

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE CONSTRUCTION OF GENERALIZED JACOBIANS
Autor: Fu, Lei
Anhang: Appendix: Proof of Proposition 3.1
DOI: <https://doi.org/10.5169/seals-64440>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 18.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

APPENDIX: PROOF OF PROPOSITION 3.1

We start with some lemmas.

LEMMA A.1. *Let A be a commutative ring with identity on which a finite group G acts, let A^G be the invariant subring, and let B be a flat A^G -algebra. Then G acts on $B \otimes_{A^G} A$ through its action on the second factor and the invariant subring of this action is B .*

Proof. We have an exact sequence

$$0 \rightarrow A^G \rightarrow A \rightarrow \prod_{g \in G} A,$$

where $\prod_{g \in G} A$ is the direct product of $|G|$ copies of A , and $A \rightarrow \prod_{g \in G} A$ is defined by $a \mapsto (ga - a)$. Since B is a flat A^G -algebra, the tensor product of B with the above sequence remains exact, that is, the sequence

$$0 \rightarrow B \rightarrow B \otimes_{A^G} A \rightarrow \prod_{g \in G} B \otimes_{A^G} A$$

is exact. Hence $B = (B \otimes_{A^G} A)^G$.

Let A be a finitely generated k -algebra on which a finite group G acts. Then A is finite over A^G . For every prime ideal \mathfrak{q} of A^G , let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all the prime ideals of A lying over \mathfrak{q} . It is known that G acts transitively on $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Fix a $\mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Let $G_d = \{g \in G \mid g\mathfrak{p} = \mathfrak{p}\}$ be the decomposition group at \mathfrak{p} .

LEMMA A.2. *Notation as above. Let $\widehat{A^G_{\mathfrak{q}}}$ be the completion of the local ring $A^G_{\mathfrak{q}}$ and let $\widehat{A_{\mathfrak{p}}}$ be the completion of the local ring $A_{\mathfrak{p}}$. Then G_d acts on $\widehat{A_{\mathfrak{p}}}$ and $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G_{\mathfrak{q}}}$.*

Proof. Since $A^G_{\mathfrak{q}}$ is a flat A^G -algebra, we have $(A^G_{\mathfrak{q}} \otimes_{A^G} A)^G = A^G_{\mathfrak{q}}$ by Lemma A.1. Replacing A by $A^G_{\mathfrak{q}} \otimes_{A^G} A$ if necessary, we may thus assume that A^G is a local ring and \mathfrak{q} is the maximal ideal of A^G .

Let \widehat{A} be the completion of A with respect to the $\mathfrak{q}A$ -adic topology. Since A is a finite A^G -algebra, we have $\widehat{A} = \widehat{A^G} \otimes_{A^G} A$. On the other hand, we have $\widehat{A} = \prod_i \widehat{A_{\mathfrak{p}_i}}$. Since $\widehat{A^G}$ is a flat A^G -algebra, we have $\widehat{A^G} = (\widehat{A^G} \otimes_{A^G} A)^G$ by Lemma A.1. So we have $\widehat{A^G} = (\prod_i \widehat{A_{\mathfrak{p}_i}})^G$. Obviously $(\prod_i \widehat{A_{\mathfrak{p}_i}})^G = (\widehat{A_{\mathfrak{p}}})^{G_d}$. Therefore $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G}$.

LEMMA A.3. *Let A be a noetherian local ring, let I_i ($i = 1, \dots, n$) be some ideals of A , and let K_i be the kernel of the canonical homomorphism $\widehat{A} \rightarrow \widehat{A/I_i}$. If $I = I_1 \cdots I_n$, then the kernel of $\widehat{A} \rightarrow \widehat{A/I}$ is $K_1 \cdots K_n$.*

Proof. Since A is noetherian, we have $\ker(\widehat{A} \rightarrow \widehat{A/I_i}) = \widehat{I_i} = I_i \widehat{A}$, that is $K_i = I_i \widehat{A}$. Similarly we have $\ker(\widehat{A} \rightarrow \widehat{A/I}) = I \widehat{A} = I_1 \cdots I_n \widehat{A}$. So $\ker(\widehat{A} \rightarrow \widehat{A/I}) = K_1 \cdots K_n$.

