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RANDOM WALK OPERATORS ON GROUPS  
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For the above values,  $f$  is an eigenfunction of the operator  $P$  and satisfies the generalized Følner condition. By Theorem 3 the norm of the random walk operator on  $\mathbf{Z}_2 \star \mathbf{Z}_4$  with the generating subset as defined before is then equal to

$$\|P\| = \frac{\sqrt{33} + 7}{\sqrt{\sqrt{33} - 1}} \approx 0.98.$$

#### 4.2.2 GENERAL CASE

The idea presented for  $\mathbf{Z}_2 \star \mathbf{Z}_4$  can be used in the general case for  $\mathbf{Z}_n \star \mathbf{Z}_m$ . As the solution involves roots of some polynomial of degree  $nm$ , we will not give details.

#### 4.3 MEAN OPERATOR ON THE HYPERBOLIC PLANE

Let us consider the hyperbolic upper half-plane  $H = \{z = x + iy \in \mathbf{C}; x \in \mathbf{R}, y > 0\}$  with a Riemannian metric  $d_{Hz} = \frac{\sqrt{dx^2 + dy^2}}{y}$  which gives rise to the measure  $\mu_H = \frac{dx dy}{y^2}$ . We consider the operator  $P$ ,

$$Pf(z_0) = \int_{|z-z_0|=R} f(z) dm_R(z),$$

where  $dm_R$  is a uniform probability measure on a hyperbolic circle of radius  $R$ . We want to compute the norm of the operator  $P$  acting on  $L^2(H, d_{Hz})$ .

First of all let us remark that the function:

$$(11) \quad f(z) = \sqrt{\operatorname{Im}(z)},$$

is an eigenfunction of  $P$ . An easy way to see this is to note that  $P$  commutes with isometries of  $H$  and that the isometries consisting of horizontal translations and homotheties act transitively on  $H$ . The effect of these on the function  $f$  is that they just multiply it by a constant.

Now we would like to show that one can find a Følner sequence with respect to the function  $f$ . Let us consider a sequence  $\{A_n\}_{n=1}^{\infty}$  of rectangles (in the Euclidean sense) in  $H$ :

$$A_n = \{z \in H; e^{-n} \leq \operatorname{Im}(z) \leq 1, 0 \leq \operatorname{Re}(z) \leq n\}.$$

It is easy to see that the measure  $|\partial A_n|$  of the boundary of  $A_n$  is bounded by the measure of the following set  $B_n$  (see Figure 5):

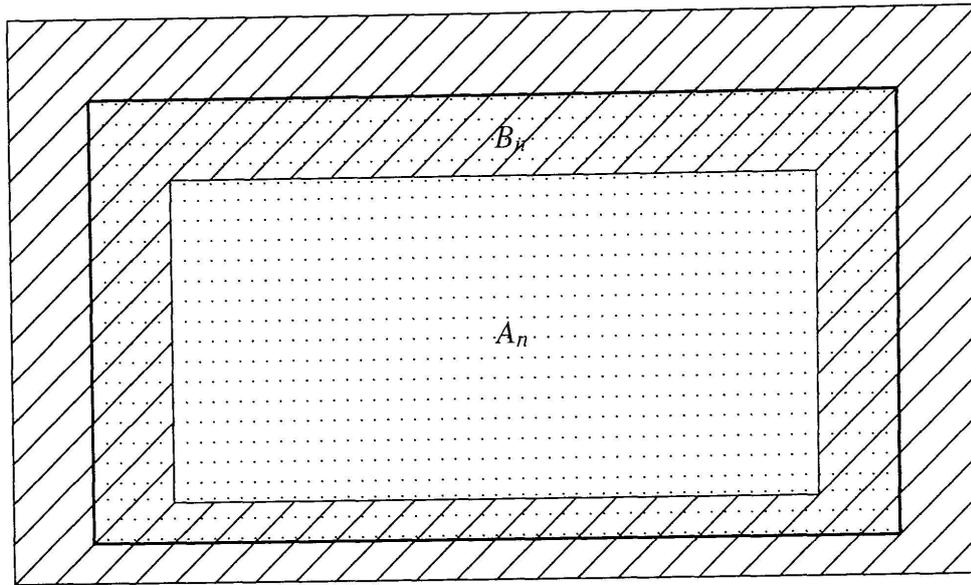


FIGURE 5  
Sets  $A_n$  and  $B_n$

$$\begin{aligned}
 B_n = & \{z \in H; -R \leq \operatorname{Re}(z) \leq R, e^R \geq \operatorname{Im}(z) \geq e^{-n-R}\} \\
 & \cup \{z \in H; -R+n \leq \operatorname{Re}(z) \leq n+R, e^R \geq \operatorname{Im}(z) \geq e^{-n-R}\} \\
 & \cup \{z \in H; -R \leq \operatorname{Re}(z) \leq n+R, e^R \geq \operatorname{Im}(z) \geq e^{-R}\} \\
 & \cup \{z \in H; -R \leq \operatorname{Re}(z) \leq n+R, e^{-n+R} \geq \operatorname{Im}(z) \geq e^{-n-R}\}.
 \end{aligned}$$

One can see that

$$|B_n|_{f^2} \approx n, \quad |A_n|_{f^2} \approx n^2.$$

This shows that  $\{A_n\}_{n=1}^\infty$  is a generalized Følner sequence. Thus

$$\|P\|_{L^2(H, d_{Hz}) \rightarrow L^2(H, d_{Hz})} = \int_{|z-i|=R} \sqrt{\operatorname{Im}(z)} \, dm_R(z).$$

#### 4.4 WREATH PRODUCTS

Let  $G$  and  $F$  be finitely generated groups. We define the wreath product  $G \wr F$  of these groups as follows. Elements of  $G \wr F$  are couples  $(g, \gamma_1)$  where  $g: F \rightarrow G$  is a function such that  $g(\gamma)$  is different from the identity element  $id_G$  of  $G$  only for finitely many elements  $\gamma$  in  $F$ , and where  $\gamma_1$  is an element of  $F$ . The multiplication in  $G \wr F$  is defined as follows:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where