4. Norms of random walk operators

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4. NORMS OF RANDOM WALK OPERATORS

Now we will show how Theorem 3 can be used in the problem of computing the norm of the random walk operator P on some groups. Our strategy is as follows: we want to find a positive eigenfunction for the operator P which satisfies the generalized Følner condition. By Theorem 3 such an eigenfunction always exists and the eigenvalue corresponding to this eigenfunction is equal to the norm of the operator P. Theorem 3 is a particular case of Theorem 2 which can also be helpful in computing the norms of more general operators as shown in Section 4.3.

4.1 FREE GROUPS

First of all, as a simple illustration of this method, we will compute the norm of the simple random walk operator on free groups, which was first done by Kesten (see [9]) using a different method.

THEOREM 6 (Kesten). *Let Γ be the free group generated by the standard symmetric set of generators S. The norm of the simple random walk operator P associated to (Γ, S) is equal to*

$$\|P\| = \frac{2\sqrt{\#S - 1}}{\#S}.$$

Proof. The Cayley graph of (Γ, S) is a homogeneous tree T_k of degree $k = \#S$. We draw the tree T_k with level lines as in Figure 3 (level lines are marked by dotted lines). Let us choose arbitrarily a line as the line of level 0. We construct a function on vertices of this tree which depends only on the level of the vertex. For a vertex $v \in T_k$ we denote by $|v|$ its level. We define $f: T_k \to \mathbf{R}_+$ as follows

$$f(v) = \left(\frac{1}{\sqrt{k-1}}\right)^{|v|}.$$

One has

$$Pf = \frac{2\sqrt{k-1}}{k} f.$$

Let A_n be the set of vertices in T_k consisting of a chosen vertex e from the level 0 and the vertices lying below e up to the level n (in Figure 3 the vertices of A_2 are marked with circles). Then

$$\sum_{v \in A_n} f^2(v) = n + 1\,,$$

$$\sum_{v \in \partial A_n} f^2(v) = 2\,.$$

This shows that $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence and by Theorem 3

$$\|P\| = \frac{2\sqrt{k-1}}{k}\,. \qquad \Box$$

4.1.1 Remarks on generalized growth

Let Γ be a group generated by a finite, symmetric set S. For $id \neq \gamma \in \Gamma$ we define its length $|\gamma|$ as the minimal number of generators from S needed to represent γ, i.e.

$$|\gamma| = \min\{n\,;\ \gamma = s_{i_1} \dots s_{i_n}\,,\ s_{i_j} \in S\}\,,$$

and we declare $|id| = 0$.

The growth function (see [10], [18]) of the pair (Γ, S) associates to each integer $n \geq 0$ the number $\beta(\Gamma, S)(n)$ of elements $\gamma \in \Gamma$ such that $|\gamma| \leq n$, i.e.

$$\beta(\Gamma, S)(n) = \#\{\gamma \in \Gamma;\ |\gamma| \leq n\}\,.$$

One is often interested only in the type of the growth function. For instance, we say that the group Γ is of polynomial growth if there exist constants c and D such that

$$c^{-1} n^D \leq \beta(\Gamma, S)(n) \leq c n^D\,.$$

The exponent D does not depend on the set of generators S. If the growth function is bounded by a polynomial, it is known (see [6]) that Γ is of polynomial growth and D is an integer. For a group of polynomial growth with the exponent D, it is known (see [19]) that there exists a constant c such that

$$c^{-1} n^{-\frac{D}{2}} \leq P^{2n}(id, id) \leq c n^{-\frac{D}{2}}\,, \tag{5}$$

where $P^{2n}(id, id)$ is the probability of the return to the identity element of the simple random walk after $2n$ steps.

It seems natural to define a generalized growth function, using an eigenfunction of P. Let f be a positive eigenfunction of P corresponding to the eigenvalue equal to the norm of P, i.e.

$$Pf = \|P\| f\,.$$

The generalized growth function $\beta(\Gamma, S, f)$ associates to each positive integer n the number

$$\beta(\Gamma, S, f)(n) = \sum_{\gamma \in \Gamma, |\gamma| \leq n} f^2(\gamma),$$

i.e., each element in the ball of radius n is counted with weight f^2.