Let T be a k -scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_m \times T & \longrightarrow & X_m \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k). \end{array}$$

We have the following

LEMMA A.4. *Let $s: T \rightarrow X_m \times T$ be a section of q . Then s is a closed immersion and the closed subscheme D defined by s is a relative effective Cartier divisor on $X_m \times T/T$.*

Proof. Since $qs = \text{id}$ is a closed immersion and since q is separated, s is also a closed immersion. The closed subscheme D defined by s is flat because $qs = \text{id}$. Let \mathcal{I} be the sheaf of $\mathcal{O}_{X_m \times T}$ -ideals defining D . We have an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{X_m \times T} \rightarrow \mathcal{O}_D \rightarrow 0.$$

For any $t \in T$, since \mathcal{O}_D is \mathcal{O}_T flat, the following sequence is exact:

$$0 \rightarrow \mathcal{I} \otimes_{\mathcal{O}_T} k(t) \rightarrow \mathcal{O}_{X_m \times T} \otimes_{\mathcal{O}_T} k(t) \rightarrow \mathcal{O}_{D_t} \rightarrow 0,$$

where D_t is the fiber of $D \rightarrow T$ at t . Hence $\mathcal{I} \otimes_{\mathcal{O}_T} k(t)$ is the ideal defining the closed subscheme D_t of $q^{-1}(t) \cong X_m$. Obviously D_t defines a divisor of X_m . So for every point $x \in q^{-1}(t)$, the ideal $\mathcal{I}_x \otimes_{\mathcal{O}_T} k(t)$ of $\mathcal{O}_{X_m \times T, x} \otimes_{\mathcal{O}_T, t} k(t)$ is generated by an element which is not a zero divisor. By Nakayama's lemma, the ideal \mathcal{I}_x of $\mathcal{O}_{X_m \times T, x}$ is generated by one element whose image in $\mathcal{O}_{X_m \times T, x} \otimes_{\mathcal{O}_T, t} k(t)$ is not a zero divisor. By Lemma 2.3, D is a relative effective Cartier divisor.

Consider the sections

$$s_i: (X - S)^n \rightarrow X_m \times (X - S)^n, \quad (P_1, \dots, P_n) \mapsto (P_i, P_1, \dots, P_n).$$

Denote the relative effective Cartier divisors defined by s_i also by s_i , and let $D = s_1 + \dots + s_n$. The relative effective Cartier divisor D can also be regarded as a closed subscheme of $X_m \times (X - S)^n$. The n -th symmetric group \mathfrak{S}_n acts on $(X - S)^n$ by permuting the factors. It acts on $X_m \times (X - S)^n$ through its action on the second factor. Obviously D is stable under this action. Let \mathcal{D} be the quotient of D by \mathfrak{S}_n .

PROPOSITION A.5.

- (a) *The quotient of $X_m \times (X - S)^n$ by \mathfrak{S}_n is $X_m \times (X - S)^{(n)}$.*
 (b) *The closed immersion $D \rightarrow X_m \times (X - S)^n$ induces a closed immersion $\mathcal{D} \rightarrow X_m \times (X - S)^{(n)}$ and \mathcal{D} is a relative effective Cartier divisor on $(X_m \times (X - S)^{(n)})/(X - S)^{(n)}$. Moreover D is the pull-back of \mathcal{D} .*

Proof. (a) We have a Cartesian square

$$\begin{array}{ccc} X_m \times (X - S)^n & \longrightarrow & X_m \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ (X - S)^n & \longrightarrow & (X - S)^{(n)}. \end{array}$$

The morphism $X_m \times (X - S)^{(n)} \rightarrow (X - S)^{(n)}$ is flat since it is obtained from the flat morphism $X_m \rightarrow \text{spec}(k)$ through the base extension $(X - S)^{(n)} \rightarrow \text{spec}(k)$. Our assertion then follows directly from Lemma A.1.

(b) Consider the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_m \times (X - S)^n & \longrightarrow & X_m \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ (X - S)^n & \longrightarrow & (X - S)^{(n)}. \end{array}$$

One can easily show that $\mathcal{D} \rightarrow X_m \times (X - S)^{(n)}$ is a finite morphism and induces a homeomorphism of \mathcal{D} with a closed subset of $X_m \times (X - S)^{(n)}$. We are going to show that for any point $y \in \mathcal{D}$, the homomorphism $\mathcal{O}_{X_m \times (X - S)^{(n)}, y} \rightarrow \mathcal{O}_{\mathcal{D}, y}$ is surjective and the homomorphism $\mathcal{O}_{(X - S)^{(n)}, t} \rightarrow \mathcal{O}_{\mathcal{D}, y}$ is flat, where t is the image of y in $(X - S)^{(n)}$. If this is done, then $\mathcal{D} \rightarrow X_m \times (X - S)^{(n)}$

is a closed immersion and $\mathcal{D} \rightarrow (X - S)^{(n)}$ is flat. Obviously the fibers of $\mathcal{D} \rightarrow (X - S)^{(n)}$ are effective divisors. As in the proof of Lemma A.4, one can then use Nakayama's lemma and Lemma 2.3 to show that \mathcal{D} is a relative effective Cartier divisor.

