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Autor: Fu, Lei
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APPENDIX: PROOF OF PROPOSITION 3.1

We start with some lemmas.

LEMMA A.1. *Let A be a commutative ring with identity on which a finite group G acts, let A^G be the invariant subring, and let B be a flat A^G-algebra. Then G acts on $B \otimes_{A^G} A$ through its action on the second factor and the invariant subring of this action is B.*

Proof. We have an exact sequence

$$0 \to A^G \to A \to \prod_{g \in G} A,$$

where $\prod_{g \in G} A$ is the direct product of $|G|$ copies of A, and $A \to \prod_{g \in G} A$ is defined by $a \mapsto (ga - a)$. Since B is a flat A^G-algebra, the tensor product of B with the above sequence remains exact, that is, the sequence

$$0 \to B \to B \otimes_{A^G} A \to \prod_{g \in G} B \otimes_{A^G} A$$

is exact. Hence $B = (B \otimes_{A^G} A)^G$.

Let A be a finitely generated k-algebra on which a finite group G acts. Then A is finite over A^G. For every prime ideal $\mathfrak{q}$ of A^G, let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all the prime ideals of A lying over $\mathfrak{q}$. It is known that G acts transitively on $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Fix a $\mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Let $G_d = \{g \in G \mid g\mathfrak{p} = \mathfrak{p}\}$ be the decomposition group at $\mathfrak{p}$.

LEMMA A.2. *Notation as above. Let $\widehat{A^G_{\mathfrak{q}}}$ be the completion of the local ring $A^G_{\mathfrak{q}}$ and let $\widehat{A_{\mathfrak{p}}}$ be the completion of the local ring $A_{\mathfrak{p}}$. Then G_d acts on $\widehat{A_{\mathfrak{p}}}$ and $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G_{\mathfrak{q}}}$.*

Proof. Since $A^G_{\mathfrak{q}}$ is a flat A^G-algebra, we have $(A^G_{\mathfrak{q}} \otimes_{A^G} A)^G = A^G_{\mathfrak{q}}$ by Lemma A.1. Replacing A by $A^G_{\mathfrak{q}} \otimes_{A^G} A$ if necessary, we may thus assume that A^G is a local ring and $\mathfrak{q}$ is the maximal ideal of A^G.

Let $\widehat{A}$ be the completion of A with respect to the $\mathfrak{q}A$-adic topology. Since A is a finite A^G-algebra, we have $\widehat{A} = \widehat{A^G} \otimes_{A^G} A$. On the other hand, we have $\widehat{A} = \prod_i \widehat{A_{\mathfrak{p}_i}}$. Since $\widehat{A^G}$ is a flat A^G-algebra, we have $\widehat{A^G} = (\widehat{A^G} \otimes_{A^G} A)^G$ by Lemma A.1. So we have $\widehat{A^G} = (\prod_i \widehat{A_{\mathfrak{p}_i}})^G$. Obviously $(\prod_i \widehat{A_{\mathfrak{p}_i}})^G = (\widehat{A_{\mathfrak{p}}})^{G_d}$. Therefore $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G}$.

LEMMA A.3. *Let A be a noetherian local ring, let I_i $(i = 1, \dots, n)$ be some ideals of A, and let K_i be the kernel of the canonical homomorphism $\widehat{A} \to \widehat{A/I_i}$. If $I = I_1 \cdots I_n$, then the kernel of $\widehat{A} \to \widehat{A/I}$ is $K_1 \cdots K_n$.*

Proof. Since A is noetherian, we have $\ker(\widehat{A} \to \widehat{A/I_i}) = \widehat{I_i} = I_i\widehat{A}$, that is $K_i = I_i\widehat{A}$. Similarly we have $\ker(\widehat{A} \to \widehat{A/I}) = I\widehat{A} = I_1 \cdots I_n\widehat{A}$. So $\ker(\widehat{A} \to \widehat{A/I}) = K_1 \cdots K_n$.