Let us compute the generalized growth function in a particular case. Let P be the simple random walk operator on the free group with the standard set of generators of cardinality k as in Section 4.1. Let g be the unique radial eigenfunction of P corresponding to the eigenvalue $\|P\|$ and such that $g(id) = 1$. Explicitly we have:

$$g(\gamma) = \left(\frac{k-2}{k}|\gamma| + 1\right)\left(\frac{1}{\sqrt{k-1}}\right)^{|\gamma|}.$$

Then we have

$$\sum_{\gamma \in \Gamma, |\gamma| \leq n} g^2(\gamma) =$$

$$n^3\left(\frac{k^2 - 4k + 4}{3k^2 - 3k}\right) + n^2\left(\frac{3k^2 - 8k + 4}{2k^2 - 2k}\right) + n\left(\frac{7k^2 - 16k + 4}{6k^2 - 6k}\right) + 1.$$

This shows that the generalized growth is like n^3. In particular the sequence of balls is a generalized Følner sequence.

By analogy to (5) we conjecture that the fact that the generalized growth function for the free groups is like n^3 explains that for the free groups one has (see [16]):

$$c^{-1}\lambda^{2n}n^{-\frac{3}{2}} \leq P^{2n}(id, id) \leq c\lambda^{2n}n^{-\frac{3}{2}},$$

where c is a constant and λ is the norm of P.

4.2 Free products of finite groups

Random walks on free products of finite groups were already considered in [1], [3], [17] and [21].

Let us consider the group $\mathbf{Z}_m \star \mathbf{Z}_n$ with the following generating set:

- if $m \neq 2$ we take $\{\pm 1\}$ as generators of $\mathbf{Z}_m = \{0, 1, \ldots, m-1\}$;
- we take $\{1\}$ as a generator of $\mathbf{Z}_2 = \{0, 1\}$.

In Figure 2 we represent the Cayley graph for $\mathbf{Z}_2 \star \mathbf{Z}_4$ with the above set of generators. In general the Cayley graph for $\mathbf{Z}_m \star \mathbf{Z}_n$ with the generating set defined above has the following construction:

- m-gons and n-gons are attached to each other;
- at each vertex of an n-gon there is one m-gon attached and at each vertex of an m-gon there is one n-gon attached.

4.2.1 $\mathbf{Z}_2 \star \mathbf{Z}_4$

We will present our method in the special case for $\mathbf{Z}_2 \star \mathbf{Z}_4$. The Cayley graph for this group is represented in Figure 2. Our aim is to construct the eigenfunction f of the random walk operator satisfying the generalized Følner condition. By Theorem 3, the eigenvalue corresponding to this eigenfunction is equal to the norm of a random walk operator. We will construct f in two steps.