One can show that

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong \mathcal{O}_{\mathcal{D},y} \otimes_{\mathcal{O}_{X_m \times (X-S)^{(n)},y}} \widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)},y}.$$

Note that $\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)},y}$ is a faithfully flat $\mathcal{O}_{X_m \times (X-S)^{(n)},y}$ -algebra. Thus to show that $\mathcal{O}_{X_m \times (X-S)^{(n)},y} \rightarrow \mathcal{O}_{\mathcal{D},y}$ is surjective, it is enough to show that $\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)},y} \rightarrow \widehat{\mathcal{O}}_{\mathcal{D},y}$ is surjective; and to show that $\mathcal{O}_{(X-S)^{(n)},t} \rightarrow \mathcal{O}_{\mathcal{D},y}$ is flat, it is enough to show that $\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \rightarrow \widehat{\mathcal{O}}_{\mathcal{D},y}$ is flat.

Assume $t = n_1 P_1 + \cdots + n_l P_l \in (X - S)^{(n)}$, where the P_i are distinct points of $X - S$, $n_i > 0$ and $\sum_i n_i = n$. Then $y = (P_{i_0}, t) \in X_m \times (X - S)^{(n)}$ for some $i_0 \in \{1, \dots, l\}$. Let $t' = (P_1, \dots, P_1, \dots, P_l, \dots, P_l) \in (X - S)^n$, where the first n_1 components of t' are P_1, \dots , and the last n_l components are P_l . The point t' is a point in $(X - S)^n$ lying over $t \in (X - S)^{(n)}$. Let y' be the point (P_{i_0}, t') in $X_m \times (X - S)^n$. It lies over y . Note that y' is also a point in D . With respect to the actions of \mathfrak{S}_n on $(X - S)^n$, on $X_m \times (X - S)^n$, and on D , the decomposition groups at $t' \in (X - S)^n$, at $y' \in X_m \times (X - S)^n$, and at $y' \in D$ are all $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$. We have

$$\widehat{\mathcal{O}}_{(X-S)^n,t} \cong k[[x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

and the decomposition group $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{(X-S)^n,t}$ by permuting x_{i1}, \dots, x_{in_i} for each i . We have

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n,y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

and the decomposition group $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{X_m \times (X-S)^n,y'}$ by fixing x and permuting x_{i1}, \dots, x_{in_i} for each i .

For each $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \dots, n_1 + \cdots + n_{i_0}\}$, the section

$$s_i: (X - S)^n \rightarrow X_m \times (X - S)^n, \quad (P_1, \dots, P_n) \mapsto (P_i, P_1, \dots, P_n)$$

induces a homomorphism

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n,y'} \rightarrow \widehat{\mathcal{O}}_{(X-S)^n,t'}.$$

Through the isomorphism

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n,y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]]$$

and the isomorphism

$$\widehat{\mathcal{O}}_{(X-S)^n, t'} \cong k[[x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

this homomorphism induced by s_i is

$$\begin{aligned} k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]] &\rightarrow k[[x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]], \\ x &\mapsto x_{i_0j}, \quad x_{\alpha\beta} \mapsto x_{\alpha\beta} \quad (\alpha = 1, \dots, l, \beta = 1, \dots, n_\alpha), \end{aligned}$$

where $j \in \{1, \dots, n_{i_0}\}$ is uniquely determined by $n_1 + \dots + n_{i_0-1} + j = i$. The kernel of this homomorphism is the ideal $(x - x_{i_0j})$. By Lemma A.3, the kernel of the homomorphism $\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \rightarrow \widehat{\mathcal{O}}_{D, y'}$ is identified with the ideal $\left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})\right)$ through the isomorphism

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]].$$

Hence

$$\widehat{\mathcal{O}}_{D, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})\right),$$

and the decomposition group $\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{D, y'}$ by fixing x and permuting x_{i1}, \dots, x_{in_i} for each i . Let $\sigma_{i1}, \dots, \sigma_{in_i}$ be the elementary symmetric functions in x_{i1}, \dots, x_{in_i} . By Lemma A.2, we have

$$\widehat{\mathcal{O}}_{D, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})\right),$$