Let T be a k-scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathrm{m}} \times T & \longrightarrow & X_{\mathrm{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \mathrm{spec}(k)\,. \end{array}$$

We have the following

LEMMA A.4. *Let $s\colon T \to X_{\mathrm{m}} \times T$ be a section of q. Then s is a closed immersion and the closed subscheme D defined by s is a relative effective Cartier divisor on $X_{\mathrm{m}} \times T/T$.*

Proof. Since qs=id is a closed immersion and since q is separated, s is also a closed immersion. The closed subscheme D defined by s is flat because qs=id. Let $\mathcal{I}$ be the sheaf of $\mathcal{O}_{X_{\mathrm{m}} \times T}$-ideals defining D. We have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X_{\mathrm{m}} \times T} \to \mathcal{O}_D \to 0\,.$$

For any $t \in T$, since $\mathcal{O}_D$ is $\mathcal{O}_T$ flat, the following sequence is exact:

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_T} k(t) \to \mathcal{O}_{X_{\mathrm{m}} \times T} \otimes_{\mathcal{O}_T} k(t) \to \mathcal{O}_{D_t} \to 0\,,$$

where D_t is the fiber of $D \to T$ at t. Hence $\mathcal{I} \otimes_{\mathcal{O}_T} k(t)$ is the ideal defining the closed subscheme D_t of $q^{-1}(t) \cong X_{\mathrm{m}}$. Obviously D_t defines a divisor of X_{m}. So for every point $x \in q^{-1}(t)$, the ideal $\mathcal{I}_x \otimes_{\mathcal{O}_T} k(t)$ of $\mathcal{O}_{X_{\mathrm{m}} \times T, x} \otimes_{\mathcal{O}_{T,t}} k(t)$ is generated by an element which is not a zero divisor. By Nakayama's lemma, the ideal $\mathcal{I}_x$ of $\mathcal{O}_{X_{\mathrm{m}} \times T, x}$ is generated by one element whose image in $\mathcal{O}_{X_{\mathrm{m}} \times T, x} \otimes_{\mathcal{O}_{T,t}} k(t)$ is not a zero divisor. By Lemma 2.3, D is a relative effective Cartier divisor.

Consider the sections

$$s_i\colon (X-S)^n \to X_{\mathfrak{m}} \times (X-S)^n, \quad (P_1,\ldots,P_n) \mapsto (P_i,P_1,\ldots,P_n).$$

Denote the relative effective Cartier divisors defined by s_i also by s_i, and let $D = s_1+\cdots+s_n$. The relative effective Cartier divisor D can also be regarded as a closed subscheme of $X_{\mathfrak{m}} \times (X-S)^n$. The n-th symmetric group $\mathfrak{S}_n$ acts on $(X-S)^n$ by permuting the factors. It acts on $X_{\mathfrak{m}} \times (X-S)^n$ through its action on the second factor. Obviously D is stable under this action. Let $\mathcal{D}$ be the quotient of D by $\mathfrak{S}_n$.

PROPOSITION A.5.

(a) *The quotient of* $X_{\mathfrak{m}} \times (X-S)^n$ *by* $\mathfrak{S}_n$ *is* $X_{\mathfrak{m}} \times (X-S)^{(n)}$.

(b) *The closed immersion* $D \to X_{\mathfrak{m}} \times (X-S)^n$ *induces a closed immersion* $\mathcal{D} \to X_{\mathfrak{m}} \times (X-S)^{(n)}$ *and* $\mathcal{D}$ *is a relative effective Cartier divisor on* $(X_{\mathfrak{m}} \times (X-S)^{(n)})/(X-S)^{(n)}$. *Moreover* D *is the pull-back of* $\mathcal{D}$.

Proof. (a) We have a Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times (X-S)^n & \longrightarrow & X_{\mathfrak{m}} \times (X-S)^{(n)} \\ \downarrow & & \downarrow \\ (X-S)^n & \longrightarrow & (X-S)^{(n)}. \end{array}$$

The morphism $X_{\mathfrak{m}} \times (X-S)^{(n)} \to (X-S)^{(n)}$ is flat since it is obtained from the flat morphism $X_{\mathfrak{m}} \to \operatorname{spec}(k)$ through the base extension $(X-S)^{(n)} \to \operatorname{spec}(k)$. Our assertion then follows directly from Lemma A.1.