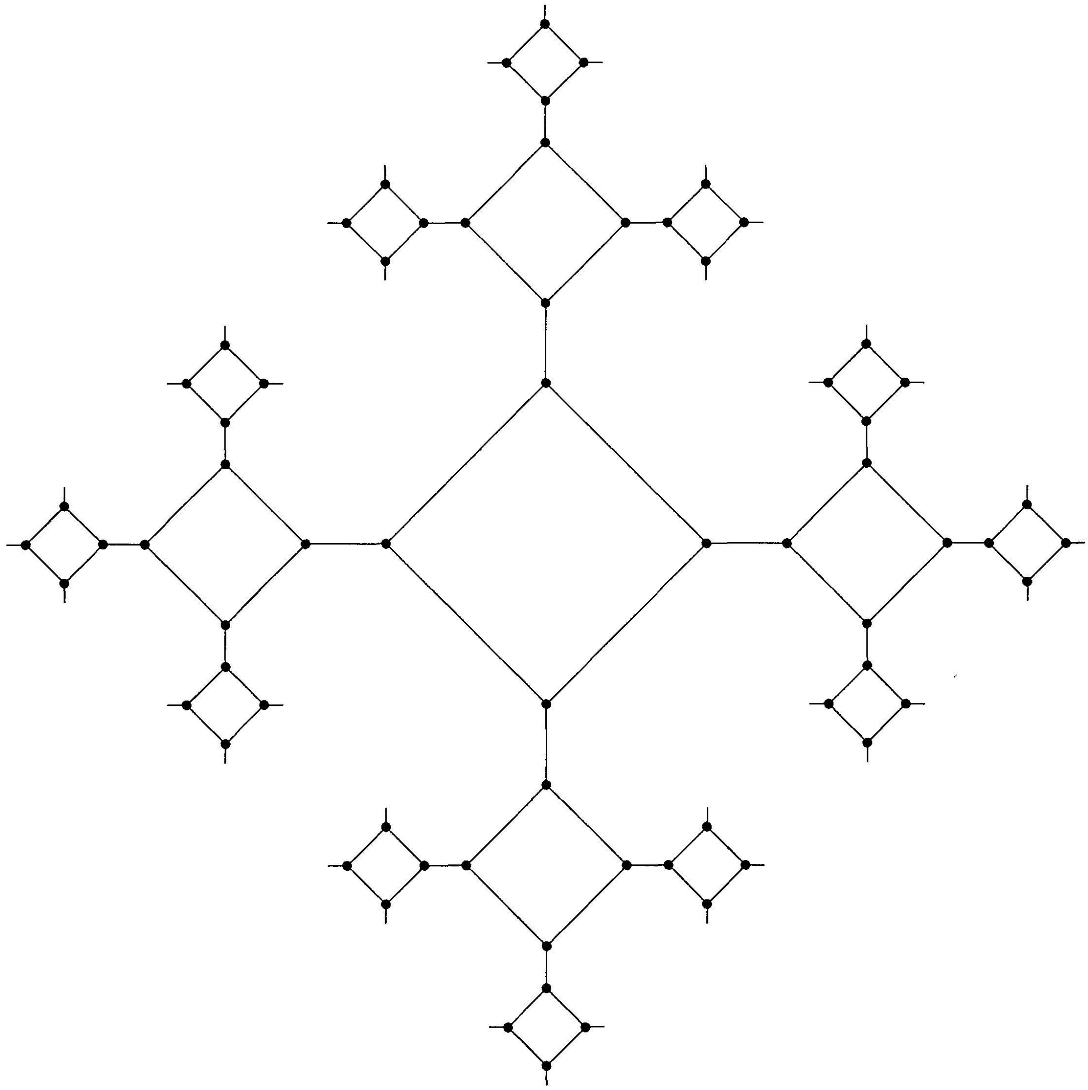

FIGURE 2
Cayley graph for $\mathbf{Z}_2 \star \mathbf{Z}_4$

STEP 1. If we contract the squares to points, the Cayley graph for $\mathbf{Z}_2 \star \mathbf{Z}_4$ is deformed to the homogeneous tree T_4 of order 4 (each vertex has 4 neighbors), which is represented in Figure 3. First of all we construct a function on vertices of T_4 satisfying the generalized Følner condition.

We draw the graph T_4 as in Figure 3, i.e. with one point set apart at infinity. The level lines or horocycles are marked by dotted lines. Each vertex of T_4 has one neighbor above and three neighbors below.

Let us fix two positive numbers r, s and define the positive function g on the vertices of the tree T_4

$$g\colon (\text{vertices of } T_4) \to \mathbf{R}_+$$

as follows:

if w is a neighbor of v lying below v then (see Figure 4)

(1) $g(w) = rg(v)$ if w is the right or left neighbor;

(2) $g(w) = sg(v)$ if w is the middle neighbor.

The above defines the function g up to a constant. Let us fix one vertex e (for instance lying on the horocycle of level 0) and put $g(e) = 1$.