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^{(m)}, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]],$$

$$\widehat{\mathcal{O}}_{(X-S)^{(m)}, t} \cong k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

Now it is easy to see that $\widehat{\mathcal{O}}_{X_m \times (X-S)^{(m)}, y} \rightarrow \widehat{\mathcal{O}}_{D, y}$ is surjective and $\widehat{\mathcal{O}}_{(X-S)^{(m)}, t} \rightarrow \widehat{\mathcal{O}}_{D, y}$ is flat. This proves \mathcal{D} is a relative effective Cartier divisor. We also have

$$\widehat{\mathcal{O}}_{D, y'} = \widehat{\mathcal{O}}_{D, y} \widehat{\otimes}_{\widehat{\mathcal{O}}_{(X-S)^{(m)}, t}} \widehat{\mathcal{O}}_{(X-S)^n, t'}.$$

This implies that $D = \mathcal{D} \times_{(X-S)^{(m)}} (X-S)^n$, that is, D is the pull-back of \mathcal{D} . This completes the proof of the proposition.

The relative effective Cartier divisor \mathcal{D} is the universal relative effective Cartier divisor.

For each $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \dots, n_1 + \cdots + n_{i_0}\}$, the section s_i induces a homomorphism $\widehat{\mathcal{O}}_{X_m \times T, z} \rightarrow \widehat{\mathcal{O}}_{T, t}$, i.e., $A[[x]] \rightarrow A$. Denote the image of x under this homomorphism by $a_{i_0 j}$, where $j \in \{1, \dots, n_{i_0}\}$ is uniquely determined by $n_1 + \cdots + n_{i_0-1} + j = i$. Then by Lemma A.3, we have

$$\widehat{\mathcal{O}}_{D, z} \cong A[[x]] / \left(\prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right).$$

Keep the notations in the proof of Proposition A.5. We have

$$\widehat{\mathcal{O}}_{D, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right).$$

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^{m_i}, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

$$\widehat{\mathcal{O}}_{(X-S)^{m_i}, f(t)} \cong k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

We have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{D, z} & \longleftarrow & \widehat{\mathcal{O}}_{D, y} \\ \uparrow & & \uparrow \\ \widehat{\mathcal{O}}_{X_m \times T, z} & \longleftarrow & \widehat{\mathcal{O}}_{X_m \times (X-S)^{m_i}, y} \\ \uparrow & & \uparrow \\ \widehat{\mathcal{O}}_{T, t} & \longleftarrow & \widehat{\mathcal{O}}_{(X-S)^{m_i}, f(t)}. \end{array}$$

It is isomorphic to

$$\begin{array}{ccc} A[[x]] / \left(\prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right) & \longleftarrow & k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right) \\ \uparrow & & \uparrow \\ A[[x]] & \longleftarrow & k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \\ \uparrow & & \uparrow \\ A & \longleftarrow & k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]]. \end{array}$$

In order for this last diagram to commute, it is necessary that $\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})$ be mapped to $\prod_{j=1}^{n_{i_0}} (x - a_{i_0j})$ under the homomorphism

$$k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{n_l}]] \rightarrow A[[x]].$$

So the image of σ_{i_0j} under the homomorphism

$$k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{n_l}]] \rightarrow A$$

is necessarily the value at $(a_{i_01}, \dots, a_{i_0n_{i_0}})$ of σ_{i_0j} considered as a function on $A^{n_{i_0}}$. We see that this is true for any indices i_0 and j if we let z go over the points in D above t . Therefore the homomorphism $k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{n_l}]] \rightarrow A$ is uniquely determined, that is, the homomorphism $\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \rightarrow \widehat{\mathcal{O}}_{T,t}$ is uniquely determined. This concludes the proof of the lemma.

LEMMA A.7. *Let T be a k -scheme and let D be a relative effective Cartier divisor on $(X_m \times T)/T$ supported on $(X_m - Q) \times T$ with degree n . Then there exist a flat morphism $T' \rightarrow T$ and sections $s_i: T' \rightarrow X_m \times T'$ ($i = 1, \dots, n$) of the projection $X_m \times T' \rightarrow T'$ such that the pull-back of D to $X_m \times T'$ is equal to $s_1 + \dots + s_n$.*

