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APPENDIX: PROOF OF PROPOSITION 3.1

We start with some lemmas.

LEMMA A.1. Let A be a commutative ring with identity on which a finite group G acts, let  $A^G$  be the invariant subring, and let B be a flat  $A^G$ -algebra. Then G acts on  $B \otimes_{A^G} A$  through its action on the second factor and the invariant subring of this action is B.

*Proof.* We have an exact sequence

$$0 \to A^G \to A \to \prod_{g \in G} A,$$

where  $\prod_{g \in G} A$  is the direct product of |G| copies of A, and  $A \to \prod_{g \in G} A$  is defined by  $a \mapsto (ga - a)$ . Since B is a flat  $A^G$ -algebra, the tensor product of B with the above sequence remains exact, that is, the sequence

$$0 \to B \to B \otimes_{A^G} A \to \prod_{g \in G} B \otimes_{A^G} A$$

is exact. Hence  $B = (B \otimes_{A^G} A)^G$ .

Let A be a finitely generated k-algebra on which a finite group G acts. Then A is finite over  $A^G$ . For every prime ideal  $\mathfrak{q}$  of  $A^G$ , let  $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$  be all the prime ideals of A lying over  $\mathfrak{q}$ . It is known that G acts transitively on  $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ . Fix a  $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ . Let  $G_d = \{g \in G \mid g\mathfrak{p} = \mathfrak{p}\}$  be the decomposition group at  $\mathfrak{p}$ .

LEMMA A.2. Notation as above. Let  $\widehat{A^G}_{\mathfrak{q}}$  be the completion of the local ring  $A^G_{\mathfrak{q}}$  and let  $\widehat{A}_{\mathfrak{p}}$  be the completion of the local ring  $A_{\mathfrak{p}}$ . Then  $G_d$  acts on  $\widehat{A}_{\mathfrak{p}}$  and  $(\widehat{A}_{\mathfrak{p}})^{G_d} = \widehat{A^G}_{\mathfrak{q}}$ .

*Proof.* Since  $A^G_{\mathfrak{q}}$  is a flat  $A^G$ -algebra, we have  $(A^G_{\mathfrak{q}} \otimes_{A^G} A)^G = A^G_{\mathfrak{q}}$  by Lemma A.1. Replacing A by  $A^G_{\mathfrak{q}} \otimes_{A^G} A$  if necessary, we may thus assume that  $A^G$  is a local ring and  $\mathfrak{q}$  is the maximal ideal of  $A^G$ .

Let  $\widehat{A}$  be the completion of A with respect to the  $\mathfrak{q}A$ -adic topology. Since A is a finite  $A^G$ -algebra, we have  $\widehat{A} = \widehat{A^G} \otimes_{A^G} A$ . On the other hand, we have  $\widehat{A} = \prod_i \widehat{A_{\mathfrak{p}_i}}$ . Since  $\widehat{A^G}$  is a flat  $A^G$ -algebra, we have  $\widehat{A^G} = (\widehat{A^G} \otimes_{A^G} A)^G$ 

by Lemma A.1. So we have  $\widehat{A^G} = (\prod_i \widehat{A_{\mathfrak{p}_i}})^G$ . Obviously  $(\prod_i \widehat{A_{\mathfrak{p}_i}})^G = (\widehat{A_{\mathfrak{p}}})^{G_d}$ .

Therefore  $(\widehat{A}_{\mathfrak{p}})^{G_d} = \widehat{A}^{G}$ .

LEMMA A.3. Let A be a noetherian local ring, let  $I_i$  (i = 1, ..., n) be some ideals of A, and let  $K_i$  be the kernel of the canonical homomorphism  $\widehat{A} \to \widehat{A/I_i}$ . If  $I = I_1 \cdots I_n$ , then the kernel of  $\widehat{A} \to \widehat{A/I}$  is  $K_1 \cdots K_n$ .

*Proof.* Since A is noetherian, we have  $\ker(\widehat{A} \to \widehat{A/I_i}) = \widehat{I_i} = I_i \widehat{A}$ , that is  $K_i = I_i \widehat{A}$ . Similarly we have  $\ker(\widehat{A} \to \widehat{A/I}) = I\widehat{A} = I_1 \cdots I_n \widehat{A}$ . So  $\ker(\widehat{A} \to \widehat{A/I}) = K_1 \cdots K_n$ .

