

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CLASS NUMBER FORMULAE FOR IMAGINARY QUADRATIC NUMBER FIELDS $\mathbb{Q}(\sqrt{-n})$ WITH n SQUAREFREE AND $n \equiv 1 \pmod{4}$ OR $n \equiv 2 \pmod{4}$
Autor: HUDSON, Richard H. / Judge, Charles J. / TEKER, Turker
DOI: <https://doi.org/10.5169/seals-64455>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 25.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

CLASS NUMBER FORMULAE FOR IMAGINARY QUADRATIC
NUMBER FIELDS $\mathbf{Q}(\sqrt{-n})$ WITH n SQUAREFREE AND
 $n \equiv 1 \pmod{4}$ OR $n \equiv 2 \pmod{4}$

by Richard H. HUDSON, Charles J. JUDGE and Turker TEKER

1. INTRODUCTION AND SUMMARY

Let $\mathbf{Q}(\sqrt{-n})$ denote an imaginary quadratic number field where throughout n will always be a positive, squarefree integer and let $h(-n)$ denote its class number. Berndt and Chowla [2] showed that if $p \equiv 3 \pmod{4}$, then the Legendre symbol $\left(\frac{a}{p}\right)$ summed over certain subintervals of $(0, p)$ is equal to zero. The result leads immediately to interesting class number formulae in terms of the remaining subintervals of $(0, p)$ using Dirichlet's classical results ([3], [4]), and the results are easily generalized to composite moduli $n \equiv 3 \pmod{4}$. Berndt and Chowla remark that it would be interesting to obtain similar results for $p \equiv 1 \pmod{4}$. In this paper we show that a simple and elementary modification of Berndt and Chowla's method, when used in conjunction with the Jacobi symbol $\left(\frac{-4n}{a}\right)$ in subintervals of $(0, 2n)$, as suggested by Dirichlet [3], [4], leads to class number formulae relating values of $\left(\frac{-4n}{a}\right)$ in subintervals of $(0, 2n)$ to $h(-n)$ for either $n \equiv 1 \pmod{4}$ or $n \equiv 2 \pmod{4}$. In particular, in section two we prove the following theorem (throughout $[x]$ denotes the greatest integer $\leq x$).

THEOREM. *Let n be a positive, squarefree integer with either $n \equiv 1 \pmod{4}$ or $n \equiv 2 \pmod{4}$ and with $(a, 2n) = 1$, and let j be a positive integer with $(j, 2n) = 1$ and $1 \leq j \leq n$. Then if $\left(\frac{-4n}{j}\right) = +1$, we have*

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} \sum_{a=\left[\frac{4in}{j}\right]+1}^{\left[\frac{(4i+2)n}{j}\right]} \left(\frac{-4n}{a}\right),$$

and if $\left(\frac{-4n}{j}\right) = -1$, then we have

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} \sum_{a=\left[\frac{(4i-2)n}{j}\right]+1}^{\left[\frac{4in}{j}\right]} \left(\frac{-4n}{a}\right).$$

If $j = 1$, the result is due to Dirichlet [3], [4]. We illustrate the theorem when $n = 13$ and $j = 3$. Then $\left(\frac{-52}{3}\right) = \left(\frac{-1}{3}\right) = -1$. Thus

$$h(-13) = \frac{1}{2} \sum_{a=9}^{17} \left(\frac{-52}{a}\right).$$

Now $\left(\frac{-52}{9}\right) = \left(\frac{-52}{11}\right) = \left(\frac{-52}{15}\right) = \left(\frac{-52}{17}\right) = +1$, and so $h(-13) = \frac{1}{2}(4) = 2$. The study of class numbers relating values of the Jacobi symbol $\left(\frac{a}{n}\right)$ to $h(-n)$ when $n \equiv 3 \pmod{4}$ in subintervals other than $(0, \frac{n}{2})$ has been given by numerous authors. These include among others, Berndt [1], Berndt and Chowla [2], Dirichlet [3]–[4], Holden [5]–[11], Hudson and Williams [12], Johnson and Mitchell [13], Karpinski [14], and Lerch [15]–[16]. A partial summary of these results appears in [12].