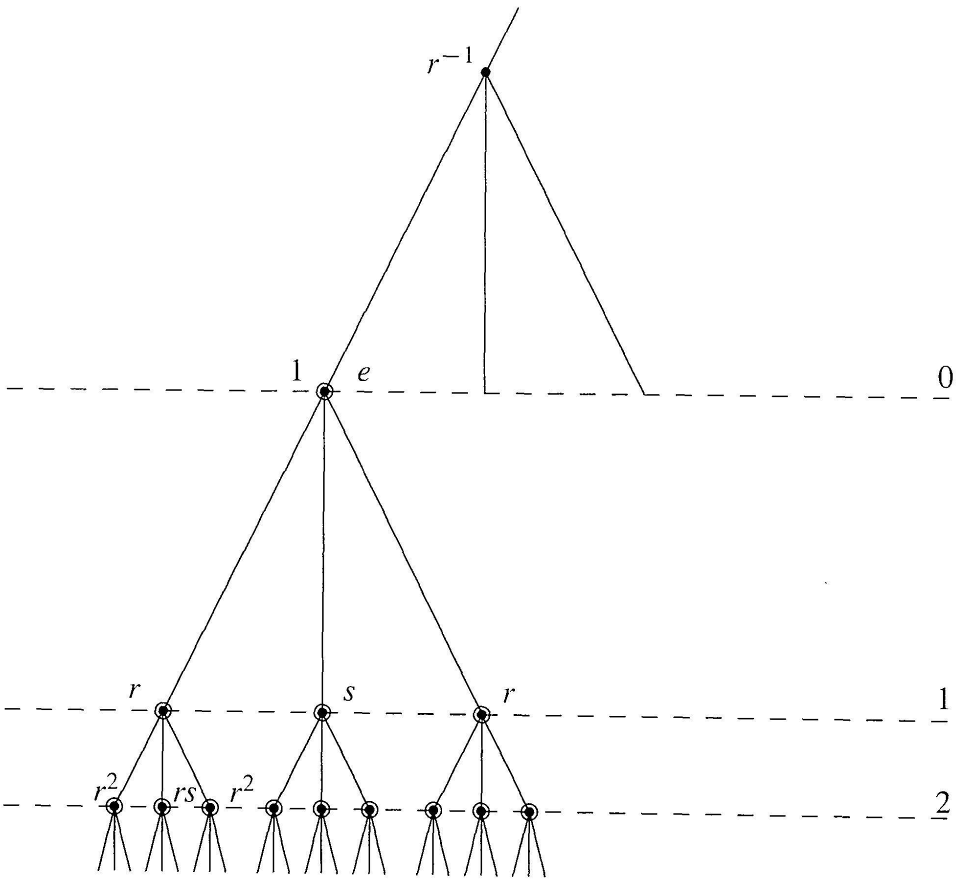

FIGURE 3
Tree T_4 of order 4

Now we need

LEMMA 4. *For* $2r^2 + s^2 = 1$ *the function* g *satisfies the generalized Følner condition, i.e. there exists a sequence* $\{A_n\}_{n=1}^{\infty}$ *of finite subsets of* T_4 *such that*

$$\frac{\sum_{v \in \partial A_n} g^2(v)}{\sum_{v \in A_n} g^2(v)} \to_{n \to \infty} 0 .$$

Proof. Let A_n be the subset of vertices of the tree T_4 consisting of e and the vertices lying below e up to the level n (in Figure 3 the vertices of A_2 are marked with circles).

One can easily see that

$$\sum_{v \in A_n} g^2(v) = n + 1 ,$$

$$\sum_{v \in \partial A_n} g^2(v) = 2 .$$

Thus $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence for P corresponding to g. $\square$