Let T be a k-scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \operatorname{spec}(k) \ . \end{array}$$

We have the following

LEMMA A.4. Let  $s: T \to X_{\mathfrak{m}} \times T$  be a section of q. Then s is a closed immersion and the closed subscheme D defined by s is a relative effective Cartier divisor on  $X_{\mathfrak{m}} \times T/T$ .

*Proof.* Since qs=id is a closed immersion and since q is separated, s is also a closed immersion. The closed subscheme D defined by s is flat because qs=id. Let  $\mathcal{I}$  be the sheaf of  $\mathcal{O}_{X_m \times T}$ -ideals defining D. We have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X_{\mathfrak{m}} \times T} \to \mathcal{O}_D \to 0$$
.

For any  $t \in T$ , since  $\mathcal{O}_D$  is  $\mathcal{O}_T$  flat, the following sequence is exact:

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_T} k(t) \to \mathcal{O}_{X_{\mathfrak{m}} \times T} \otimes_{\mathcal{O}_T} k(t) \to \mathcal{O}_{D_t} \to 0,$$

where  $D_t$  is the fiber of  $D \to T$  at t. Hence  $\mathcal{I} \otimes_{\mathcal{O}_T} k(t)$  is the ideal defining the closed subscheme  $D_t$  of  $q^{-1}(t) \cong X_{\mathfrak{m}}$ . Obviously  $D_t$  defines a divisor of  $X_{\mathfrak{m}}$ . So for every point  $x \in q^{-1}(t)$ , the ideal  $\mathcal{I}_x \otimes_{\mathcal{O}_T} k(t)$  of  $\mathcal{O}_{X_{\mathfrak{m}} \times T, x} \otimes_{\mathcal{O}_{T,t}} k(t)$  is generated by an element which is not a zero divisor. By Nakayama's lemma, the ideal  $\mathcal{I}_x$  of  $\mathcal{O}_{X_{\mathfrak{m}} \times T, x}$  is generated by one element whose image in  $\mathcal{O}_{X_{\mathfrak{m}} \times T, x} \otimes_{\mathcal{O}_{T,t}} k(t)$  is not a zero divisor. By Lemma 2.3, D is a relative effective Cartier divisor.

Consider the sections

$$s_i: (X-S)^n \to X_{\mathfrak{m}} \times (X-S)^n, \quad (P_1,\ldots,P_n) \mapsto (P_i,P_1,\ldots,P_n).$$

Denote the relative effective Cartier divisors defined by  $s_i$  also by  $s_i$ , and let  $D = s_1 + \cdots + s_n$ . The relative effective Cartier divisor D can also be regarded as a closed subscheme of  $X_m \times (X - S)^n$ . The n-th symmetric group  $\mathfrak{S}_n$  acts on  $(X - S)^n$  by permuting the factors. It acts on  $X_m \times (X - S)^n$  through its action on the second factor. Obviously D is stable under this action. Let D be the quotient of D by  $\mathfrak{S}_n$ .

# PROPOSITION A.5.

- (a) The quotient of  $X_{\mathfrak{m}} \times (X-S)^n$  by  $\mathfrak{S}_n$  is  $X_{\mathfrak{m}} \times (X-S)^{(n)}$ .
- (b) The closed immersion  $D \to X_{\mathfrak{m}} \times (X S)^n$  induces a closed immersion  $\mathcal{D} \to X_{\mathfrak{m}} \times (X S)^{(n)}$  and  $\mathcal{D}$  is a relative effective Cartier divisor on  $(X_{\mathfrak{m}} \times (X S)^{(n)})/(X S)^{(n)}$ . Moreover D is the pull-back of  $\mathcal{D}$ .

Proof. (a) We have a Cartesian square

$$X_{\mathfrak{m}} \times (X - S)^{n} \longrightarrow X_{\mathfrak{m}} \times (X - S)^{(n)}$$

$$\downarrow \qquad \qquad \downarrow$$

$$(X - S)^{n} \longrightarrow (X - S)^{(n)}.$$

The morphism  $X_{\mathfrak{m}} \times (X-S)^{(n)} \to (X-S)^{(n)}$  is flat since it is obtained from the flat morphism  $X_{\mathfrak{m}} \to \operatorname{spec}(k)$  through the base extension  $(X-S)^{(n)} \to \operatorname{spec}(k)$ . Our assertion then follows directly from Lemma A.1.