2. PROOF OF THE THEOREM

We first note that j is an odd, positive integer with $(j, n) = 1$. We write

$$\sum_{\substack{a=1 \\ (a, 2n)=1}}^{2n-1} \left(\frac{-4n}{a}\right) = \sum_{r=0}^{j-1} S_r$$

where

$$S_r = \sum_{\substack{a=1 \\ a \equiv r \pmod{j} \\ (a,2n)=1}}^{2n-1} \left(\frac{-4n}{a} \right).$$

If $1 \leq r \leq j-1$, then there exists a unique integer k such that $1 \leq k \leq j-1$ and $2kn \equiv r \pmod{j}$ because $(j, n) = 1$. If $a \equiv r \pmod{j}$ with $1 \leq a \leq 2n-1$ and $(a, 2n) = 1$, then we observe that $2kn - a \equiv 0 \pmod{j}$. Now

$$\left(\frac{-4n}{a} \right) = \left(\frac{-4n}{2kn - a} \right)$$

if k is odd, and

$$\left(\frac{-4n}{a} \right) = - \left(\frac{-4n}{2kn - a} \right)$$

if k is even. Thus,

$$\begin{aligned} S_r &= \sum_{\substack{a=(2k-2)n \\ a \equiv 0 \pmod{j} \\ (a,2n)=1}}^{2kn} \left(\frac{-4n}{2kn - a} \right) \\ &= \pm \sum_{\substack{a=(2k-2)n \\ a \equiv 0 \pmod{j} \\ (a,2n)=1}}^{2kn} \left(\frac{-4n}{a} \right) \\ &= \pm \left(\frac{-4n}{j} \right) \sum_{\substack{a=[\frac{(2k-2)n}{j}] + 1 \\ (a,2n)=1}}^{[\frac{2kn}{j}]} \left(\frac{-4n}{a} \right) \end{aligned}$$

where the plus sign holds if k is odd and the minus sign holds if k is even. Thus we have for each j ,

$$\begin{aligned} 0 &= \left(\frac{-4n}{j} \right) \sum_{\substack{k=1 \\ (k,2)=2}}^j \sum_{\substack{a=[\frac{(2k-2)n}{j}] + 1 \\ (a,2n)=1}}^{[\frac{2kn}{j}]} \left(\frac{-4n}{a} \right) \\ &+ \left(\frac{-4n}{j} \right) \sum_{\substack{k=1 \\ (k,2)=1}}^j \sum_{\substack{a=[\frac{(2k-2)n}{j}] + 1 \\ (a,2n)=1}}^{[\frac{2kn}{j}]} \left(\frac{-4n}{a} \right) - \sum_{\substack{a=1 \\ (a,2n)=1}}^{2n-1} \left(\frac{-4n}{a} \right). \end{aligned}$$

It then follows that

$$0 = -2 \sum_{\substack{k=1 \\ (k,2)=2}}^j \sum_{\substack{a=\lfloor \frac{(2k-2)n}{j} \rfloor + 1 \\ (a,2n)=1}}^{\lfloor \frac{2kn}{j} \rfloor} \left(\frac{-4n}{a} \right)$$

if $\left(\frac{-4n}{j}\right) = +1$, and

$$0 = -2 \sum_{\substack{k=1 \\ (k,2)=1}}^j \sum_{\substack{a=\lfloor \frac{(2k-2)n}{j} \rfloor + 1 \\ (a,2n)=1}}^{\lfloor \frac{2kn}{j} \rfloor} \left(\frac{-4n}{a} \right)$$

if $\left(\frac{-4n}{j}\right) = -1$. In the case that $\left(\frac{-4n}{j}\right) = +1$, we are only considering those k which are even, and so we may write $k = 2i$. In the case that $\left(\frac{-4n}{j}\right) = -1$, we are only considering those k which are odd, and so we may write $k = 2i + 1$.

Thus we have proven that for each j ,

$$0 = \sum_{i=1}^{\lfloor \frac{j}{2} \rfloor} \sum_{\substack{a=\lfloor \frac{(4i-2)n}{j} \rfloor + 1 \\ (a,2n)=1}}^{\lfloor \frac{4in}{j} \rfloor} \left(\frac{-4n}{a} \right)$$

if $\left(\frac{-4n}{j}\right) = +1$, and

$$0 = \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{\substack{a=\lfloor \frac{4in}{j} \rfloor + 1 \\ (a,2n)=1}}^{\lfloor \frac{(4i+2)n}{j} \rfloor} \left(\frac{-4n}{a} \right)$$

if $\left(\frac{-4n}{j}\right) = -1$. These subintervals clearly cover $[1, 2n - 1]$ and are non-overlapping. Now Dirichlet [3], [4] showed that

$$\sum_{\substack{a=1 \\ (a,2n)=1}}^{2n-1} \left(\frac{-4n}{a} \right) = 2h(-n).$$

It follows at once that

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} \sum_{a=\lfloor \frac{4i}{j} \rfloor + 1}^{\lfloor \frac{(4i+2)n}{j} \rfloor} \left(\frac{-4n}{a} \right)$$

if $\left(\frac{-4n}{j} \right) = +1$, and

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} \sum_{a=\lfloor \frac{(4i-2)n}{j} \rfloor + 1}^{\lfloor \frac{4in}{j} \rfloor} \left(\frac{-4n}{a} \right)$$

if $\left(\frac{-4n}{j} \right) = -1$.

