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CLASS NUMBER FORMULAE FOR IMAGINARY QUADRATIC  
NUMBER FIELDS  $\mathbf{Q}(\sqrt{-n})$  WITH  $n$  SQUAREFREE AND  
 $n \equiv 1 \pmod{4}$  OR  $n \equiv 2 \pmod{4}$

by Richard H. HUDSON, Charles J. JUDGE and Turker TEKER

1. INTRODUCTION AND SUMMARY

Let  $\mathbf{Q}(\sqrt{-n})$  denote an imaginary quadratic number field where throughout  $n$  will always be a positive, squarefree integer and let  $h(-n)$  denote its class number. Berndt and Chowla [2] showed that if  $p \equiv 3 \pmod{4}$ , then the Legendre symbol  $\left(\frac{a}{p}\right)$  summed over certain subintervals of  $(0, p)$  is equal to zero. The result leads immediately to interesting class number formulae in terms of the remaining subintervals of  $(0, p)$  using Dirichlet's classical results ([3], [4]), and the results are easily generalized to composite moduli  $n \equiv 3 \pmod{4}$ . Berndt and Chowla remark that it would be interesting to obtain similar results for  $p \equiv 1 \pmod{4}$ . In this paper we show that a simple and elementary modification of Berndt and Chowla's method, when used in conjunction with the Jacobi symbol  $\left(\frac{-4n}{a}\right)$  in subintervals of  $(0, 2n)$ , as suggested by Dirichlet [3], [4], leads to class number formulae relating values of  $\left(\frac{-4n}{a}\right)$  in subintervals of  $(0, 2n)$  to  $h(-n)$  for either  $n \equiv 1 \pmod{4}$  or  $n \equiv 2 \pmod{4}$ . In particular, in section two we prove the following theorem (throughout  $[x]$  denotes the greatest integer  $\leq x$ ).

**THEOREM.** *Let  $n$  be a positive, squarefree integer with either  $n \equiv 1 \pmod{4}$  or  $n \equiv 2 \pmod{4}$  and with  $(a, 2n) = 1$ , and let  $j$  be a positive integer with  $(j, 2n) = 1$  and  $1 \leq j \leq n$ . Then if  $\left(\frac{-4n}{j}\right) = +1$ , we have*

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} \sum_{a=\left[\frac{4in}{j}\right]+1}^{\left[\frac{(4i+2)n}{j}\right]} \left(\frac{-4n}{a}\right),$$

*and if  $\left(\frac{-4n}{j}\right) = -1$ , then we have*

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} \sum_{a=\left[\frac{(4i-2)n}{j}\right]+1}^{\left[\frac{4in}{j}\right]} \left(\frac{-4n}{a}\right).$$

If  $j = 1$ , the result is due to Dirichlet [3], [4]. We illustrate the theorem when  $n = 13$  and  $j = 3$ . Then  $\left(\frac{-52}{3}\right) = \left(\frac{-1}{3}\right) = -1$ . Thus

$$h(-13) = \frac{1}{2} \sum_{a=9}^{17} \left(\frac{-52}{a}\right).$$

Now  $\left(\frac{-52}{9}\right) = \left(\frac{-52}{11}\right) = \left(\frac{-52}{15}\right) = \left(\frac{-52}{17}\right) = +1$ , and so  $h(-13) = \frac{1}{2}(4) = 2$ . The study of class numbers relating values of the Jacobi symbol  $\left(\frac{a}{n}\right)$  to  $h(-n)$  when  $n \equiv 3 \pmod{4}$  in subintervals other than  $(0, \frac{n}{2})$  has been given by numerous authors. These include among others, Berndt [1], Berndt and Chowla [2], Dirichlet [3]–[4], Holden [5]–[11], Hudson and Williams [12], Johnson and Mitchell [13], Karpinski [14], and Lerch [15]–[16]. A partial summary of these results appears in [12].

## 2. PROOF OF THE THEOREM

We first note that  $j$  is an odd, positive integer with  $(j, n) = 1$ . We write

$$\sum_{\substack{a=1 \\ (a, 2n)=1}}^{2n-1} \left(\frac{-4n}{a}\right) = \sum_{r=0}^{j-1} S_r$$

where