(b) Consider the commutative diagram

$$D \longrightarrow \mathcal{D}$$

$$\downarrow \qquad \qquad \downarrow$$

$$X_{\mathfrak{m}} \times (X - S)^{n} \longrightarrow X_{\mathfrak{m}} \times (X - S)^{(n)}$$

$$\downarrow \qquad \qquad \downarrow$$

$$(X - S)^{n} \longrightarrow (X - S)^{(n)}.$$

One can easily show that  $\mathcal{D} \to X_{\mathfrak{m}} \times (X-S)^{(n)}$  is a finite morphism and induces a homeomorphism of  $\mathcal{D}$  with a closed subset of  $X_{\mathfrak{m}} \times (X-S)^{(n)}$ . We are going to show that for any point  $y \in \mathcal{D}$ , the homomorphism  $\mathcal{O}_{X_{\mathfrak{m}} \times (X-S)^{(n)}, y} \to \mathcal{O}_{\mathcal{D}, y}$  is surjective and the homomorphism  $\mathcal{O}_{(X-S)^{(n)}, t} \to \mathcal{O}_{\mathcal{D}, y}$  is flat, where t is the image of y in  $(X-S)^{(n)}$ . If this is done, then  $\mathcal{D} \to X_{\mathfrak{m}} \times (X-S)^{(n)}$ 

is a closed immersion and  $\mathcal{D} \to (X - S)^{(n)}$  is flat. Obviously the fibers of  $\mathcal{D} \to (X - S)^{(n)}$  are effective divisors. As in the proof of Lemma A.4, one can then use Nakayama's lemma and Lemma 2.3 to show that  $\mathcal{D}$  is a relative effective Cartier divisor.

One can show that

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong \mathcal{O}_{\mathcal{D},y} \otimes_{\mathcal{O}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y}} \widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y}.$$

Note that  $\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times (X-S)^{(n)},y}$  is a faithfully flat  $\mathcal{O}_{X_{\mathfrak{m}}\times (X-S)^{(n)},y}$ -algebra. Thus to show that  $\mathcal{O}_{X_{\mathfrak{m}}\times (X-S)^{(n)},y}\to \mathcal{O}_{\mathcal{D},y}$  is surjective, it is enough to show that  $\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times (X-S)^{(n)},y}\to \widehat{\mathcal{O}}_{\mathcal{D},y}$  is surjective; and to show that  $\mathcal{O}_{(X-S)^{(n)},t}\to \mathcal{O}_{\mathcal{D},y}$  is flat, it is enough to show that  $\widehat{\mathcal{O}}_{(X-S)^{(n)},t}\to \widehat{\mathcal{O}}_{\mathcal{D},y}$  is flat.

Assume  $t = n_1 P_1 + \cdots + n_l P_l \in (X - S)^{(n)}$ , where the  $P_i$  are distinct points of X - S,  $n_i > 0$  and  $\sum_i n_i = n$ . Then  $y = (P_{i_0}, t) \in X_{\mathfrak{m}} \times (X - S)^{(n)}$  for some  $i_0 \in \{1, \ldots, l\}$ . Let  $t' = (P_1, \ldots, P_1, \ldots, P_l, \ldots, P_l) \in (X - S)^n$ , where the first  $n_1$  components of t' are  $P_1, \ldots$ , and the last  $n_l$  components are  $P_l$ . The point t' is a point in  $(X - S)^n$  lying over  $t \in (X - S)^{(n)}$ . Let y' be the point  $(P_{i_0}, t')$  in  $X_{\mathfrak{m}} \times (X - S)^n$ . It lies over y. Note that y' is also a point in D. With respect to the actions of  $\mathfrak{S}_n$  on  $(X - S)^n$ , on  $X_{\mathfrak{m}} \times (X - S)^n$ , and on D, the decomposition groups at  $t' \in (X - S)^n$ , at  $y' \in X_{\mathfrak{m}} \times (X - S)^n$ , and at  $y' \in D$  are all  $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ . We have

