be deformed into each other by a homotopy which misses the $(n-3)$-skeleton of X and crosses the $(n-2)$-skeleton of X transversally.

For $g \in \Gamma$ set $h_i(g) = f_i(g(p_0))$. Since the chosen system of coorientations is invariant under the action of Γ', the maps $h_i \colon \Gamma' \to \mathbf{Z}$ are homomorphisms. We finish the proof of Theorem 3 by showing that the image of $h = (h_1, \dots, h_n)$ is of finite index in $\mathbf{Z}^n$.

We need to show that the image of h contains n linearly independent vectors. To that end, we show that the image contains non-zero vectors which span arbitrarily small angles with the unit vectors e_i in $\mathbf{R}^n$, $1 \leq i \leq n$. Let σ be a unit speed geodesic ray with $\sigma(0) = p_0$ which is perpendicular in P_0 to the wall with label i. By the choice of p_0, the ray σ does not meet the $(n-2)$-skeleton of X and is perpendicular to all $(n-1)$-faces and walls with label i which it intersects. We have

$$f_j(\sigma(m)) = \delta_{ij} \cdot m, \quad m \geq 1 .$$

By the cocompactness of the action of Γ', there is an integer $k \geq 1$ such that for any $m \geq 1$ there is a $g_m \in \Gamma'$ with $d(\sigma(m), g_m(p_0)) \leq k$. By the definition of h_j this implies $|h_j(g_m) - f_j(\sigma(m))| \leq k$. Theorem 3 follows.

REFERENCES

[AB] ALEXANDER, S. and R. BISHOP. The Hadamard-Cartan theorem in locally convex metric spaces. *L'Enseignement Math. (2) 36* (1990), 309–320.

[Ba] BALLMANN, W. *Lectures on Spaces of Nonpositive Curvature.* DMV Seminar 25. Birkhäuser Verlag, 1995.

[BB] BALLMANN, W. and M. BRIN. Polygonal complexes and combinatorial group theory. *Geom. Dedicata 50* (1994), 165–191.

[BS] BALLMANN, W. and J. ŚWIĄTKOWSKI. On L^2-cohomology and property (T) for automorphism groups of polyhedral cell complexes. *GAFA 7* (1997), 615–645.

[Bar] BARRE, S. Immeubles de Tits triangulaires exotiques. Preprint, 1998.

[Be] BENAKLI, N. Polygonal complexes I: Combinatorial and geometric properties. *J. Pure Appl. Algebra 97* (1994), 247–263.

[B-Z] BJÖRNER, B., M. LAS VERGNAS, B. STURMFELS, N. WHITE and G. ZIEGLER. *Oriented Matroids.* Cambridge University Press, 1993.

[B1] BRIDSON, M. Geodesics and curvature in metric simplicial complexes. In: *Group theory from a geometrical viewpoint.* E. Ghys, A. Haefliger, A. Verjovsky, eds. Proceedings ICTP, Trieste. World Scientific, 1991.

[B2] —— On the semisimplicity of polyhedral isometries. Preprint, 1997.

[BH] BRIDSON, M. and A. HAEFLIGER. *Metric Spaces of Non-Positive Curvature.* Book preprint.

[Br] BRØNDSTED, A. *An Introduction to Convex Polytopes.* Graduate Texts in Math. 90. Springer, New York, 1983.

[CD] R. CHARNEY, R. and M. DAVIS. Strict hyperbolization. *Topology 34* (1995), 329–350.

[CE] CHEEGER, J. and D. EBIN. *Comparison Theorems in Riemannian Geometry.* North Holland, Amsterdam-New York, 1975.

[Da] DAVIS, M. Buildings are *CAT*(0). In: *Proc. LMS Durham Symposium on Geometry and Cohomology in Group Theory.* P. Kropholler, R. Stohr, eds. LMS Lecture Notes, Cambridge University Press (to appear).

[DJ1] DAVIS, M. and T. JANUSZKIEWICZ. Hyperbolization of polyhedra. *J. Differential Geom. 34* (1991), 347–388.

[DJ2] DAVIS, M. and T. JANUSZKIEWICZ. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J. 62* (1991), 417–451.

[DJS] DAVIS, M., T. JANUSZKIEWICZ and P. SCOTT. Nonpositive curvature of blow-ups. *Selecta Math.* (to appear).

[Go] GONCIULEA, C. Infinite Coxeter groups virtually surject onto **Z**. Preprint, Ohio State University, 1996.

[Gr] GROMOV, M. Hyperbolic groups. In: *Essays in Group Theory,* S. M. Gersten ed. *M.S.R.I. Publ. 8.* Springer Verlag, 1987.

[H] HAGLUND, F. Les polyèdres de Gromov. *C. R. Acad. Sci. Paris 313* (1991), 603–606.

[HV] DE LA HARPE, P. and A. VALETTE. La propriété (T) de Kazhdan pour les groupes localement compacts. *Astérisque 175.* Soc. Math. France, 1989.

[Mo] MOUSSONG, G. Hyperbolic Coxeter groups. PhD thesis (Ohio State University), 1988.

[NR] NIBLO G. and L. REEVES. Groups acting on $CAT(0)$ cube complexes. *Geom. Topol. 1* (1997), 1–7.

[PV] PAYS, I. and A. VALETTE. Sous-groupes libres dans les groupes d'automorphismes d'arbres. *L'Enseignement Math. (2) 37* (1991), 151–174.

[Sw] ŚWIĄTKOWSKI, J. Trivalent polygonal complexes of nonpositive curvature and Platonic symmetry. *Geom. Dedicata 70* (1998), 87–110.

(Reçu le 14 mai 1998)

Werner Ballmann

Mathematisches Institut der Universität Bonn
Wegelerstrasse 10
53115 Bonn
Germany
e-mail: ballmann@math.uni-bonn.de

Jacek Świątkowski

Instytut Matematyczny
Uniwersytet Wrocławski
pl. Grunwaldzki 2/4
50-384 Wrocław
Poland
e-mail: swiatkow@math.uni.wroc.pl

vide-leer-empty