(b) Consider the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_{\mathfrak{m}} \times (X-S)^n & \longrightarrow & X_{\mathfrak{m}} \times (X-S)^{(n)} \\ \downarrow & & \downarrow \\ (X-S)^n & \longrightarrow & (X-S)^{(n)}. \end{array}$$

One can easily show that $\mathcal{D} \to X_{\mathfrak{m}} \times (X-S)^{(n)}$ is a finite morphism and induces a homeomorphism of $\mathcal{D}$ with a closed subset of $X_{\mathfrak{m}} \times (X-S)^{(n)}$. We are going to show that for any point $y \in \mathcal{D}$, the homomorphism $\mathcal{O}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y} \to \mathcal{O}_{\mathcal{D},y}$ is surjective and the homomorphism $\mathcal{O}_{(X-S)^{(n)},t} \to \mathcal{O}_{\mathcal{D},y}$ is flat, where t is the image of y in $(X-S)^{(n)}$. If this is done, then $\mathcal{D} \to X_{\mathfrak{m}} \times (X-S)^{(n)}$

is a closed immersion and $\mathcal{D} \to (X-S)^{(n)}$ is flat. Obviously the fibers of $\mathcal{D} \to (X-S)^{(n)}$ are effective divisors. As in the proof of Lemma A.4, one can then use Nakayama's lemma and Lemma 2.3 to show that $\mathcal{D}$ is a relative effective Cartier divisor.

One can show that

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong \mathcal{O}_{\mathcal{D},y} \otimes_{\mathcal{O}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y}} \widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y} .$$

Note that $\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y}$ is a faithfully flat $\mathcal{O}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y}$-algebra. Thus to show that $\mathcal{O}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y} \to \mathcal{O}_{\mathcal{D},y}$ is surjective, it is enough to show that $\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y} \to \widehat{\mathcal{O}}_{\mathcal{D},y}$ is surjective; and to show that $\mathcal{O}_{(X-S)^{(n)},t} \to \mathcal{O}_{\mathcal{D},y}$ is flat, it is enough to show that $\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \to \widehat{\mathcal{O}}_{\mathcal{D},y}$ is flat.

Assume $t = n_1P_1 + \cdots + n_lP_l \in (X-S)^{(n)}$, where the P_i are distinct points of $X-S$, $n_i > 0$ and $\sum_i n_i = n$. Then $y = (P_{i_0}, t) \in X_{\mathfrak{m}} \times (X-S)^{(n)}$ for some $i_0 \in \{1, \ldots, l\}$. Let $t' = (P_1, \ldots, P_1, \ldots, P_l, \ldots, P_l) \in (X-S)^n$, where the first n_1 components of t' are P_1, ... , and the last n_l components are P_l. The point t' is a point in $(X-S)^n$ lying over $t \in (X-S)^{(n)}$. Let y' be the point (P_{i_0}, t') in $X_{\mathfrak{m}} \times (X-S)^n$. It lies over y. Note that y' is also a point in D. With respect to the actions of $\mathfrak{S}_n$ on $(X-S)^n$, on $X_{\mathfrak{m}} \times (X-S)^n$, and on D, the decomposition groups at $t' \in (X-S)^n$, at $y' \in X_{\mathfrak{m}} \times (X-S)^n$, and at $y' \in D$ are all $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$. We have

$$\widehat{\mathcal{O}}_{(X-S)^n,t} \cong k[[x_{11}, \ldots, x_{1n_1}, \ldots, x_{l1}, \ldots, x_{ln_l}]] ,$$

and the decomposition group $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{(X-S)^n,t}$ by permuting $x_{i1}, \ldots, x_{in_i}$ for each i. We have

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X-S)^n,y'} \cong k[[x, x_{11}, \ldots, x_{1n_1}, \ldots, x_{l1}, \ldots, x_{ln_l}]] ,$$

and the decomposition group $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X-S)^n,y'}$ by fixing x and permuting $x_{i1}, \ldots, x_{in_i}$ for each i.