$$\widehat{\mathcal{O}}_{(X-S)^n,t} \cong k[[x_{11},\ldots,x_{1n_1},\ldots,x_{l1},\ldots,x_{ln_l}]],$$

and the decomposition group  $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$  acts on  $\widehat{\mathcal{O}}_{(X-S)^n,t}$  by permuting  $x_{i1}, \ldots, x_{in_i}$  for each i. We have

$$\widehat{\mathcal{O}}_{X_{\mathbf{m}}\times(X-S)^n,y'}\cong k[[x,x_{11},\ldots,x_{1n_1},\ldots,x_{l1},\ldots,x_{ln_l}]],$$

and the decomposition group  $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$  acts on  $\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X-S)^n, y'}$  by fixing x and permuting  $x_{i1}, \ldots, x_{in_i}$  for each i.

For each  $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \dots, n_1 + \cdots + n_{i_0}\}$ , the section

$$s_i: (X-S)^n \to X_m \times (X-S)^n, \quad (P_1,\ldots,P_n) \mapsto (P_i,P_1,\ldots,P_n)$$

induces a homomorphism

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times (X-S)^n,y'} \to \widehat{\mathcal{O}}_{(X-S)^n,t'}$$
.

Through the isomorphism

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times (X-S)^{n},y'}\cong k[[x,x_{11},\ldots,x_{1n_{1}},\ldots,x_{l1},\ldots,x_{ln_{l}}]]$$

and the isomorphism

$$\widehat{\mathcal{O}}_{(X-S)^n,t'} \cong k[[x_{11},\ldots,x_{1n_1},\ldots,x_{l1},\ldots,x_{ln_l}]],$$

this homomorphism induced by  $s_i$  is

$$k[[x, x_{11}, \ldots, x_{1n_1}, \ldots, x_{l1}, \ldots, x_{ln_l}]] \to k[[x_{11}, \ldots, x_{1n_1}, \ldots, x_{l1}, \ldots, x_{ln_l}]],$$
  
 $x \mapsto x_{l\alpha l}, \ x_{\alpha\beta} \mapsto x_{\alpha\beta} \ (\alpha = 1, \ldots, l, \ \beta = 1, \ldots, n_{\alpha}),$ 

where  $j \in \{1, \ldots, n_{i_0}\}$  is uniquely determined by  $n_1 + \cdots + n_{i_0-1} + j = i$ . The kernel of this homomorphism is the ideal  $(x - x_{i_0j})$ . By Lemma A.3, the kernel of the homomorphism  $\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X - S)^n, y'} \to \widehat{\mathcal{O}}_{D, y'}$  is identified with the ideal  $\left(\prod_{i=1}^{n_{i_0}} (x - x_{i_0j})\right)$  through the isomorphism

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times (X-S)^{n}, \mathbf{y}'} \cong k[[x, x_{11}, \ldots, x_{1n_{1}}, \ldots, x_{l1}, \ldots, x_{ln_{l}}]].$$

Hence

$$\widehat{\mathcal{O}}_{D,y'} \cong k[[x,x_{11},\ldots,x_{1n_1},\ldots,x_{l1},\ldots,x_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x-x_{i_0j})\right),$$

and the decomposition group  $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$  acts on  $\widehat{\mathcal{O}}_{D,y'}$  by fixing x and permuting  $x_{i1}, \ldots, x_{in_i}$  for each i. Let  $\sigma_{i1}, \ldots, \sigma_{in_i}$  be the elementary symmetric functions in  $x_{i1}, \ldots, x_{in_i}$ . By Lemma A.2, we have

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong k[[x,\sigma_{11},\ldots,\sigma_{1n_1},\ldots,\sigma_{l1},\ldots,\sigma_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x-x_{i_0j})\right),$$

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times(X-S)^{(n)},y} \cong k[[x,\sigma_{11},\ldots,\sigma_{1n_1},\ldots,\sigma_{l1},\ldots,\sigma_{ln_l}]],$$

$$\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \cong k[[\sigma_{11},\ldots,\sigma_{1n_1},\ldots,\sigma_{l1},\ldots,\sigma_{ln_l}]].$$

