Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 45 (1999)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CLASS NUMBER FORMULAE FOR IMAGINARY QUADRATIC NUMBER FIELDS $Q(\sqrt{-n})$ WITH n SQUAREFREE AND n 1 (mod 4) OR n 2 (mod 4)

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Kapitel: 1. Introduction and summary

DOI: https://doi.org/10.5169/seals-64455

L'Enseignement Mathématique, t. 45 (1999), p. 349–355

CLASS NUMBER FORMULAE FOR IMAGINARY QUADRATIC NUMBER FIELDS $\mathbf{Q}(\sqrt{-n})$ WITH n SQUAREFREE AND $n \equiv 1 \pmod 4$ OR $n \equiv 2 \pmod 4$

by Richard H. HUDSON, Charles J. JUDGE and Turker TEKER

1. INTRODUCTION AND SUMMARY

Let $\mathbf{Q}(\sqrt{-n})$ denote an imaginary quadratic number field where throughout n will always be a positive, squarefree integer and let $h(-n)$ denote its class number. Berndt and Chowla [2] showed that if $p \equiv 3 \pmod 4$, then the Legendre symbol $\left(\frac{a}{p}\right)$ summed over certain subintervals of $(0,p)$ is equal to zero. The result leads immediately to interesting class number formulae in terms of the remaining subintervals of $(0,p)$ using Dirichlet's classical results ([3], [4]), and the results are easily generalized to composite moduli $n \equiv 3 \pmod 4$. Berndt and Chowla remark that it would be interesting to obtain similar results for $p \equiv 1 \pmod 4$. In this paper we show that a simple and elementary modification of Berndt and Chowla's method, when used in conjunction with the Jacobi symbol $\left(\frac{-4n}{a}\right)$ in subintervals of $(0,2n)$, as suggested by Dirichlet [3], [4], leads to class number formulae relating values of $\left(\frac{-4n}{a}\right)$ in subintervals of $(0,2n)$ to $h(-n)$ for either $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$. In particular, in section two we prove the following theorem (throughout $[x]$ denotes the greatest integer $\leq x$).

THEOREM. *Let n be a positive, squarefree integer with either $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and with $(a, 2n) = 1$, and let j be a positive integer with $(j, 2n) = 1$ and $1 \leq j \leq n$. Then if $\left(\frac{-4n}{j}\right) = +1$, we have*

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} \sum_{a=\left[\frac{4in}{j}\right]+1}^{\left[\frac{(4i+2)n}{j}\right]} \left(\frac{-4n}{a}\right),$$

and if $\left(\frac{-4n}{j}\right) = -1$, then we have

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} \sum_{a=\left[\frac{(4i-2)n}{j}\right]+1}^{\left[\frac{4in}{j}\right]} \left(\frac{-4n}{a}\right).$$

If $j = 1$, the result is due to Dirichlet [3], [4]. We illustrate the theorem when $n = 13$ and $j = 3$. Then $\left(\frac{-52}{3}\right) = \left(\frac{-1}{3}\right) = -1$. Thus

$$h(-13) = \frac{1}{2} \sum_{a=9}^{17} \left(\frac{-52}{a}\right).$$

Now $\left(\frac{-52}{9}\right) = \left(\frac{-52}{11}\right) = \left(\frac{-52}{15}\right) = \left(\frac{-52}{17}\right) = +1$, and so $h(-13) = \frac{1}{2}(4) = 2$. The study of class numbers relating values of the Jacobi symbol $\left(\frac{a}{n}\right)$ to $h(-n)$ when $n \equiv 3 \pmod 4$ in subintervals other than $(0, \frac{n}{2})$ has been given by numerous authors. These include among others, Berndt [1], Berndt and Chowla [2], Dirichlet [3]–[4], Holden [5]–[11], Hudson and Williams [12], Johnson and Mitchell [13], Karpinski [14], and Lerch [15]–[16]. A partial summary of these results appears in [12].

2. PROOF OF THE THEOREM

We first note that j is an odd, positive integer with $(j, n) = 1$. We write

$$\sum_{\substack{a=1 \\ (a,2n)=1}}^{2n-1} \left(\frac{-4n}{a}\right) = \sum_{r=0}^{j-1} S_r$$

where