For each $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \ldots, n_1 + \cdots + n_{i_0}\}$, the section

$$s_i \colon (X-S)^n \to X_{\mathfrak{m}} \times (X-S)^n, \quad (P_1, \ldots, P_n) \mapsto (P_i, P_1, \ldots, P_n)$$

induces a homomorphism

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X-S)^n,y'} \to \widehat{\mathcal{O}}_{(X-S)^n,t'} .$$

Through the isomorphism

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times (X-S)^n,y'} \cong k[[x, x_{11}, \ldots, x_{1n_1}, \ldots, x_{l1}, \ldots, x_{ln_l}]]$$

and the isomorphism

$$\widehat{\mathcal{O}}_{(X-S)^n,t'} \cong k[[x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]],$$

this homomorphism induced by s_i is

$$k[[x,x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]] \longrightarrow k[[x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]],$$
$$x \mapsto x_{i_0 j},\ x_{\alpha\beta} \mapsto x_{\alpha\beta} \quad (\alpha = 1,\dots,l,\ \beta = 1,\dots,n_\alpha),$$

where $j \in \{1,\dots,n_{i_0}\}$ is uniquely determined by $n_1 + \cdots + n_{i_0-1} + j = i$. The kernel of this homomorphism is the ideal $(x - x_{i_0 j})$. By Lemma A.3, the kernel of the homomorphism $\widehat{\mathcal{O}}_{X_\mathrm{m}\times(X-S)^n,y'} \longrightarrow \widehat{\mathcal{O}}_{D,y'}$ is identified with the ideal $\left(\prod_{j=1}^{n_{i_0}}(x - x_{i_0 j})\right)$ through the isomorphism

$$\widehat{\mathcal{O}}_{X_\mathrm{m}\times(X-S)^n,y'} \cong k[[x,x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]].$$

Hence

$$\widehat{\mathcal{O}}_{D,y'} \cong k[[x,x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]] \Big/ \left(\prod_{j=1}^{n_{i_0}}(x - x_{i_0 j})\right),$$

and the decomposition group $\mathfrak{S}_{n_1}\times\cdots\times\mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{D,y'}$ by fixing x and permuting $x_{i1},\dots,x_{in_i}$ for each i. Let $\sigma_{i1},\dots,\sigma_{in_i}$ be the elementary symmetric functions in $x_{i1},\dots,x_{in_i}$. By Lemma A.2, we have

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong k[[x,\sigma_{11},\dots,\sigma_{1n_1},\dots,\sigma_{l1},\dots,\sigma_{ln_l}]] \Big/ \left(\prod_{j=1}^{n_{i_0}}(x - x_{i_0 j})\right),$$

$$\widehat{\mathcal{O}}_{X_\mathrm{m}\times(X-S)^{(n)},y} \cong k[[x,\sigma_{11},\dots,\sigma_{1n_1},\dots,\sigma_{l1},\dots,\sigma_{ln_l}]],$$

$$\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \cong k[[\sigma_{11},\dots,\sigma_{1n_1},\dots,\sigma_{l1},\dots,\sigma_{ln_l}]].$$

Now it is easy to see that $\widehat{\mathcal{O}}_{X_\mathrm{m}\times(X-S)^{(n)},y} \longrightarrow \widehat{\mathcal{O}}_{\mathcal{D},y}$ is surjective and $\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \longrightarrow \widehat{\mathcal{O}}_{\mathcal{D},y}$ is flat. This proves $\mathcal{D}$ is a relative effective Cartier divisor. We also have

$$\widehat{\mathcal{O}}_{D,y'} = \widehat{\mathcal{O}}_{\mathcal{D},y}\widehat{\otimes}_{\widehat{\mathcal{O}}_{(X-S)^{(n)},t}}\widehat{\mathcal{O}}_{(X-S)^n,t'}.$$

This implies that $D = \mathcal{D}\times_{(X-S)^{(n)}}(X-S)^n$, that is, D is the pull-back of $\mathcal{D}$. This completes the proof of the proposition.