Now it is easy to see that  $\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times (X-S)^{(n)},y}\to \widehat{\mathcal{O}}_{\mathcal{D},y}$  is surjective and  $\widehat{\mathcal{O}}_{(X-S)^{(n)},t}\to \widehat{\mathcal{O}}_{\mathcal{D},y}$  is flat. This proves  $\mathcal{D}$  is a relative effective Cartier divisor. We also have

$$\widehat{\mathcal{O}}_{D,y'} = \widehat{\mathcal{O}}_{\mathcal{D},y} \widehat{\otimes}_{\widehat{\mathcal{O}}_{(X-S)^{(n)},t}} \widehat{\mathcal{O}}_{(X-S)^n,t'}$$
.

This implies that  $D = \mathcal{D} \times_{(X-S)^{(n)}} (X-S)^n$ , that is, D is the pull-back of  $\mathcal{D}$ . This completes the proof of the proposition.

The relative effective Cartier divisor  $\mathcal{D}$  is the universal relative effective Cartier divisor.

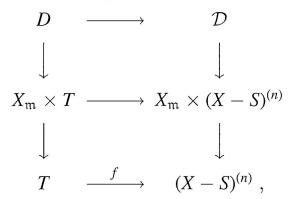
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LEMMA A.6. Let T be a k-scheme and let  $s_i \colon T \to X_{\mathfrak{m}} \times T$   $(i = 1, \ldots, n)$  be some sections of the projection  $q \colon X_{\mathfrak{m}} \times T \to T$ . Assume the images of  $s_i$  lie in  $(X_{\mathfrak{m}} - Q) \times T$ . Then there is a unique morphism of schemes  $f \colon T \to (X - S)^{(n)}$  such that the pull-back by  $id \times f$  of the universal relative effective Cartier divisor  $\mathcal{D}$  to  $X_{\mathfrak{m}} \times T$  is  $s_1 + \cdots + s_n$ .

*Proof.* Let  $p: X_{\mathfrak{m}} \times T \to X_{\mathfrak{m}}$  be the projection. The morphisms  $ps_i \colon T \to X_{\mathfrak{m}}$  induce  $(ps_1, \ldots, ps_n) \colon T \to X_{\mathfrak{m}}^n$ . Since the images of  $s_i$  lie in  $(X_{\mathfrak{m}} - Q) \times T$ , we actually get a morphism  $(ps_1, \ldots, ps_n) \colon T \to (X - S)^n$ . Composing with the canonical morphism  $(X - S)^n \to (X - S)^{(n)}$ , we get  $f: T \to (X - S)^{(n)}$  so that the pull-back of  $\mathcal{D}$  by  $\mathrm{id} \times f$  is  $s_1 + \cdots + s_n$ . This proves the existence of f.

To prove the uniqueness of f, we first note that  $f: T \to (X - S)^{(n)}$  is uniquely determined as a map on the underlying topological space. Indeed, for every point  $t \in T$ , f(t) is necessarily the point in  $(X - S)^{(n)}$  corresponding to the effective divisor  $(s_1 + \cdots + s_n)_t$  on  $q^{-1}(t) = X_m$ . To prove f is unique as a morphism of schemes, it is enough to prove that the homomorphism on local rings  $\mathcal{O}_{(X-S)^{(n)},f(t)} \to \mathcal{O}_{T,t}$  induced by f is uniquely determined. It suffices to prove that  $\widehat{\mathcal{O}}_{(X-S)^{(n)},f(t)} \to \widehat{\mathcal{O}}_{T,t}$  is uniquely determined.

Consider the commutative diagram



where D is the closed subscheme of  $X_{\mathfrak{m}} \times T$  corresponding to the divisor  $s_1 + \cdots + s_n$ . Let  $A = \widehat{\mathcal{O}}_{T,t}$ , let  $z \in D$  be a point lying over  $t \in T$ , and let  $y \in \mathcal{D}$  be the image of z. We have  $\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times T, z} \cong A[[x]]$ .

Without loss of generality, assume

where  $n_i > 0$  (i = 1, ..., l),  $n_1 + \cdots + n_l = n$ , and the  $P_i$  are distinct points in X - S. Then we have  $z = (P_{i_0}, t) \in X_{\mathfrak{m}} \times T$  for some  $i_0 \in \{1, ..., l\}$ .