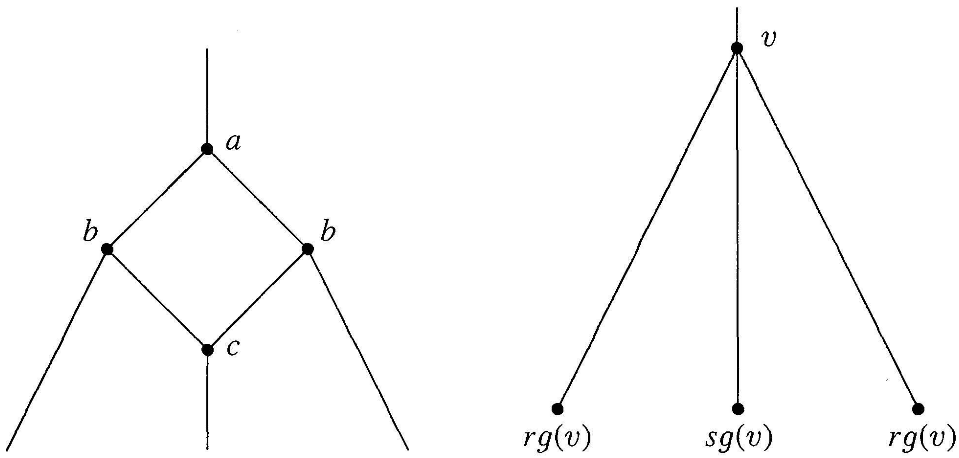

FIGURE 4

Labelling of vertices and the definition of the function g

STEP 2. The second step consists of labelling the vertices of the Cayley graph of $\mathbf{Z}_2 \star \mathbf{Z}_4$ with a, b or c (the precise values of the numbers a, b and c are given later). The vertices of each square are labelled as in Figure 4. This defines the unique labelling if we bear in mind the way we have drawn the tree T_4 obtained by contracting the squares (see Figure 3).

Now we can define the positive function f on $\mathbf{Z}_2 \star \mathbf{Z}_4$ as follows. If v is the vertex of type t ($t = a$, b or c) of the square which corresponds to the vertex w of the tree T_4 then

$$f(v) = tg(w).$$

We want to find a, b, c, r, s and λ so that f is an eigenfunction of the random walk operator P with the eigenvalue λ.

Let us write the equation

$$Pf = \lambda f$$

for vertices of type a, b and c. On a vertex of type a, the function f has to satisfy the following

$$\frac{b + 2br}{3} = \lambda ar, \tag{6}$$

$$\frac{c + 2bs}{3} = \lambda as. \tag{7}$$

For a vertex of type b, function f has to satisfy

$$\frac{a + c + ar}{3} = \lambda b \tag{8}$$

and for a vertex of type c, function f has to satisfy

$$\frac{2b + as}{3} = \lambda c. \tag{9}$$

If f satisfies the above conditions it is an eigenfunction of P with the eigenvalue λ. For $2r^2 + s^2 = 1$, by Lemma 4 the function g satisfies the generalized Følner condition and so does f. So we want to have a condition

$$2r^2 + s^2 = 1. \tag{10}$$

After solving equations (6)-(10) we obtain the following values for a, b, c, r, s and λ (a, b and c are determined up to a constant so we suppose a=1):

$$a = 1; \qquad b = \frac{u\sqrt{1 - 2u^2}}{-1 + 4u^2}; \qquad c = \frac{1 - 2u^2}{-1 + 4u^2};$$

$$r = u; \qquad s = \sqrt{1 - 2u^2}; \qquad \lambda = \frac{-1 + 2u + 4u^2}{3\sqrt{1 - 2u^2}};$$

where

$$u = \frac{\sqrt{33} - 1}{8}.$$

For the above values, f is an eigenfunction of the operator P and satisfies the generalized Følner condition. By Theorem 3 the norm of the random walk operator on $\mathbf{Z}_2 \star \mathbf{Z}_4$ with the generating subset as defined before is then equal to

$$\|P\| = \frac{\sqrt{33}+7}{\sqrt{\sqrt{33}-1}} \approx 0.98\,.$$

4.2.2 General case

The idea presented for $\mathbf{Z}_2 \star \mathbf{Z}_4$ can be used in the general case for $\mathbf{Z}_n \star \mathbf{Z}_m$. As the solution involves roots of some polynomial of degree nm, we will not give details.

4.3 Mean operator on the hyperbolic plane

Let us consider the hyperbolic upper half-plane $H = \{z = x + iy \in \mathbf{C}\,;\, x \in \mathbf{R},\ y > 0\}$ with a Riemannian metric $d_H z = \frac{\sqrt{dx^2+dy^2}}{y}$ which gives rise to the measure $\mu_H = \frac{dxdy}{y^2}$. We consider the operator P,

$$Pf(z_0) = \int_{|z-z_0|=R} f(z)\, dm_R(z)\,,$$

where dm_R is a uniform probability measure on a hyperbolic circle of radius R. We want to compute the norm of the operator P acting on $L^2(H, d_H z)$.

First of all let us remark that the function:

$$f(z) = \sqrt{\mathrm{Im(z)}}, \tag{11}$$

is an eigenfunction of P. An easy way to see this is to note that P commutes with isometries of H and that the isometries consisting of horizontal translations and homotheties act transitively on H. The effect of these on the function f is that they just multiply it by a constant.

