Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE CONSTRUCTION OF GENERALIZED JACOBIANS
Autor: Fu, Lei
Kapitel: 6. Generalized jacobians and Picard schemes
DOI: https://doi.org/10.5169/seals-64440

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

THEOREM 1. *There is a morphism of algebraic varieties $\theta\colon X - S \to J_{\mathfrak{m}}$ satisfying the following properties:*

(a) *The extension of θ to the group of divisors on X prime to S induces, by passing to quotient, an isomorphism between the group $C_{\mathfrak{m}}^0$ of classes of divisors of degree zero with respect to $\mathfrak{m}$-equivalence and the group $J_{\mathfrak{m}}$.*

(b) *The extension of θ to $(X-S)^{(\pi)}$ induces a birational map from $X^{(\pi)}$ to $J_{\mathfrak{m}}$.*

The following theorem characterizes $J_{\mathfrak{m}}$ by a universal property:

THEOREM 2. *Let $f\colon X \to G$ be a rational map from X to a commutative algebraic group G and assume $\mathfrak{m}$ is a modulus for f. Then there is a unique homomorphism $F\colon J_{\mathfrak{m}} \to G$ of algebraic groups such that $f = F\circ\theta + f(P_0)$.*

Proof. Replacing f by $f - f(P_0)$, we may assume $f(P_0) = 0$. Since $\mathfrak{m}$ is a modulus for f, the extension of f to the group of divisors of X prime to S induces a homomorphism $C_{\mathfrak{m}}^0 \to G$ by passing to quotient. By Theorem 1 (a) we have $J_{\mathfrak{m}} \cong C_{\mathfrak{m}}^0$ as groups. So we have a homomorphism of groups $F\colon J_{\mathfrak{m}} \to G$ such that $f = F\theta$. It remains to prove F is a morphism of algebraic varieties. By Theorem 1 (b) we have a birational map $\theta\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$. Denote the extension of f to $(X-S)^{(\pi)}$ by f'. Then $F\theta = f'$. Since θ is birational, it induces an isomorphism between an open subvariety of $(X-S)^{(\pi)}$ and an open subvariety of $J_{\mathfrak{m}}$. Moreover f' is a morphism of algebraic varieties. Hence F is a morphism of algebraic varieties when restricted to some open subset of $J_{\mathfrak{m}}$. The whole $J_{\mathfrak{m}}$ can be obtained from this open subset by translation. So F is a morphism of algebraic varieties.

6. GENERALIZED JACOBIANS AND PICARD SCHEMES

In this section we prove $J_{\mathfrak{m}}$ is the Picard scheme of $X_{\mathfrak{m}}$.

Let T be a k-scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \operatorname{spec}(k)\,. \end{array}$$

We have $q_*\mathcal{O}_{X_{\mathfrak{m}}\times T} = \mathcal{O}_T$ by [EGA] III, § 1.4.15, the fact $H^0(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}}) = k$, and the fact that $T \to \operatorname{spec}(k)$ is flat. The morphism q has a section $s\colon T \to X_{\mathfrak{m}} \times T$, $t \mapsto (P_0, t)$.

LEMMA 6.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invertible sheaves on $X_{\mathfrak{m}} \times T$. Assume $\mathcal{L}_1 \cong \mathcal{L}_2$. Then the canonical map* $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \to \mathrm{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_2)$ *induced by s is bijective.*

Proof. Since $\mathcal{L}_1 \cong \mathcal{L}_2$, it is enough to show that the canonical map $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_1) \to \mathrm{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$ is bijective. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X_{\mathfrak{m}} \times T}(X_{\mathfrak{m}} \times T) & \longrightarrow & \mathcal{O}_T(T) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_1) & \longrightarrow & \mathrm{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)\,, \end{array}$$

where the horizontal arrows are induced by s. We have

$$\begin{aligned} \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_1) &\cong \mathrm{Hom}(\mathcal{O}_{X_{\mathfrak{m}} \times T}, \mathcal{L}_1 \otimes \mathcal{L}_1^{-1}) \\ &\cong \mathrm{Hom}(\mathcal{O}_{X_{\mathfrak{m}} \times T}, \mathcal{O}_{X_{\mathfrak{m}} \times T}) \cong \mathcal{O}_{X_{\mathfrak{m}} \times T}(X_{\mathfrak{m}} \times T)\,. \end{aligned}$$

Hence the left vertical arrow in the above diagram is bijective. Similarly the right vertical arrow is also bijective. Since $q_*\mathcal{O}_{X_{\mathfrak{m}} \times T} = \mathcal{O}_T$, we have $\mathcal{O}_{X_{\mathfrak{m}} \times T}(X_{\mathfrak{m}} \times T) \cong \mathcal{O}(T)$, and the upper horizontal arrow is bijective. Hence $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_1) \cong \mathrm{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$ by the commutativity of the above diagram.