For each  $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \ldots, n_1 + \cdots + n_{i_0}\}$ , the section  $s_i$  induces a homomorphism  $\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times T, z} \to \widehat{\mathcal{O}}_{T, t}$ , i.e.,  $A[[x]] \to A$ . Denote the image of x under this homomorphism by  $a_{i_0j}$ , where  $j \in \{1, \ldots, n_{i_0}\}$  is uniquely determined by  $n_1 + \cdots + n_{i_0-1} + j = i$ . Then by Lemma A.3, we have

$$\widehat{\mathcal{O}}_{D,z} \cong A[[x]] / \left( \prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right).$$

Keep the notations in the proof of Proposition A.5. We have

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong k[[x,\sigma_{11},\ldots,\sigma_{1n_{1}},\ldots,\sigma_{l1},\ldots,\sigma_{ln_{l}}]] / \left(\prod_{j=1}^{n_{i_{0}}} (x-x_{i_{0}j})\right).$$

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times(X-S)^{(n)},y} \cong k[[x,\sigma_{11},\ldots,\sigma_{1n_{1}},\ldots,\sigma_{l1},\ldots,\sigma_{ln_{l}}]].$$

$$\widehat{\mathcal{O}}_{(X-S)^{(n)},f(t)} \cong k[[\sigma_{11},\ldots,\sigma_{1n_{1}},\ldots,\sigma_{l1},\ldots,\sigma_{ln_{l}}]].$$

We have a commutative diagram

It is isomorphic to

In order for this last diagram to commute, it is necessary that  $\prod_{j=1}^{n_{i_0}} (x - x_{i_0 j})$ 

be mapped to  $\prod_{j=1}^{n_{i_0}} (x - a_{i_0 j})$  under the homomorphism

$$k[[x, \sigma_{11}, \ldots, \sigma_{1n_1}, \ldots, \sigma_{l1}, \ldots, \sigma_{n_l}]] \rightarrow A[[x]].$$

So the image of  $\sigma_{i_0j}$  under the homomorphism

$$k[[\sigma_{11},\ldots,\sigma_{1n_1},\ldots,\sigma_{l1},\ldots,\sigma_{ln_l}]] \to A$$

is necessarily the value at  $(a_{i_01},\ldots,a_{i_0n_{i_0}})$  of  $\sigma_{i_0j}$  considered as a function on  $A^{n_{i_0}}$ . We see that this is true for any indices  $i_0$  and j if we let z go over the points in D above t. Therefore the homomorphism  $k[[\sigma_{11},\ldots,\sigma_{1n_1},\ldots,\sigma_{l1},\ldots,\sigma_{ln_l}]] \to A$  is uniquely determined, that is, the homomorphism  $\widehat{\mathcal{O}}_{(X-S)^{(n)},f(t)} \to \widehat{\mathcal{O}}_{T,t}$  is uniquely determined. This concludes the proof of the lemma.

LEMMA A.7. Let T be a k-scheme and let D be a relative effective Cartier divisor on  $(X_{\mathfrak{m}} \times T)/T$  supported on  $(X_{\mathfrak{m}} - Q) \times T$  with degree n. Then there exist a flat morphism  $T' \to T$  and sections  $s_i \colon T' \to X_{\mathfrak{m}} \times T'$   $(i = 1, \ldots, n)$  of the projection  $X_{\mathfrak{m}} \times T' \to T'$  such that the pull-back of D to  $X_{\mathfrak{m}} \times T'$  is equal to  $s_1 + \cdots + s_n$ .