3. REMARKS

In Bruce Berndt's paper "Classical Theorems on Quadratic Residues" [1], he uses the following notation:

$$S_{ji} = \sum_{\frac{(i-1)k}{j} < n < \frac{ik}{j}} \chi(n).$$

Using this notation, we can rewrite the class number formulae as follows:

1. If $\left(\frac{-4n}{j} \right) = +1$, then we have

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} S_{j,2i+1}.$$

2. If $\left(\frac{-4n}{j} \right) = -1$, then we have

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} S_{j,2i}.$$

4. ACKNOWLEDGEMENTS

We would like to thank Zekeriya Tufekci at Clemson University for numerical data which assisted us in obtaining the results in this paper.

REFERENCES

- [1] BERNDT, B. C. Classical theorems on quadratic residues. *L'Enseignement Math.* (2) 22 (1976), 261–304.
- [2] BERNDT, B. C. and S. CHOWLA. Zero sums of the Legendre symbol. *Nordisk Matematisk Tidsskrift* 22 (1974), 5–8.
- [3] DIRICHLET, G. L. Recherches sur diverses applications de l'analyse infinitésimale à la théorie des nombres (Première Partie). *J. reine angew. Math.* 19 (1839), 324–369.
- [4] — Recherches sur diverses applications de l'analyse infinitésimale à la théorie des nombres (Seconde Partie). *J. reine angew. Math.* 21 (1840), 134–155.
- [5] HOLDEN, H. On various expressions for h , the number of properly primitive classes for a determinant $-p$, where p is a prime of the form $4n + 3$ (First Paper). *The Messenger of Mathematics* 35 (1905/1906), 73–80.
- [6] — On various expressions for h , the number of properly primitive classes for a determinant $-p$, where p is a prime of the form $4n + 3$, and is a prime or the product of different primes (Second Paper). *The Messenger of Mathematics* 35 (1905/1906), 102–110.
- [7] — On various expressions for h , the number of properly primitive classes for any negative determinant, not involving a square factor (Third Paper). *The Messenger of Mathematics* 35 (1905/1906), 110–117.
- [8] — On various expressions for h , the number of properly primitive classes for a negative determinant (Fourth Paper). *The Messenger of Mathematics* 36 (1906/1907), 69–75.
- [9] — On various expressions for h , the number of properly primitive classes for a determinant $-p$, where p is a prime of the form $4n + 3$, and is a prime or the product of different primes (Addition to the Second Paper). *The Messenger of Mathematics* 36 (1906/1907), 75–77.
- [10] — On various expressions for h , the number of properly primitive classes for a negative determinant not containing a square factor (Fifth Paper). *The Messenger of Mathematics* 36 (1906/1907), 126–134.
- [11] — On various expressions for h , the number of properly primitive classes for any negative determinant not containing a square factor (Sixth Paper). *The Messenger of Mathematics* 37 (1907/1908), 13–16.
- [12] HUDSON, R. H. and K. S. WILLIAMS. Class number formulae of Dirichlet type. *Mathematics of Computation* 39 (1982), 725–732.
- [13] JOHNSON, W. and K. J. MITCHELL. Symmetries for sums of the Legendre symbol. *Pacific Journal of Mathematics* 69 (1977), 117–124.
- [14] KARPINSKI, L. C. Über die Verteilung der quadratischen Reste. *J. reine angew. Math.* 127 (1904), 1–19.

- [15] LERCH, M. Essais sur le calcul du nombre des classes de formes quadratiques binaires aux coefficients entiers. *Acta Mathematica* 29 (1905), 333–424.
- [16] ——— Essais sur le calcul du nombre des classes de formes quadratiques binaires aux coefficients entiers. *Acta Mathematica* 30 (1906), 203–293.

(Reçu le 6 octobre 1998)

Richard H. Hudson

Charles J. Judge

Turker Teker

University of South Carolina

Columbia, SC 29208-0103

U. S. A.

Vide-leer-empty