The relative effective Cartier divisor $\mathcal{D}$ is the universal relative effective Cartier divisor.

LEMMA A.6. *Let T be a k-scheme and let $s_i\colon T \to X_{\mathrm{m}} \times T$ $(i = 1, \ldots, n)$ be some sections of the projection $q\colon X_{\mathrm{m}} \times T \to T$. Assume the images of s_i lie in $(X_{\mathrm{m}} - Q) \times T$. Then there is a unique morphism of schemes $f\colon T \to (X-S)^{(n)}$ such that the pull-back by $\mathrm{id} \times f$ of the universal relative effective Cartier divisor $\mathcal{D}$ to $X_{\mathrm{m}} \times T$ is $s_1 + \cdots + s_n$.*

Proof. Let $p\colon X_{\mathrm{m}} \times T \to X_{\mathrm{m}}$ be the projection. The morphisms $ps_i\colon T \to X_{\mathrm{m}}$ induce $(ps_1, \ldots, ps_n)\colon T \to X_{\mathrm{m}}^n$. Since the images of s_i lie in $(X_{\mathrm{m}} - Q) \times T$, we actually get a morphism $(ps_1, \ldots, ps_n)\colon T \to (X - S)^n$. Composing with the canonical morphism $(X - S)^n \to (X - S)^{(n)}$, we get $f\colon T \to (X - S)^{(n)}$ so that the pull-back of $\mathcal{D}$ by $\mathrm{id} \times f$ is $s_1 + \cdots + s_n$. This proves the existence of f.

To prove the uniqueness of f, we first note that $f\colon T \to (X - S)^{(n)}$ is uniquely determined as a map on the underlying topological space. Indeed, for every point $t \in T$, $f(t)$ is necessarily the point in $(X - S)^{(n)}$ corresponding to the effective divisor $(s_1 + \cdots + s_n)_t$ on $q^{-1}(t) = X_m$. To prove f is unique as a morphism of schemes, it is enough to prove that the homomorphism on local rings $\mathcal{O}_{(X-S)^{(n)}, f(t)} \to \mathcal{O}_{T,t}$ induced by f is uniquely determined. It suffices to prove that $\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \to \widehat{\mathcal{O}}_{T,t}$ is uniquely determined.

Consider the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_{\mathrm{m}} \times T & \longrightarrow & X_{\mathrm{m}} \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & (X - S)^{(n)}, \end{array}$$

where D is the closed subscheme of $X_{\mathrm{m}} \times T$ corresponding to the divisor $s_1 + \cdots + s_n$. Let $A = \widehat{\mathcal{O}}_{T,t}$, let $z \in D$ be a point lying over $t \in T$, and let $y \in \mathcal{D}$ be the image of z. We have $\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times T, z} \cong A[[x]]$.

Without loss of generality, assume

$$ps_1(t) = \cdots = ps_{n_1}(t) = P_1\,,$$
$$ps_{n_1+1}(t) = \cdots = ps_{n_1+n_2}(t) = P_2\,,$$
$$\cdots\cdots\cdots$$
$$ps_{n_1+\cdots+n_{l-1}+1}(t) = \cdots = ps_{n_1+\cdots+n_l}(t) = P_l\,,$$

where $n_i > 0$ $(i = 1, \ldots, l)$, $n_1 + \cdots + n_l = n$, and the P_i are distinct points in $X - S$. Then we have $z = (P_{i_0}, t) \in X_{\mathrm{m}} \times T$ for some $i_0 \in \{1, \ldots, l\}$.

