Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: GENERALIZED FØLNER CONDITION AND THE NORMS OF RANDOM WALK OPERATORS ON GROUPS
Autor: UK, Andrzej
Kapitel: 4.4 Wreath products
DOI: https://doi.org/10.5169/seals-64454

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 27.04.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

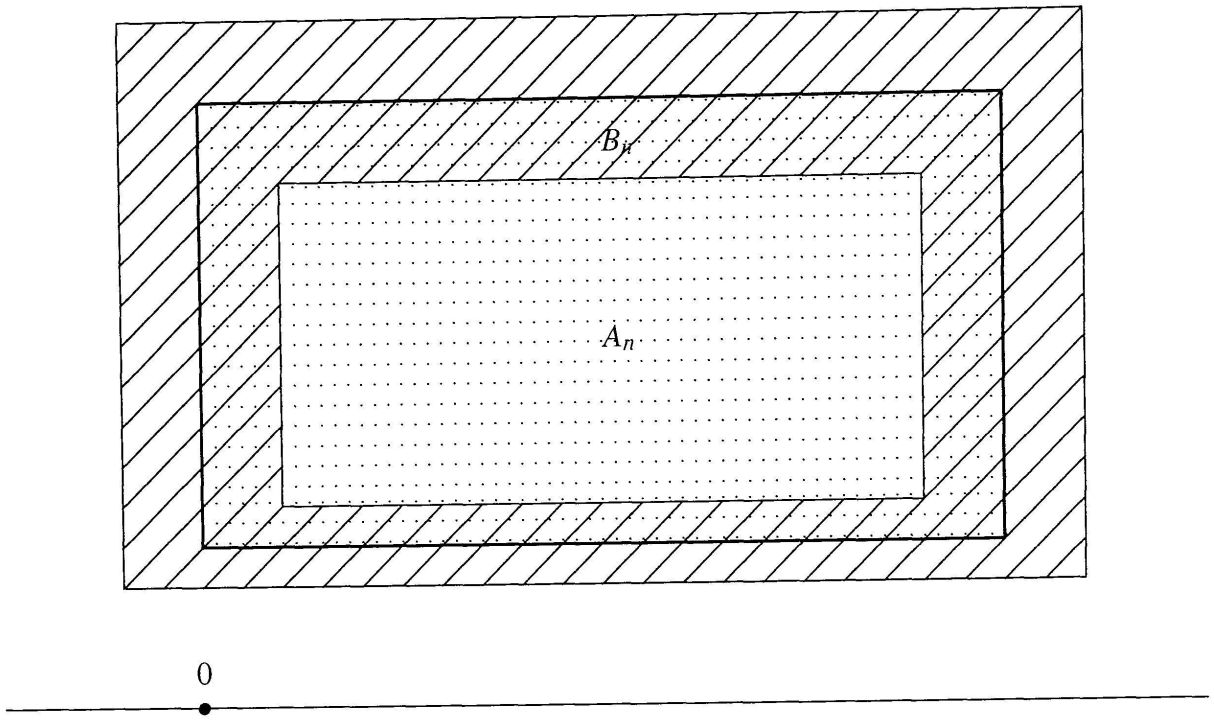

FIGURE 5
Sets A_n and B_n

$$\begin{aligned} B_n = &\{z \in H;\ -R \le \mathrm{Re}(z) \le R,\ e^R \ge \mathrm{Im}(z) \ge e^{-n-R}\} \\ &\cup \{z \in H;\ -R+n \le \mathrm{Re}(z) \le n+R,\ e^R \ge \mathrm{Im}(z) \ge e^{-n-R}\} \\ &\cup \{z \in H;\ -R \le \mathrm{Re}(z) \le n+R,\ e^R \ge \mathrm{Im}(z) \ge e^{-R}\} \\ &\cup \{z \in H;\ -R \le \mathrm{Re}(z) \le n+R,\ e^{-n+R} \ge \mathrm{Im}(z) \ge e^{-n-R}\}. \end{aligned}$$