LEMMA 6.2. *Let $\{U_i\}$ be an open covering of T and let $\mathcal{L}_i$ be invertible sheaves on $X_{\mathfrak{m}} \times U_i$. Assume $s^*\mathcal{L}_i \cong \mathcal{O}_{U_i}$ and $\mathcal{L}_i\,|_{X_{\mathfrak{m}} \times (U_i \cap U_j)} \cong \mathcal{L}_j\,|_{X_{\mathfrak{m}} \times (U_i \cap U_j)}$. Then there exists an invertible sheaf $\mathcal{L}$ on $X_{\mathfrak{m}} \times T$ such that $\mathcal{L}\,|_{X_{\mathfrak{m}} \times U_i} \cong \mathcal{L}_i$ and $s^*\mathcal{L} \cong \mathcal{O}_T$. Moreover $\mathcal{L}$ is unique up to isomorphism.*

Proof. Fix an isomorphism $\alpha_i \colon s^*\mathcal{L}_i \to \mathcal{O}_{U_i}$ for each i. Let

$$\alpha_{ij} \colon s^*\mathcal{L}_i|_{U_i \cap U_j} \to s^*\mathcal{L}_j|_{U_i \cap U_j}$$

be the isomorphism $(\alpha_j|_{U_i \cap U_j})^{-1} \circ (\alpha_i|_{U_i \cap U_j})$. By Lemma 6.1 the canonical map

$$\mathrm{Hom}(\mathcal{L}_i|_{X_{\mathfrak{m}} \times (U_i \cap U_j)}, \mathcal{L}_j|_{X_{\mathfrak{m}} \times (U_i \cap U_j)}) \to \mathrm{Hom}(s^*\mathcal{L}_i|_{U_i \cap U_j}, s^*\mathcal{L}_j|_{U_i \cap U_j})$$

is bijective. So α_{ij} can be lifted uniquely to an isomorphism

$$A_{ij} \colon \mathcal{L}_i\,|_{X_{\mathfrak{m}} \times (U_i \cap U_j)} \to \mathcal{L}_j\,|_{X_{\mathfrak{m}} \times (U_i \cap U_j)}\,.$$

By the uniqueness of the lifting and the fact that $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$ on $U_i \cap U_j \cap U_k$, we have $A_{jk}A_{ij} = A_{ik}$ on $X_{\mathfrak{m}} \times (U_i \cap U_j \cap U_k)$. So A_{ij} defines glueing data and we can glue the $\mathcal{L}_i$ together to get an invertible sheaf $\mathcal{L}$ on $X_{\mathfrak{m}} \times T$. By the construction of $\mathcal{L}$ we have $s^*\mathcal{L} \cong \mathcal{O}_T$. This proves the existence of $\mathcal{L}$. Similarly using Lemma 6.1 one can prove $\mathcal{L}$ is unique up to isomorphism.

LEMMA 6.3. *Assume T is integral. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invertible sheaves on $X_{\mathrm{m}} \times T$ satisfying $\mathcal{L}_{1_t} \cong \mathcal{L}_{2_t}$ for all $t \in T$. Then there is an invertible sheaf $\mathcal{M}$ on T such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes q^*\mathcal{M}$.*

Proof. Let $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$. Then $\mathcal{L}_t \cong \mathcal{O}_{X_{\mathrm{m}}}$. It suffices to show that $\mathcal{L} \cong q^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on T. We have $H^0(X_{\mathrm{m}}, \mathcal{L}_t) = H^0(X_{\mathrm{m}}, \mathcal{O}_{X_{\mathrm{m}}}) = k$. By Theorem 1.1 (c), the sheaf $q_*\mathcal{L}$ is invertible and $q_*\mathcal{L} \otimes k(t) = H^0(X_{\mathrm{m}}, \mathcal{L}_t)$. So the restriction $(q^*q_*\mathcal{L})_t \to \mathcal{L}_t$ of the canonical map $q^*q_*\mathcal{L} \to \mathcal{L}$ to the fiber of q at $t \in T$ is $H^0(X_{\mathrm{m}}, \mathcal{L}_t) \otimes \mathcal{O}_{X_{\mathrm{m}}} \to \mathcal{L}_t$, which is an isomorphism since $\mathcal{L}_t \cong \mathcal{O}_{X_{\mathrm{m}}}$. By Nakayama's Lemma, the canonical map $q^*q_*\mathcal{L} \to L$ is surjective. But since it is a homomorphism of invertible sheaves, it must be bijective. Hence $\mathcal{L} \cong q^*q_*\mathcal{L}$.

