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RANDOM WALK OPERATORS ON GROUPS  
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*Proof.* There are several proofs of this theorem. In [20] one can find the proof where the analogue of Perron-Frobenius theory is developed and in [11] the truncation method is used.  $\square$

### 3.2 EIGENFUNCTIONS IN $l^2$

One can ask whether the positive eigenfunctions of the random walk operator are in  $l^2(X, N)$ . The answer is no in the case when  $X$  is the Cayley graph of an infinite group  $\Gamma$  (see Theorem 5). But in the general case there are examples of eigenfunctions which are in  $l^2(X, N)$  (see Proposition 2).

#### 3.2.1 THE CASE OF GROUPS

**THEOREM 5.** *Let  $f$  be a positive eigenfunction of the simple random walk operator  $P$  on the group  $\Gamma$  generated by a finite symmetric set  $S$ , i.e.  $Pf = \lambda f$ . If  $\Gamma$  is infinite then*

$$\sum_{\gamma \in \Gamma} f^2(\gamma) = +\infty.$$

*Proof.* Suppose the contrary, i.e. that there is a positive eigenfunction  $f$  of the operator  $P$  for which the  $l^2$  norm is finite:

$$\begin{aligned} Pf_0 &= \lambda f_0, \\ \sum_{\gamma \in \Gamma} f_0^2(\gamma) &< +\infty. \end{aligned}$$

The second condition implies that  $f_0$  is not constant and so there are  $\gamma_0, \gamma_1 \in \Gamma$  such that

$$f_0(\gamma_0) < f_0(\gamma_1).$$

Let us define the function  $f_1$  as a translation of  $f_0$  by  $\gamma_0\gamma_1^{-1}$ , i.e.

$$f_1(\gamma) = f_0(\gamma_0\gamma_1^{-1}\gamma).$$

The function  $f_1$ , being the translation of  $f_0$ , is an eigenfunction of  $P$ , i.e.

$$Pf_1 = \lambda f_1.$$

So the function  $\tilde{f}$  defined as follows:

$$\tilde{f}(\gamma) = \max\{f_0(\gamma), f_1(\gamma)\},$$

satisfies

$$P\tilde{f} \geq \lambda\tilde{f}.$$

As  $f_0$  and  $f_1$  are in  $l^2(\Gamma)$ , the function  $\tilde{f}$  is in  $l^2(\Gamma)$  as well. The functions  $f_0$  and  $f_1$  have the same  $l^2$  norms and

$$f_1(\gamma_1) = f_0(\gamma_0) < f_0(\gamma_1),$$

so there exists  $\gamma_2 \in \Gamma$  such that

$$f_1(\gamma_2) > f_0(\gamma_2).$$

Note that these two inequalities imply that  $\tilde{f} \geq f_0$  with equality at some points and strict inequality at some other points. Thus  $g = \tilde{f} - f_0$  satisfies  $g \geq 0$ ,  $g \neq 0$ ,  $g$  vanishes at some points and  $Pg \geq \lambda g$ . Let us prove that this implies  $P\tilde{f} \neq \lambda\tilde{f}$ . Indeed, if we had equality then  $Pg = \lambda g$  as well and thus  $P^n g = \lambda^n g$ . Taking  $n$  large enough makes  $P^n g$  non-zero at points where  $g$  vanishes, a contradiction. We have thus shown that  $P\tilde{f} \geq \lambda\tilde{f}$  with  $P\tilde{f} \neq \lambda\tilde{f}$ .

This means that

$$\|P\tilde{f}\|_{l^2(\Gamma)} > \lambda\|\tilde{f}\|_{l^2(\Gamma)}.$$

Hence

$$\|P\| > \lambda.$$

But this provides the desired contradiction because by Theorem 4 there are no positive eigenfunctions of  $P$  with an eigenvalue smaller than the norm of  $P$ .  $\square$

### 3.2.2 THE GENERAL CASE

It will be shown that there are examples of the infinite graph  $X$  and the simple random walk operator  $P$  for which there is a positive eigenfunction in  $l^2(X, N)$ . It was pointed out to us by the referee that when  $P$  is the adjacency operator, examples of infinite graphs with positive eigenvalues in  $l^2$  can be found for instance in [5] (page 232).

Let  $X$  be a uniform tree (*i.e.* a simply connected graph) of degree 3. By a theorem of Kesten (see [9]) one knows that  $\|P\| = \frac{2}{3}\sqrt{2} < 1$ . Let  $a$  and  $b$  be two neighboring vertices in  $X$ . Now let  $X_n$  be a graph which is the same as the graph  $X$ , except that the edge  $(a, b)$  is subdivided into  $n$  vertices. Let  $I_n$  denote the set of vertices  $a$ ,  $b$  and added vertices which we label  $1, \dots, n$  (see Figure 1). Let  $P_n$  be the simple random walk operator on  $X_n$ . One has  $\|P_n\| \rightarrow_{n \rightarrow \infty} 1$ . In fact we will prove:

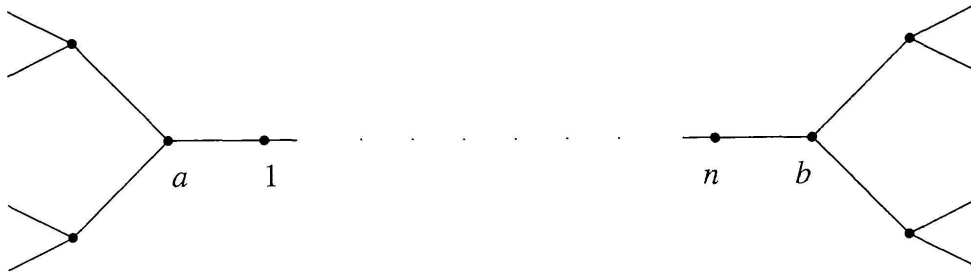


FIGURE 1  
The graph  $X_n$

PROPOSITION 2. For  $n \geq 7$  one has

$$\|P_n\| > \cos\left(\frac{\pi}{n+3}\right) > \frac{2\sqrt{2}}{3}.$$

For any  $n_0 \geq 1$  such that  $\|P_{n_0}\| > \frac{2}{3}\sqrt{2}$  the eigenfunctions of  $P_{n_0}$  corresponding to the eigenvalue  $\|P_{n_0}\|$  are in  $l^2(X_{n_0}, N)$ .