Now we would like to show that one can find a Følner sequence with respect to the function f. Let us consider a sequence $\{A_n\}_{n=1}^{\infty}$ of rectangles (in the Euclidean sense) in H:

$$A_n = \{z \in H\ ;\ e^{-n} \le \mathrm{Im}(z) \le 1,\ 0 \le \mathrm{Re}(z) \le n\}\,.$$

It is easy to see that the measure $|\partial A_n|$ of the boundary of A_n is bounded by the measure of the following set B_n (see Figure 5):

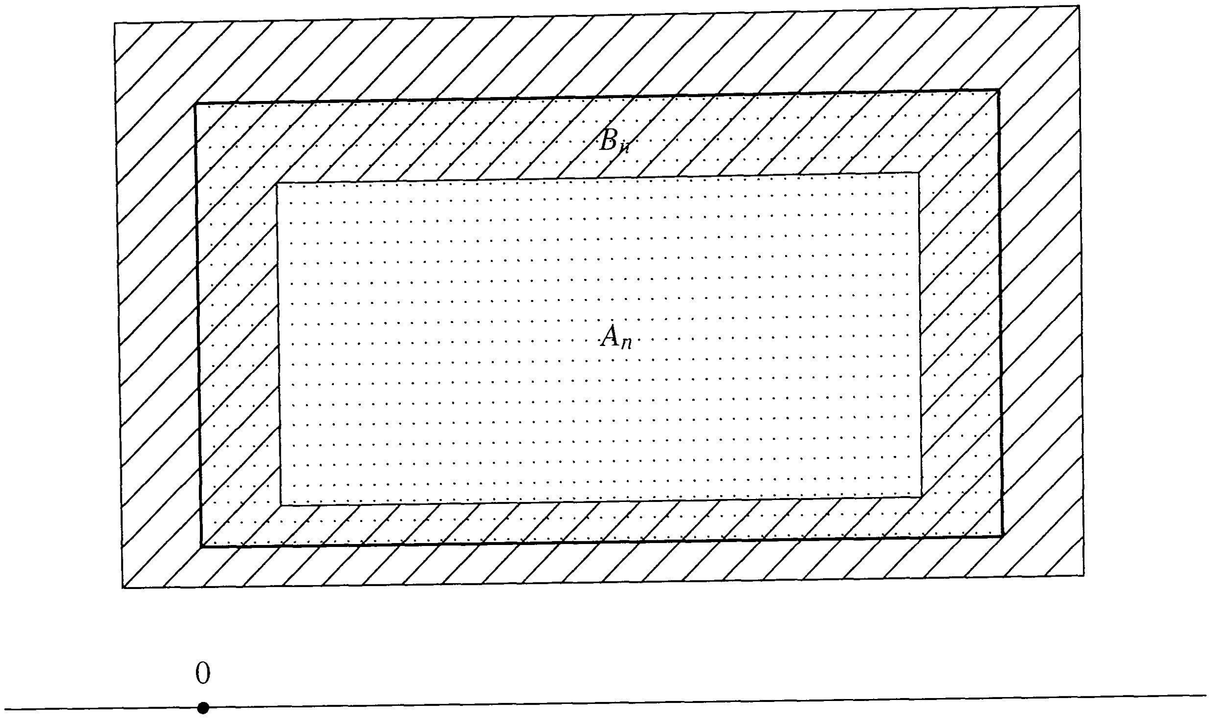

FIGURE 5
Sets A_n and B_n

$$\begin{aligned} B_n = &\{z \in H;\ -R \leq \mathrm{Re}(z) \leq R,\ e^R \geq \mathrm{Im}(z) \geq e^{-n-R}\} \\ &\cup \{z \in H;\ -R+n \leq \mathrm{Re}(z) \leq n+R,\ e^R \geq \mathrm{Im}(z) \geq e^{-n-R}\} \\ &\cup \{z \in H;\ -R \leq \mathrm{Re}(z) \leq n+R,\ e^R \geq \mathrm{Im}(z) \geq e^{-R}\} \\ &\cup \{z \in H;\ -R \leq \mathrm{Re}(z) \leq n+R,\ e^{-n+R} \geq \mathrm{Im}(z) \geq e^{-n-R}\}. \end{aligned}$$