Proof. By the definition of relative effective Cartier divisors, D is flat over T . On the other hand, $D \rightarrow T$ is proper and has finite fibers. So D is finite over T by [EGA] III, §4.4.2. Take $T_1 = D$. Then we have a finite flat morphism $T_1 \rightarrow T$. Consider the commutative diagram

$$\begin{array}{ccccc} D \times_T T_1 & \xrightarrow{p'} & D & & \\ i' \downarrow & & i \downarrow & & \\ X_m \times T_1 & \xrightarrow{p} & X_m \times T & \longrightarrow & X_m \\ q' \downarrow & & q \downarrow & & \downarrow \\ D = T_1 & \xrightarrow{qi} & T & \longrightarrow & \text{spec}(k). \end{array}$$

Let $\Delta: D \rightarrow D \times_T D = D \times_T T_1$ be the diagonal map. It is a closed immersion since the morphism qi is separated. Take $s_1 = i'\Delta$. This is a section of q' . Hence it defines a relative effective Cartier divisor on $(X_m \times T_1)/T_1$. The pull-back D_1 of the relative effective Cartier divisor D to $X_m \times T_1$ is the closed subscheme defined by i' . Let \mathcal{I}_{D_1} and \mathcal{I}_s be the ideal sheaves of the

closed subschemes defined by i' and s_1 , respectively. Since s_1 factors through i' , we have $\mathcal{I}_{D_1} \subset \mathcal{I}_s$. Hence $D_1 - s$ is a relative effective Cartier divisor on $(X_m \times T_1)/T_1$ by Lemma 2.2 (b), that is, there exists a relative effective Cartier divisor D_1' such that $D_1 = s_1 + D_1'$. Now we take $T_2 = D_1'$. We then have a finite flat morphism $T_2 \rightarrow T_1$, a section $s_2: T_2 \rightarrow X_m \times T_2$ of the projection $X_m \times T_2 \rightarrow T_2$, and a relative effective Cartier divisor D_2' on $(X_m \times T_2)/T_2$ such that the pull-back of D_1' to $X_m \times T_2$ is equal to $s_2 + D_2'$. Then we take $T_3 = D_2'$, \dots . In this way we get finite flat morphisms $T_i \rightarrow T_{i-1}$ ($i = 1, \dots, n$), sections $s_i: T_i \rightarrow X_m \times T_i$, such that the pull-back of D to $X_m \times T_n$ is equal to $s_1 + \dots + s_n$, where the s_i denote the relative effective Cartier divisors on $(X_m \times T_n)/T_n$ induced by the sections s_i . This proves our lemma.

Finally we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. By Lemma A.7, there exist a finite flat morphism $\pi: T' \rightarrow T$ and sections $s_i: T' \rightarrow X_m \times T'$ ($i = 1, \dots, n$) of the projection $X_m \times T' \rightarrow T'$ such that the pull-back π^*D of D to $X_m \times T'$ is equal to $s_1 + \dots + s_n$. By Lemma A.6, there exists a unique morphism of schemes $f': T' \rightarrow (X - S)^{(n)}$ such that the pull-back $f'^*\mathcal{D}$ of the universal relative effective Cartier divisor \mathcal{D} to $X_m \times T'$ is $s_1 + \dots + s_n$. Let $p_1, p_2: T' \times_T T' \rightarrow T'$ be the projections. We have

$$(f'p_1)^*(\mathcal{D}) = p_1^*f'^*\mathcal{D} = p_1^*(s_1 + \dots + s_n) = p_1^*\pi^*D = p_2^*\pi^*D = \dots = (f'p_2)^*(\mathcal{D}).$$

that is, $(f'p_1)^*(\mathcal{D}) = (f'p_2)^*(\mathcal{D})$. By Lemma A.6 we have $f'p_1 = f'p_2$. By the theory of descent, ([SGA 1] VIII, Theorem 5.2), there exists a unique morphism of schemes $f: T \rightarrow (X_m - Q)^{(n)}$ such that $f' = f\pi$, and the pull-back of \mathcal{D} to $X_m \times T$ is D .

REFERENCES

- [A] ARTIN, M. *Néron Models*. Arithmetic Geometry, edited by Cornell and Silverman, Springer-Verlag, 1986.
- [BLR] BOSCH, S., W. LÜTKEBOHMERT and M. RAYNAUD. *Néron Models*. Springer-Verlag, 1990.
- [EGA] GROTHENDIECK, A. *Éléments de Géométrie Algébrique* (rédigés avec la collaboration de J. Dieudonné). Chap. 0 à IV. *Publ. Math. IHES* 4, 8, 11, 17, 20, 24, 28, 32 (1960–1967).
- [SGA 1] ——— *Revêtements étales et groupe fondamental*. Lecture Notes in Mathematics 224, Springer-Verlag, 1971.