*Proof.* For  $n \geq 7$  let  $t = \sin\left(\frac{\pi}{n+3}\right) / \sin\left(\frac{2\pi}{n+3}\right)$  so that  $0 < t < 1$ .

For  $x \in X \setminus I_n$  let  $|x|$  be the minimum of its distances from  $a$  and  $b$ . We define the function  $f_n$  on  $X_n$  as follows:

$$f_n(y) = \begin{cases} t^{|y|} & \text{for } y \in X \setminus I_n \\ \sin\left(\frac{\pi(y+1)}{n+3}\right) / \sin\left(\frac{\pi}{n+3}\right) & \text{for } y = 1, \dots, n \\ 1 & \text{for } y = a, b. \end{cases}$$

We verify that

$$P_n f_n(i) = \cos\left(\frac{\pi}{n+3}\right) f_n(i) \quad \text{for } i = 1, \dots, n$$

$$P_n f_n(x) = \frac{1}{3} \left( \cos^{-1}\left(\frac{\pi}{n+3}\right) + 2 \cos\left(\frac{\pi}{n+3}\right) \right) f_n(x) \quad \text{for } x \in X_n \setminus \{1, \dots, n\}.$$

On the other hand for  $n \geq 7$  we have  $t < \frac{1}{\sqrt{3}}$  and

$$\sum_{x \in X_n \setminus I_n} f_n^2(x) N(x) = 2 \sum_{k=1}^{\infty} 2 \cdot 3^{k-1} (t^k)^2 \cdot 3 < \infty.$$

Thus  $f_n$  is in  $l^2(X_n, N)$  and

$$P_n f_n \geq \cos\left(\frac{\pi}{n+3}\right) f_n.$$

So we have proved the first part of Proposition 2.

Let  $n_0$  be such that

$$\|P_{n_0}\|_{l^2(X_{n_0}, N) \rightarrow l^2(X_{n_0}, N)} = \sigma > \frac{2\sqrt{2}}{3}.$$

Now let  $f$  be an eigenfunction of the operator  $P_{n_0}$  with the eigenvalue  $\sigma$ , i.e.

$$P_{n_0}f = \sigma f.$$

We want to show that  $f \in l^2(X_{n_0}, N)$ . Suppose this is not true, i.e.

$$\sum_{x \in X_{n_0}} f^2(x)N(x) = +\infty.$$

By Theorem 2, there exists a sequence of subsets of  $X_{n_0}$ ,  $A_k \subset X_{n_0}$  such that

$$(3) \quad \frac{\sum_{x \in \partial A_k} f^2(x)N(x)}{\sum_{x \in A_k} f^2(x)N(x)} \rightarrow_{k \rightarrow \infty} 0.$$

As  $I_{n_0}$  is a fixed finite set, the sequence  $C_k = A_k \setminus I_{n_0}$  is non-empty for  $k$  sufficiently large. We need the following:

LEMMA 3. *One has*

$$\frac{\sum_{x \in \partial C_k} f^2(x)N(x)}{\sum_{x \in C_k} f^2(x)N(x)} \rightarrow_{k \rightarrow \infty} 0.$$

*Proof.* If  $\sum_{x \in A_k} f^2(x)N(x) \rightarrow_{k \rightarrow \infty} \infty$  then the statement of the lemma is clear. Suppose then that for all  $k$

$$(4) \quad \sum_{x \in A_k} f^2(x)N(x) \leq \alpha < \infty.$$

If  $A_k \cap I_{n_0} = \emptyset$  then  $A_k$  and  $C_k$  coincide. So we are interested only in those  $k$  for which  $A_k \cap I_{n_0} \neq \emptyset$ . Let us consider the ball  $B_R$  of radius  $R$  centered in  $a \in I_{n_0}$  (i.e. those vertices in  $X_{n_0}$  for which at most  $R$  edges are needed to connect them to  $a$ ).

Because of (3) and (4) we have that for  $k$  sufficiently large  $\partial A_k \cap B_R = \emptyset$  which, by the fact that  $A_k \cap I_{n_0} \neq \emptyset$ , implies that  $B_R \subset A_k$ . But  $R$  can be chosen arbitrarily large and as  $f$  is not in  $l^2(X, N)$  we get

$$\sum_{x \in A_k} f^2(x)N(x) \rightarrow_{k \rightarrow \infty} \infty,$$

which contradicts (4). This completes the proof of the lemma.  $\square$

On the subsets  $C_k$  the graphs  $X$  and  $X_{n_0}$  coincide. This implies:

$$\|P\|_{l^2(X, N) \rightarrow l^2(X, N)} \geq \sigma > \frac{2\sqrt{2}}{3},$$

which yields the desired contradiction. This ends the proof of Proposition 2.  $\square$