For each $i \in \{n_1 + \cdots + n_{i_0 - 1} + 1, \ldots, n_1 + \cdots + n_{i_0}\}$, the section s_i induces a homomorphism $\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times T, z} \to \widehat{\mathcal{O}}_{T, t}$, i.e., $A[[x]] \to A$. Denote the image of x under this homomorphism by $a_{i_0 j}$, where $j \in \{1, \ldots, n_{i_0}\}$ is uniquely determined by $n_1 + \cdots + n_{i_0 - 1} + j = i$. Then by Lemma A.3, we have

$$\widehat{\mathcal{O}}_{D, z} \cong A[[x]] \Big/ \left(\prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right) .$$

Keep the notations in the proof of Proposition A.5. We have

$$\widehat{\mathcal{O}}_{\mathcal{D}, y} \cong k[[x, \sigma_{11}, \ldots, \sigma_{1n_1}, \ldots, \sigma_{l1}, \ldots, \sigma_{ln_l}]] \Big/ \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right).$$

$$\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^{(n)}, y} \cong k[[x, \sigma_{11}, \ldots, \sigma_{1n_1}, \ldots, \sigma_{1l}, \ldots, \sigma_{ln_l}]] .$$

$$\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \cong k[[\sigma_{11}, \ldots, \sigma_{1n_1}, \ldots, \sigma_{l1}, \ldots, \sigma_{ln_l}]] .$$

We have a commutative diagram

$$\begin{array}{ccc}
\widehat{\mathcal{O}}_{D, z} & \longleftarrow & \widehat{\mathcal{O}}_{\mathcal{D}, y} \\
\uparrow & & \uparrow \\
\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times T, z} & \longleftarrow & \widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^{(n)}, y} \\
\uparrow & & \uparrow \\
\widehat{\mathcal{O}}_{T, t} & \longleftarrow & \widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} .
\end{array}$$

It is isomorphic to

$$\begin{array}{ccc}
A[[x]] \Big/ \left(\prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right) & \longleftarrow & k[[x, \sigma_{11}, \ldots, \sigma_{1n_1}, \ldots, \sigma_{l1}, \ldots, \sigma_{ln_l}]] \Big/ \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right) \\
\uparrow & & \uparrow \\
A[[x]] & \longleftarrow & k[[x, \sigma_{11}, \ldots, \sigma_{1n_1}, \ldots, \sigma_{l1}, \ldots, \sigma_{ln_l}]] \\
\uparrow & & \uparrow \\
A & \longleftarrow & k[[\sigma_{11}, \ldots, \sigma_{1n_1}, \ldots, \sigma_{l1}, \ldots, \sigma_{n_l}]] .
\end{array}$$

In order for this last diagram to commute, it is necessary that $\prod_{j=1}^{n_{i_0}}(x - x_{i_0 j})$ be mapped to $\prod_{j=1}^{n_{i_0}}(x - a_{i_0 j})$ under the homomorphism

$$k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{n_l}]] \to A[[x]] .$$

So the image of $\sigma_{i_0 j}$ under the homomorphism

$$k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \to A$$

is necessarily the value at $(a_{i_0 1}, \dots, a_{i_0 n_{i_0}})$ of $\sigma_{i_0 j}$ considered as a function on $A^{n_{i_0}}$. We see that this is true for any indices i_0 and j if we let z go over the points in D above t. Therefore the homomorphism $k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \to A$ is uniquely determined, that is, the homomorphism $\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \to \widehat{\mathcal{O}}_{T,t}$ is uniquely determined. This concludes the proof of the lemma.

LEMMA A.7. *Let T be a k-scheme and let D be a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times T)/T$ supported on $(X_{\mathfrak{m}} - Q) \times T$ with degree n. Then there exist a flat morphism $T' \to T$ and sections $s_i \colon T' \to X_{\mathfrak{m}} \times T'$ $(i = 1, \dots, n)$ of the projection $X_{\mathfrak{m}} \times T' \to T'$ such that the pull-back of D to $X_{\mathfrak{m}} \times T'$ is equal to $s_1 + \cdots + s_n$.*