One can see that

$$|B_n|_{f^2} \approx n, \quad |A_n|_{f^2} \approx n^2.$$

This shows that $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence. Thus

$$||P||_{L^2(H,d_H z)\to L^2(H,d_H z)} = \int_{|z-i|=R} \sqrt{\mathrm{Im(z)}}\, dm_R(z).$$

4.4 WREATH PRODUCTS

Let G and F be finitely generated groups. We define the wreath product $G \wr F$ of these groups as follows. Elements of $G \wr F$ are couples (g, γ_1) where $g\colon F \to G$ is a function such that $g(\gamma)$ is different from the identity element id_G of G only for finitely many elements γ in F, and where γ_1 is an element of F. The multiplication in $G \wr F$ is defined as follows:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where

$$g_3(\gamma) = g_1(\gamma) g_2(\gamma\gamma_1) \quad \text{for } \gamma \in F\,.$$

If S_G and S_F are generators of G and F respectively then

$$\{(g,\gamma)\ ;\ (g(F) = id_G,\ \gamma \in S_F) \text{ or } (g(F \setminus id_F) = id_G,\ g(id_F) \in S_G,\ \gamma = id_F)\}$$

is a generating subset for $G \wr F$.

Let μ and ν be symmetric, finitely supported probability measures on F and G respectively.

As there is a natural embedding of F and G into $G \wr F$, one can view the measures μ and ν as measures on $G \wr F$. More precisely:

$$\nu(g,\gamma) = \begin{cases} \nu(g(id_F)) & \text{if } \gamma = id_F \text{ and } g(F \setminus id_F) = id_G \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu(g,\gamma) = \begin{cases} \mu(\gamma) & \text{if } g(F) = id_G \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu \star \nu \star \mu$ is a symmetric measure on $G \wr F$. Explicitly we have:

$$\mu \star \nu \star \mu(g,\gamma) = \begin{cases} \mu(\gamma(\gamma_0)^{-1})\mu(\gamma_0)\nu(g(\gamma_0)) & \text{if } g(F \setminus \gamma_0) = id_G \\ 0 & \text{otherwise}\,. \end{cases}$$

We want to prove:

THEOREM 7. *Let F and G be finitely generated groups. If F is amenable then the spectral radius of ν on G is the same as the spectral radius of $\mu \star \nu \star \mu$ on $G \wr F$.*

Proof. We will prove Theorem 7 by constructing on $G \wr F$ a positive function $\widetilde{f}$ which is an eigenfunction for the convolution by $\mu \star \nu \star \mu$ with eigenvalue $\|\nu\|_{l^2(G) \to l^2(G)}$ and for which there exists a generalized Følner sequence.

Let f be a positive eigenfunction for the operator which is a convolution on $l^2(G)$ by ν, corresponding to the eigenvalue $\|\nu\|$, i.e.

$$f \star \nu = \|\nu\| f\,. \tag{12}$$

We can normalize f so that

$$f(id_G) = 1\,. \tag{13}$$

By Theorem 3 (and the remark after its proof) there exists a sequence of finite subsets $A_n \subset G$, such that

$$\frac{\sum_{\gamma\in\partial A_n} f^2(\gamma)}{\sum_{\gamma\in A_n} f^2(\gamma)} \to_{n\to\infty} 0\,.$$

As the group F is amenable there exists a sequence of finite subsets $B_n \subset F$, such that

$$\frac{\#\partial B_n}{\#B_n} \to_{n\to\infty} 0\,.$$

For technical reasons let us choose the sequences B_n and A_n in such a way that

$$\frac{\#\partial B_n}{\#B_n} < \frac{1}{n} \quad\text{and}\quad \frac{\sum_{\gamma\in\partial A_n} f^2(\gamma)}{\sum_{\gamma\in A_n} f^2(\gamma)} < \frac{1}{n(\#B_n)}\,. \tag{14}$$