One can see that

$$|B_n|_{f^2} \approx n, \quad |A_n|_{f^2} \approx n^2.$$

This shows that $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence. Thus

$$\|P\|_{L^2(H,d_H z)\to L^2(H,d_H z)} = \int_{|z-i|=R} \sqrt{\mathrm{Im}(z)}\, dm_R(z).$$

4.4 WREATH PRODUCTS

Let G and F be finitely generated groups. We define the wreath product $G \wr F$ of these groups as follows. Elements of $G \wr F$ are couples (g, γ_1) where $g: F \to G$ is a function such that $g(\gamma)$ is different from the identity element id_G of G only for finitely many elements γ in F, and where γ_1 is an element of F. The multiplication in $G \wr F$ is defined as follows:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where

$$g_3(\gamma) = g_1(\gamma)g_2(\gamma\gamma_1) \quad \text{for } \gamma \in F\,.$$

If S_G and S_F are generators of G and F respectively then

$$\{(g,\gamma)\,;\ (g(F) = id_G,\ \gamma \in S_F) \text{ or } (g(F \setminus id_F) = id_G,\ g(id_F) \in S_G,\ \gamma = id_F)\}$$

is a generating subset for $G \wr F$.

Let μ and ν be symmetric, finitely supported probability measures on F and G respectively.

As there is a natural embedding of F and G into $G \wr F$, one can view the measures μ and ν as measures on $G \wr F$. More precisely:

$$\nu(g,\gamma) = \begin{cases} \nu(g(id_F)) & \text{if } \gamma = id_F \text{ and } g(F \setminus id_F) = id_G \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu(g,\gamma) = \begin{cases} \mu(\gamma) & \text{if } g(F) = id_G \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu \star \nu \star \mu$ is a symmetric measure on $G \wr F$. Explicitly we have:

$$\mu \star \nu \star \mu(g,\gamma) = \begin{cases} \mu(\gamma(\gamma_0)^{-1})\mu(\gamma_0)\nu(g(\gamma_0)) & \text{if } g(F \setminus \gamma_0) = id_G \\ 0 & \text{otherwise.} \end{cases}$$

We want to prove:

THEOREM 7. *Let F and G be finitely generated groups. If F is amenable then the spectral radius of ν on G is the same as the spectral radius of $\mu \star \nu \star \mu$ on $G \wr F$.*

Proof. We will prove Theorem 7 by constructing on $G \wr F$ a positive function $\widetilde{f}$ which is an eigenfunction for the convolution by $\mu \star \nu \star \mu$ with eigenvalue $\|\nu\|_{l^2(G)\to l^2(G)}$ and for which there exists a generalized Følner sequence.

Let f be a positive eigenfunction for the operator which is a convolution on $l^2(G)$ by ν, corresponding to the eigenvalue $\|\nu\|$, i.e.

$$f \star \nu = \|\nu\| f\,. \tag{12}$$

We can normalize f so that

$$f(id_G) = 1\,. \tag{13}$$

By Theorem 3 (and the remark after its proof) there exists a sequence of finite subsets $A_n \subset G$, such that

$$\frac{\sum_{\gamma\in\partial A_n} f^2(\gamma)}{\sum_{\gamma\in A_n} f^2(\gamma)} \to_{n\to\infty} 0\,.$$

As the group F is amenable there exists a sequence of finite subsets $B_n \subset F$, such that

$$\frac{\#\partial B_n}{\#B_n} \to_{n\to\infty} 0\,.$$

For technical reasons let us choose the sequences B_n and A_n in such a way that

$$\frac{\#\partial B_n}{\#B_n} < \frac{1}{n} \quad \text{and} \quad \frac{\sum_{\gamma\in\partial A_n} f^2(\gamma)}{\sum_{\gamma\in A_n} f^2(\gamma)} < \frac{1}{n(\#B_n)}\,. \tag{14}$$

