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**Autor:** UK, Andrzej  
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## 5. LOWER BOUNDS

Now we will consider generalized Følner sequences for functions  $f$  such that

$$Pf \geq \|P\|f.$$

This will enable us to obtain some lower bounds on the norm of random walk operators on graphs.

As in Section 3, let  $X$  be a connected, locally finite graph and let  $P$  be the simple random walk operator on  $X$ .

In this section we will prove the following lower bound on the norm  $\|P\|$  :

**THEOREM 8.** *Let  $X$  be a graph such that at each vertex there are at most  $k$  edges. Then*

$$\|P\| \geq \frac{2\sqrt{k-1}}{k}.$$

The norm of the random walk operator  $\|P\|$  is equal to  $\frac{2\sqrt{k-1}}{k}$  for the random walk on the tree which has  $k$  edges at each vertex. In [9] Kesten proved this lower bound in the case of Cayley graphs.

*Proof of Theorem 8.* Let us consider a graph  $X$  such that at each vertex there are at most  $k$  edges. We can suppose that  $k \geq 3$  because for  $k = 2$  we obtain subgraphs of  $\mathbf{Z}$  or finite graphs, and necessarily  $\|P\| = 1$ . As it is enough to prove the desired bound for any connected component of  $X$ , we can suppose that  $X$  is connected.

In order to show that  $\|P\|$  is large enough, we will construct a sequence of functions  $f_n \in l^2(X, N)$  such that

$$\limsup_{n \rightarrow +\infty} \frac{\|Pf_n\|_{l^2(X, N)}}{\|f_n\|_{l^2(X, N)}} \geq \frac{2\sqrt{k-1}}{k}.$$

Let us endow the set of vertices of  $X$  with a metric. The distance between two vertices is the smallest number of edges needed to connect them. Let us choose a vertex  $e$  in  $X$  and for a vertex  $v$  let us denote by  $|v|$  its distance from  $e$ .

Let  $f$  be the unique (up to translations and multiplications) radial eigenfunction of  $P$  on the homogeneous tree of degree  $k$ , corresponding to the eigenvalue  $\frac{2\sqrt{k-1}}{k}$ , which is the norm of  $P$  on this tree, i.e.

$$(15) \quad f(v) = g(|v|) = \left( \frac{k-2}{k} |v| + 1 \right) \left( \frac{1}{\sqrt{k-1}} \right)^{|v|}.$$

Using (15) we can define  $f$  on  $X$ . We then prove

LEMMA 6. For any vertex  $v \in X$ ,

$$Pf(v) \geq \frac{2\sqrt{k-1}}{k} f(v).$$

*Proof.* If  $v = e$  the result is clearly true. Let us consider then a vertex  $v \in X$  such that  $n = |v| \geq 1$ . Let the number of neighbors of  $v$  which are at a distance  $n-1$  or  $n$  from  $e$  be equal respectively to  $p$  and  $q$ . So the number of neighbors of  $v$  which are at a distance  $n+1$  is equal to  $N(v) - p - q$ . Hence

$$Pf(v) = \frac{1}{N(v)} (pg(n-1) + qg(n) + (N(v) - p - q)g(n+1)).$$

As  $p \geq 1$  and  $g$  is a decreasing function,

$$Pf(v) \geq \frac{1}{N(v)} (g(n-1) + (N(v) - 1)g(n+1)).$$

As  $N(v) \leq k$  and  $g(n-1) \geq g(n+1)$ ,

$$Pf(v) \geq \frac{1}{k} (g(n-1) + (k-1)g(n+1)) = \frac{2\sqrt{k-1}}{k} g(n). \quad \square$$

Let us denote by  $S_n$  and  $B_n$  the vertices which are respectively at a distance  $n$  and less than or equal to  $n$ .

LEMMA 7.

$$\frac{\sum_{v \in S_{n+1}} f^2(v) N(v)}{\sum_{v \in B_n} f^2(v) N(v)} \rightarrow_{n \rightarrow \infty} 0.$$

*Proof.* As  $1 \leq N(v) \leq k$  it is enough to show that

$$\frac{\sum_{v \in S_{n+1}} f^2(v)}{\sum_{v \in B_n} f^2(v)} \rightarrow_{n \rightarrow \infty} 0.$$

Let us denote

$$a_n = \sum_{v \in S_n} f^2(v) = |S_n| g^2(n).$$

As  $|S_{n+1}| \leq (k-1)|S_n|$  one has

$$(16) \quad a_{n+1} = |S_{n+1}|g^2(n+1) \leq (k-1)|S_n|g^2(n+1) = \left(1 + \frac{k-2}{(k-2)n+k}\right)^2 a_n.$$

We have to show that

$$(17) \quad \frac{\sum_{v \in S_{n+1}} f^2(v)}{\sum_{v \in B_n} f^2(v)} = \frac{a_{n+1}}{a_1 + \dots + a_n} \xrightarrow{n \rightarrow \infty} 0.$$

It is a standard exercise to show that (16) implies (17).  $\square$

Let  $f_n$  be the sequence of functions which are restrictions of  $f$  to the vertices that are at a distance not greater than  $n$ :

$$f_n = f|_{B_n}.$$

By Lemma 6 and Lemma 7 it follows that

$$\limsup_{n \rightarrow +\infty} \frac{\|Pf_n\|_{l^2(X,N)}}{\|f_n\|_{l^2(X,N)}} \geq \frac{2\sqrt{k-1}}{k},$$

which proves Theorem 8.  $\square$

Some examples of upper bounds on the norm of the simple random walk operator on graphs and their comparison with the lower bound from Theorem 8 can be found in [22].

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