Now, on $G\wr F$ we define $\widetilde{f}$ as follows

$$\widetilde{f}(g,\gamma_1) = \prod_{\gamma\in F} f(g(\gamma))\,.$$

The function $\widetilde{f}$ is well defined because by (13), $f(g(\gamma))$ is different from 1 only for finitely many $\gamma \in F$. This function is of course positive and does not depend on γ_1. From (12) one has

$$\widetilde{f}\star\mu\star\nu\star\mu = \widetilde{f}\star\nu\star\mu = \|\nu\|\widetilde{f}\star\mu = \|\nu\|\widetilde{f}\,.$$

To complete the proof of Theorem 7 it is enough to construct a generalized Følner sequence $C_n \subset G\wr F$ for $\widetilde{f}$. We define C_n as follows:

$$C_n = \{(g,\gamma_1);\gamma_1 \in B_n,\ g(B_n)\subset A_n,\ g^{-1}(G\setminus id_G)\subset B_n\}\,.$$

LEMMA 5. *The sequence $C_n \subset G\wr F$ is a generalized Følner sequence for $\widetilde{f}$.*

Proof. Let us define sets D_n and ∂D_n as follows:

$$\begin{aligned} D_n &= \{g\colon F\to G;\ g(B_n)\subset A_n,\ g^{-1}(G\setminus id_G)\subset B_n\}\,,\\ \partial D_n &= \{g\colon F\to G;\ \text{there exists } \gamma_0\in B_n \text{ such that } g(\gamma_0)\in\partial A_n,\\ &\qquad\qquad g(B_n\setminus\gamma_0)\subset A_n,\ g^{-1}(G\setminus id_G)\subset B_n\}\,. \end{aligned}$$

Thus

$$\begin{aligned} C_n &= D_n\times B_n,\\ \partial C_n &= (\partial D_n\times B_n)\cup(D_n\times\partial B_n)\,. \end{aligned}$$

We have then

$$\begin{aligned}\sum_{(g,\gamma_1)\in C_n} \left(\tilde{f}(g,\gamma_1)\right)^2 &= \sum_{(g,\gamma_1)\in C_n} \Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2 \\ &= \sum_{(g,\gamma_1)\in D_n\times B_n} \Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2 = \#B_n \sum_{g\in D_n} \Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2 .\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{(g,\gamma_1)\in \partial C_n} \left(\tilde{f}(g,\gamma_1)\right)^2 &= \sum_{(g,\gamma_1)\in \partial C_n} \Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2 \\ &= \sum_{(g,\gamma_1)\subset(\partial D_n\times B_n)\cup(D_n\times\partial B_n)} \Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2 \\ &= \#\partial B_n \sum_{g\in D_n} \Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2 + \#B_n \sum_{g\in \partial D_n} \Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2 \\ &= \frac{\#\partial B_n}{\#B_n} \sum_{(g,\gamma_1)\in C_n} \left(\tilde{f}(g,\gamma_1)\right)^2 \\ &\quad + \frac{\sum_{g\in\partial D_n}\Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2}{\sum_{g\in D_n}\Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2} \sum_{(g,\gamma_1)\in C_n} \left(\tilde{f}(g,\gamma_1)\right)^2 .\end{aligned}$$

But

$$\sum_{g\in \partial D_n} \Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2 = \#B_n \frac{\sum_{\alpha\in\partial A_n} f^2(\alpha)}{\sum_{\alpha\in A_n} f^2(\alpha)} \sum_{g\in D_n} \Big(\prod_{\gamma\in F} f(g(\gamma))\Big)^2 .$$

Thus by (14)

$$\begin{aligned}\sum_{(g,\gamma_1)\in \partial C_n} \left(\tilde{f}(g,\gamma_1)\right)^2 &= \left(\frac{\#\partial B_n}{\#B_n} + \#B_n \frac{\sum_{\alpha\in\partial A_n} f^2(\alpha)}{\sum_{\alpha\in A_n} f^2(\alpha)}\right) \sum_{(g,\gamma_1)\in C_n} \left(\tilde{f}(g,\gamma_1)\right)^2 \\ &\leq \frac{2}{n} \sum_{(g,\gamma_1)\in C_n} \left(\tilde{f}(g,\gamma_1)\right)^2 ,\end{aligned}$$

which shows that C_n is a generalized Følner sequence for $\tilde{f}$. □

This ends the proof of Theorem 7.