Proof. By the definition of relative effective Cartier divisors, D is flat over T. On the other hand, $D \to T$ is proper and has finite fibers. So D is finite over T by [EGA] III, §4.4.2. Take $T_1 = D$. Then we have a finite flat morphism $T_1 \to T$. Consider the commutative diagram

$$\begin{array}{ccccc}
D \times_T T_1 & \xrightarrow{p'} & D & & \\
\downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \\
X_{\mathfrak{m}} \times T_1 & \xrightarrow{p} & X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\
\downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \downarrow \\
D = T_1 & \xrightarrow{qi} & T & \longrightarrow & \operatorname{spec}(k) .
\end{array}$$

Let $\Delta \colon D \to D \times_T D = D \times_T T_1$ be the diagonal map. It is a closed immersion since the morphism qi is separated. Take $s_1 = i'\Delta$. This is a section of q'. Hence it defines a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times T_1)/T_1$. The pull-back D_1 of the relative effective Cartier divisor D to $X_{\mathfrak{m}} \times T_1$ is the closed subscheme defined by i'. Let $\mathcal{I}_{D_1}$ and $\mathcal{I}_s$ be the ideal sheaves of the closed subschemes defined by i' and s_1, respectively. Since s_1 factors through i', we have $\mathcal{I}_{D_1} \subset \mathcal{I}_s$. Hence $D_1 - s$ is a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times T_1)/T_1$ by Lemma 2.2 (b), that is, there exists a relative effective Cartier divisor $D_1{}'$ such that $D_1 = s_1 + D_1{}'$. Now we take $T_2 = D_1{}'$. We then have a finite flat morphism $T_2 \to T_1$, a section $s_2 \colon T_2 \to X_{\mathfrak{m}} \times T_2$ of the projection $X_{\mathfrak{m}} \times T_2 \to T_2$, and a relative effective Cartier divisor $D_2{}'$ on $(X_{\mathfrak{m}} \times T_2)/T_2$ such that the pull-back of $D_1{}'$ to $X_{\mathfrak{m}} \times T_2$ is equal to $s_2 + D_2{}'$. Then we take $T_3 = D_2{}', \ldots$. In this way we get finite flat morphisms $T_i \to T_{i-1}$ $(i = 1, \ldots, n)$, sections $s_i \colon T_i \to X_{\mathfrak{m}} \times T_i$, such that the pull-back of D to $X_{\mathfrak{m}} \times T_n$ is equal to $s_1 + \cdots + s_n$, where the s_i denote the relative effective Cartier divisors on $(X_{\mathfrak{m}} \times T_n)/T_n$ induced by the sections s_i. This proves our lemma.

Finally we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. By Lemma A.7, there exist a finite flat morphism $\pi \colon T' \to T$ and sections $s_i \colon T' \to X_{\mathfrak{m}} \times T'$ $(i = 1, \ldots, n)$ of the projection $X_{\mathfrak{m}} \times T' \to T'$ such that the pull-back $\pi^* D$ of D to $X_{\mathfrak{m}} \times T'$ is equal to $s_1 + \cdots + s_n$. By Lemma A.6, there exists a unique morphism of schemes $f' \colon T' \to (X - S)^{(n)}$ such that the pull-back $f'^* \mathcal{D}$ of the universal relative effective Cartier divisor $\mathcal{D}$ to $X_{\mathfrak{m}} \times T'$ is $s_1 + \cdots + s_n$. Let $p_1, p_2 \colon T' \times_T T' \to T'$ be the projections. We have

$$(f'p_1)^*(\mathcal{D}) = p_1^* f'^* \mathcal{D} = p_1^*(s_1 + \cdots + s_n) = p_1^* \pi^* D = p_2^* \pi^* D = \ldots = (f'p_2)^*(\mathcal{D}).$$

that is, $(f'p_1)^*(\mathcal{D}) = (f'p_2)^*(\mathcal{D})$. By Lemma A.6 we have $f'p_1 = f'p_2$. By the theory of descent, ([SGA 1] VIII, Theorem 5.2), there exists a unique morphism of schemes $f \colon T \to (X_{\mathfrak{m}} - Q)^{(n)}$ such that $f' = f\pi$, and the pull-back of $\mathcal{D}$ to $X_{\mathfrak{m}} \times T$ is D.

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