*Proof.* By the definition of relative effective Cartier divisors, D is flat over T. On the other hand,  $D \to T$  is proper and has finite fibers. So D is finite over T by [EGA] III, §4.4.2. Take  $T_1 = D$ . Then we have a finite flat morphism  $T_1 \to T$ . Consider the commutative diagram

$$D \times_{T} T_{1} \xrightarrow{p'} D$$

$$\downarrow i \downarrow \qquad \qquad \downarrow \downarrow$$

$$X_{\mathfrak{m}} \times T_{1} \xrightarrow{p} X_{\mathfrak{m}} \times T \longrightarrow X_{\mathfrak{m}}$$

$$\downarrow q' \downarrow \qquad \qquad \downarrow \qquad \downarrow$$

$$D = T_{1} \xrightarrow{qi} T \longrightarrow \operatorname{spec}(k) .$$

Let  $\Delta \colon D \to D \times_T D = D \times_T T_1$  be the diagonal map. It is a closed immersion since the morphism qi is separated. Take  $s_1 = i'\Delta$ . This is a section of q'. Hence it defines a relative effective Cartier divisor on  $(X_{\mathfrak{m}} \times T_1)/T_1$ . The pull-back  $D_1$  of the relative effective Cartier divisor D to  $X_{\mathfrak{m}} \times T_1$  is the closed subscheme defined by i'. Let  $\mathcal{I}_{D_1}$  and  $\mathcal{I}_s$  be the ideal sheaves of the

closed subschemes defined by i' and  $s_1$ , respectively. Since  $s_1$  factors through i', we have  $\mathcal{I}_{D_1} \subset \mathcal{I}_s$ . Hence  $D_1 - s$  is a relative effective Cartier divisor on  $(X_{\mathfrak{m}} \times T_1)/T_1$  by Lemma 2.2 (b), that is, there exists a relative effective Cartier divisor  $D_1'$  such that  $D_1 = s_1 + D_1'$ . Now we take  $T_2 = D_1'$ . We then have a finite flat morphism  $T_2 \to T_1$ , a section  $s_2 \colon T_2 \to X_{\mathfrak{m}} \times T_2$  of the projection  $X_{\mathfrak{m}} \times T_2 \to T_2$ , and a relative effective Cartier divisor  $D_2'$  on  $(X_{\mathfrak{m}} \times T_2)/T_2$  such that the pull-back of  $D_1'$  to  $X_{\mathfrak{m}} \times T_2$  is equal to  $s_2 + D_2'$ . Then we take  $T_3 = D_2', \ldots$ . In this way we get finite flat morphisms  $T_i \to T_{i-1}$   $(i = 1, \ldots, n)$ , sections  $s_i \colon T_i \to X_{\mathfrak{m}} \times T_i$ , such that the pull-back of D to  $X_{\mathfrak{m}} \times T_n$  is equal to  $s_1 + \cdots + s_n$ , where the  $s_i$  denote the relative effective Cartier divisors on  $(X_{\mathfrak{m}} \times T_n)/T_n$  induced by the sections  $s_i$ . This proves our lemma.

Finally we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. By Lemma A.7, there exist a finite flat morphism  $\pi\colon T'\to T$  and sections  $s_i\colon T'\to X_\mathfrak{m}\times T'$   $(i=1,\ldots,n)$  of the projection  $X_\mathfrak{m}\times T'\to T'$  such that the pull-back  $\pi^*D$  of D to  $X_\mathfrak{m}\times T'$  is equal to  $s_1+\cdots+s_n$ . By Lemma A.6, there exists a unique morphism of schemes  $f'\colon T'\to (X-S)^{(n)}$  such that the pull-back  $f'^*\mathcal{D}$  of the universal relative effective Cartier divisor  $\mathcal{D}$  to  $X_\mathfrak{m}\times T'$  is  $s_1+\cdots+s_n$ . Let  $p_1,p_2\colon T'\times_T T'\to T'$  be the projections. We have

$$(f'p_1)^*(\mathcal{D}) = p_1^*f'^*\mathcal{D} = p_1^*(s_1 + \dots + s_n) = p_1^*\pi^*D = p_2^*\pi^*D = \dots = (f'p_2)^*(\mathcal{D}).$$

that is,  $(f'p_1)^*(\mathcal{D}) = (f'p_2)^*(\mathcal{D})$ . By Lemma A.6 we have  $f'p_1 = f'p_2$ . By the theory of descent, ([SGA 1] VIII, Theorem 5.2), there exists a unique morphism of schemes  $f: T \to (X_{\mathfrak{m}} - Q)^{(n)}$  such that  $f' = f\pi$ , and the pull-back of  $\mathcal{D}$  to  $X_{\mathfrak{m}} \times T$  is D.

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