Now we use the above lemmas to construct a canonical invertible sheaf on $X_{\mathrm{m}} \times J_{\mathrm{m}}$.

On $X_{\mathrm{m}} \times (X-S)^{(\pi)}$ we have the invertible sheaf corresponding to the divisor $\mathcal{D} - p^*(\pi P_0)$, where $\mathcal{D}$ is the universal relative effective Cartier divisor and $p\colon X_{\mathrm{m}} \times (X-S)^{(\pi)} \to X_{\mathrm{m}}$ is the projection. Since $\theta\colon (X-S)^{(\pi)} \to J_{\mathrm{m}}$ is birational, there exist open subsets U in $(X-S)^{(\pi)}$ and V in J_{m} such that θ induces an isomorphism $U \cong V$. Hence we can push-forward the above invertible sheaf on $X_{\mathrm{m}} \times (X-S)^{(\pi)}$ to get an invertible sheaf $\mathcal{L}_V$ on $X_{\mathrm{m}} \times V$. For each $t \in J_{\mathrm{m}}$, denote by $\mathcal{L}(t)$ the invertible sheaf on X_{m} corresponding to the divisor class in C^0_{m} that is mapped to $t \in J_{\mathrm{m}}$ under the canonical isomorphism $C^0_{\mathrm{m}} \cong J_{\mathrm{m}}$. Obviously the restriction $\mathcal{L}_{V,t}$ of $\mathcal{L}_V$ to the fiber of the projection $q\colon X_{\mathrm{m}} \times J_{\mathrm{m}} \to J_{\mathrm{m}}$ at $t \in V$ is isomorphic to $\mathcal{L}(t)$. The invertible sheaf $\mathcal{L}_V \otimes (q^*s^*\mathcal{L}_V)^{-1}$ has the same property, where $s\colon J_{\mathrm{m}} \to X_{\mathrm{m}} \times J_{\mathrm{m}}$ is the section $t \mapsto (P_0, t)$. Thus replacing $\mathcal{L}_V$ by $\mathcal{L}_V \otimes (q^*s^*\mathcal{L}_V)^{-1}$ if necessary, we may assume that $s^*\mathcal{L}_V \cong \mathcal{O}_V$.

For each $a \in J_{\mathrm{m}}$, let $T_{-a}\colon J_{\mathrm{m}} \to J_{\mathrm{m}}$ be the translation $t \mapsto t-a$. Consider the invertible sheaf $\mathcal{L}_{a+V} = (\mathrm{id} \times T_{-a})^*\mathcal{L}_V \otimes p^*\mathcal{L}(a)$ on $X_{\mathrm{m}} \otimes (a+V)$, where $p\colon X_{\mathrm{m}} \times J_{\mathrm{m}} \to X_{\mathrm{m}}$ is the projection. The restriction $\mathcal{L}_{a+V,a+t}$ of $\mathcal{L}_{a+V}$ to the fiber of q at $a+t \in a+V$ is

$$((\mathrm{id} \times T_{-a})^*\mathcal{L}_V \otimes p^*\mathcal{L}(a))_{a+t} = \mathcal{L}_{V,t} \otimes \mathcal{L}(a) = \mathcal{L}(t) \otimes \mathcal{L}(a) = \mathcal{L}(a+t),$$

that is, $\mathcal{L}_{a+V,a+t} = \mathcal{L}(a+t)$. Hence for any $t \in V \cap (a+V)$, we have $\mathcal{L}_{V,t} = \mathcal{L}_{a+V,t}$. By Lemma 6.3, we have

$$\mathcal{L}_V|_{X_{\mathrm{m}} \times (V\cap(a+V))} \cong \mathcal{L}_{a+V}|_{X_{\mathrm{m}} \times (V\cap(a+V))} \otimes q^*\mathcal{M}$$

for some invertible sheaf $\mathcal{M}$ on $V \cap (a+V)$. But since $s^*\mathcal{L}_V \cong \mathcal{O}_V$, we also have $s^*\mathcal{L}_{a+V} = \mathcal{O}_{a+V}$. Hence $\mathcal{M} \cong \mathcal{O}_{V\cap(a+V)}$. Therefore $\mathcal{L}_V|_{X_{\mathrm{m}} \times (V\cap(a+V))} \cong$