Now, on $G \wr F$ we define $\widetilde{f}$ as follows

$$\widetilde{f}(g,\gamma_1) = \prod_{\gamma\in F} f(g(\gamma))\,.$$

The function $\widetilde{f}$ is well defined because by (13), $f(g(\gamma))$ is different from 1 only for finitely many $\gamma \in F$. This function is of course positive and does not depend on γ_1. From (12) one has

$$\widetilde{f} \star \mu \star \nu \star \mu = \widetilde{f} \star \nu \star \mu = \|\nu\| \widetilde{f} \star \mu = \|\nu\| \widetilde{f}\,.$$

To complete the proof of Theorem 7 it is enough to construct a generalized Følner sequence $C_n \subset G \wr F$ for $\widetilde{f}$. We define C_n as follows:

$$C_n = \{(g,\gamma_1); \gamma_1 \in B_n,\ g(B_n) \subset A_n,\ g^{-1}(G \setminus id_G) \subset B_n\}\,.$$

LEMMA 5. *The sequence $C_n \subset G \wr F$ is a generalized Følner sequence for $\widetilde{f}$.*

Proof. Let us define sets D_n and ∂D_n as follows:

$$D_n = \{g\colon F \to G;\ g(B_n) \subset A_n,\ g^{-1}(G \setminus id_G) \subset B_n\}\,,$$
$$\partial D_n = \{g\colon F \to G;\ \text{there exists } \gamma_0 \in B_n \text{ such that } g(\gamma_0) \in \partial A_n,\ g(B_n \setminus \gamma_0) \subset A_n,\ g^{-1}(G \setminus id_G) \subset B_n\}\,.$$

Thus

$$C_n = D_n \times B_n,$$
$$\partial C_n = (\partial D_n \times B_n) \cup (D_n \times \partial B_n)\,.$$

We have then

$$\sum_{(g,\gamma_1)\in C_n}\big(\widetilde{f}(g,\gamma_1)\big)^2=\sum_{(g,\gamma_1)\in C_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2$$
$$=\sum_{(g,\gamma_1)\in D_n\times B_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2=\#B_n\sum_{g\in D_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2.$$

On the other hand

$$\sum_{(g,\gamma_1)\in\partial C_n}\big(\widetilde{f}(g,\gamma_1)\big)^2=\sum_{(g,\gamma_1)\in\partial C_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2$$
$$=\sum_{(g,\gamma_1)\subset(\partial D_n\times B_n)\cup(D_n\times\partial B_n)}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2$$
$$=\#\partial B_n\sum_{g\in D_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2+\#B_n\sum_{g\in\partial D_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2$$
$$=\frac{\#\partial B_n}{\#B_n}\sum_{(g,\gamma_1)\in C_n}\big(\widetilde{f}(g,\gamma_1)\big)^2$$
$$+\frac{\sum_{g\in\partial D_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2}{\sum_{g\in D_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2}\sum_{(g,\gamma_1)\in C_n}\big(\widetilde{f}(g,\gamma_1)\big)^2.$$

But

$$\sum_{g\in\partial D_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2=\#B_n\frac{\sum_{\alpha\in\partial A_n}f^2(\alpha)}{\sum_{\alpha\in A_n}f^2(\alpha)}\sum_{g\in D_n}\Big(\prod_{\gamma\in F}f(g(\gamma))\Big)^2.$$

Thus by (14)

$$\sum_{(g,\gamma_1)\in\partial C_n}\big(\widetilde{f}(g,\gamma_1)\big)^2=\left(\frac{\#\partial B_n}{\#B_n}+\#B_n\frac{\sum_{\alpha\in\partial A_n}f^2(\alpha)}{\sum_{\alpha\in A_n}f^2(\alpha)}\right)\sum_{(g,\gamma_1)\in C_n}\big(\widetilde{f}(g,\gamma_1)\big)^2$$
$$\le\frac{2}{n}\sum_{(g,\gamma_1)\in C_n}\big(\widetilde{f}(g,\gamma_1)\big)^2,$$

which shows that C_n is a generalized Følner sequence for $\widetilde{f}$. □

This ends the proof of Theorem 7.