$\mathcal{L}_{a+V}|_{X_\mathfrak{m}\times(V\cap(a+V))}$. By Lemma 6.2, we can glue $\mathcal{L}_{a+V}$ $(a \in J_m)$ together to get an invertible sheaf $\mathcal{L}_{J_\mathfrak{m}}$ on $X_\mathfrak{m} \times J_\mathfrak{m}$. It has the property that its restriction to the fiber of q at $t \in J_\mathfrak{m}$ is isomorphic to $\mathcal{L}(t)$ and $s^*\mathcal{L}_{J_\mathfrak{m}} \cong \mathcal{O}_{J_\mathfrak{m}}$.

Define

$$P^0(T) = \{\mathcal{L} \in \operatorname{Pic}(X_\mathfrak{m} \times T) \mid \deg(\mathcal{L}) = 0\}/q^* \operatorname{Pic}(T),$$

where $\deg(\mathcal{L})$ is defined as the leading coefficient of $\chi(\mathcal{L}_t^{\otimes n})$ as a polynomial in n. Since $s^*q^* = \mathrm{id}$, we may define

$$P^0(T) = \{\mathcal{L} \in \operatorname{Pic}(X_\mathfrak{m} \times T) \mid \deg(\mathcal{L}) = 0 \text{ and } s^*\mathcal{L} \cong \mathcal{O}_T\}$$

as well. In particular, we have $\mathcal{L}_{J_\mathfrak{m}} \in P^0(J_\mathfrak{m})$. Using the first definition of $P^0(T)$ and Lemma 6.3, one can show that the pull-back of $\mathcal{L}_{J_\mathfrak{m}}$ by $\mathrm{id} \times \theta\colon X_\mathfrak{m} \times (X-S)^{(\pi)} \to X_\mathfrak{m} \times J_\mathfrak{m}$ is the invertible sheaf on $X_\mathfrak{m} \times (X-S)^{(\pi)}$ corresponding to the divisor $\mathcal{D} - p^*(\pi P_0)$.

The following theorem says that $J_\mathfrak{m}$ is the Picard scheme of $X_\mathfrak{m}$.

THEOREM 3. *The functor $T \to P^0(T)$ is represented by $J_\mathfrak{m}$. More precisely, for any invertible sheaf $\mathcal{L}$ on $X_\mathfrak{m} \times T$ of degree 0 satisfying $s^*\mathcal{L} \cong \mathcal{O}_T$, there is one and only one morphism of schemes $f\colon T \to J_\mathfrak{m}$ such that $\mathcal{L}$ is the pull-back of $\mathcal{L}_{J_\mathfrak{m}}$ by $\mathrm{id} \times f\colon X_\mathfrak{m} \times T \to X_\mathfrak{m} \times J_\mathfrak{m}$.*

Proof. Let $V_0 = \{D \in (X-S)^{(\pi)} \mid l_\mathfrak{m}(D) = 1, \quad l(D-\mathfrak{m}) = 0\}$. By Lemma 3.3, we know V_0 is non-empty and open in $(X-S)^{(\pi)}$. Note that for every $D \in V_0$, there is one and only one effective divisor in $X_\mathfrak{m}$ that is $\mathfrak{m}$-equivalent to D. Hence the restriction $\theta|_{V_0}$ of $\theta\colon (X-S)^{(\pi)} \to J_\mathfrak{m}$ to V_0 is injective. By [EGA] III, §4.4.9, $\theta|_{V_0}$ is an open immersion.

Consider the Cartesian square

$$\begin{array}{ccc} X_\mathfrak{m} \times T & \xrightarrow{\ p\ } & X_\mathfrak{m} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \operatorname{spec}(k)\,. \end{array}$$

Let $\mathcal{L}' = \mathcal{L} \otimes p^*\mathcal{L}(\pi P_0)$, where $\mathcal{L}(\pi P_0)$ is the invertible sheaf on $X_\mathfrak{m}$ corresponding to the divisor πP_0. Let us prove the theorem under the extra assumption that for every $t \in T$, we have $\dim H^0(X_\mathfrak{m}, \mathcal{L}'_t) = 1$ and $\dim H^0(X, \mathcal{L}'_t \otimes \mathcal{L}(-\mathfrak{m})) = 0$, where $\mathcal{L}(-\mathfrak{m})$ is the invertible sheaf on X corresponding to the divisor $-\mathfrak{m}$. By the Riemann-Roch theorem, for every $t \in T$, we have $\dim H^1(X_\mathfrak{m}, \mathcal{L}'_t) = 0$. By Theorem 1.1 (d) the sheaf $q_*\mathcal{L}'$ is invertible. The canonical map $q^*q_*\mathcal{L}' \to \mathcal{L}'$ induces

$$s\colon \mathcal{O}_{X_{\mathfrak{m}}\times T} \longrightarrow \mathcal{L}' \otimes (q^*q_*\mathcal{L}')^{-1}.$$

Using Remark 2.1, one can show that the pair $(\mathcal{L}' \otimes (q^*q_*\mathcal{L}')^{-1}, s)$ defines a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times T)/T$. By Proposition 3.1, there exists a unique morphism of schemes $g\colon T \longrightarrow (X-S)^{(\pi)}$ such that the pull-back by $\mathrm{id}\times g$ of the universal relative effective Cartier divisor $\mathcal{D}$ is the divisor defined by $(\mathcal{L}' \otimes (q^*q_*\mathcal{L}')^{-1}, s)$. Let $f = \theta g$. Then the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id}\times f$ is $\mathcal{L}$. This proves the existence of f. To prove f is unique, assume $f\colon T \longrightarrow J_{\mathfrak{m}}$ is a morphism such that the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id}\times f$ is $\mathcal{L}$. By our extra assumption, we must have $\mathrm{Im}(f) \subset \theta(V_0)$. But $\theta|_{V_0}$ is an open immersion. So there exists a morphism $g\colon T \longrightarrow (X-S)^{(\pi)}$ such that $f = \theta g$. We leave it to the reader to prove that the pull-back of the universal relative effective Cartier divisor $\mathcal{D}$ by $\mathrm{id}\times g$ is the divisor defined by the pair $(\mathcal{L}' \otimes (q^*q_*\mathcal{L}')^{-1}, s)$. By Proposition 3.1, such kind of g is unique. So f is also unique.

Now let us prove the theorem. Let t_0 be a point in T. For every point $D \in (X-S)^{(\pi)}$, denote by $\mathcal{L}(D)$ the invertible sheaf on X or on $X_{\mathfrak{m}}$ corresponding to the divisor D. By Lemma 3.3, the set

$$\{D \in (X-S)^{(\pi)} \mid \dim H^0(X_{\mathfrak{m}}, \mathcal{L}_{t_0}\otimes\mathcal{L}(D)) = 1,\ \dim H^0(X, \mathcal{L}_{t_0}\otimes\mathcal{L}(D-\mathfrak{m})) = 0\}$$

is non-empty (and open). Fix an element D in this set. Consider the set

$$U_{t_0} = \{t \in T \mid \dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t \otimes \mathcal{L}(D)) = 1,\ \dim H^0(X, \mathcal{L}_t \otimes \mathcal{L}(D-\mathfrak{m})) = 0\}.$$

This set is open by the Riemann-Roch theorem and Theorem 1.1 (b). Obviously it contains t_0. So U_{t_0} is an open neighbourhood of t_0. By the theorem with the extra assumption that we have already proved, there exists a unique morphism $f'_{U_{t_0}}\colon U_{t_0} \longrightarrow J_{\mathfrak{m}}$ such that the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id}\times f'_{U_{t_0}}$ is $(\mathcal{L}\otimes p^*\mathcal{L}(D-\pi P_0))|_{X_{\mathfrak{m}}\times U_{t_0}}$. Put $f_{U_{t_0}} = f'_{U_{t_0}} + a$, where a is the point in $J_{\mathfrak{m}}$ corresponding to the divisor class $\pi P_0 - D$ in $C^0_{\mathfrak{m}}$. Obviously the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by the morphism $\mathrm{id}\times f_{U_{t_0}}$ is $\mathcal{L}\,|_{X_{\mathfrak{m}}\times U_{t_0}}$. Moreover, such an $f_{U_{t_0}}$ is unique. So we can glue $f_{U_{t_0}}$ together to get $f\colon T \longrightarrow J_{\mathfrak{m}}$.