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THEOREM 1. *There is a morphism of algebraic varieties $\theta: X - S \rightarrow J_{\mathfrak{m}}$ satisfying the following properties:*

- (a) *The extension of θ to the group of divisors on X prime to S induces, by passing to quotient, an isomorphism between the group $C_{\mathfrak{m}}^0$ of classes of divisors of degree zero with respect to \mathfrak{m} -equivalence and the group $J_{\mathfrak{m}}$.*
- (b) *The extension of θ to $(X - S)^{(\pi)}$ induces a birational map from $X^{(\pi)}$ to $J_{\mathfrak{m}}$.*

The following theorem characterizes $J_{\mathfrak{m}}$ by a universal property:

THEOREM 2. *Let $f: X \rightarrow G$ be a rational map from X to a commutative algebraic group G and assume \mathfrak{m} is a modulus for f . Then there is a unique homomorphism $F: J_{\mathfrak{m}} \rightarrow G$ of algebraic groups such that $f = F \circ \theta + f(P_0)$.*

Proof. Replacing f by $f - f(P_0)$, we may assume $f(P_0) = 0$. Since \mathfrak{m} is a modulus for f , the extension of f to the group of divisors of X prime to S induces a homomorphism $C_{\mathfrak{m}}^0 \rightarrow G$ by passing to quotient. By Theorem 1 (a) we have $J_{\mathfrak{m}} \cong C_{\mathfrak{m}}^0$ as groups. So we have a homomorphism of groups $F: J_{\mathfrak{m}} \rightarrow G$ such that $f = F\theta$. It remains to prove F is a morphism of algebraic varieties. By Theorem 1 (b) we have a birational map $\theta: (X - S)^{(\pi)} \rightarrow J_{\mathfrak{m}}$. Denote the extension of f to $(X - S)^{(\pi)}$ by f' . Then $F\theta = f'$. Since θ is birational, it induces an isomorphism between an open subvariety of $(X - S)^{(\pi)}$ and an open subvariety of $J_{\mathfrak{m}}$. Moreover f' is a morphism of algebraic varieties. Hence F is a morphism of algebraic varieties when restricted to some open subset of $J_{\mathfrak{m}}$. The whole $J_{\mathfrak{m}}$ can be obtained from this open subset by translation. So F is a morphism of algebraic varieties.

6. GENERALIZED JACOBIANS AND PICARD SCHEMES

In this section we prove $J_{\mathfrak{m}}$ is the Picard scheme of $X_{\mathfrak{m}}$.

Let T be a k -scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k) . \end{array}$$

We have $q_* \mathcal{O}_{X_{\mathfrak{m}} \times T} = \mathcal{O}_T$ by [EGA] III, § 1.4.15, the fact $H^0(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}}) = k$, and the fact that $T \rightarrow \text{spec}(k)$ is flat. The morphism q has a section $s: T \rightarrow X_{\mathfrak{m}} \times T$, $t \mapsto (P_0, t)$.

LEMMA 6.1. *Let \mathcal{L}_1 and \mathcal{L}_2 be two invertible sheaves on $X_m \times T$. Assume $\mathcal{L}_1 \cong \mathcal{L}_2$. Then the canonical map $\text{Hom}(\mathcal{L}_1, \mathcal{L}_2) \rightarrow \text{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_2)$ induced by s is bijective.*

Proof. Since $\mathcal{L}_1 \cong \mathcal{L}_2$, it is enough to show that the canonical map $\text{Hom}(\mathcal{L}_1, \mathcal{L}_1) \rightarrow \text{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$ is bijective. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X_m \times T}(X_m \times T) & \longrightarrow & \mathcal{O}_T(T) \\ \downarrow & & \downarrow \\ \text{Hom}(\mathcal{L}_1, \mathcal{L}_1) & \longrightarrow & \text{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1), \end{array}$$

where the horizontal arrows are induced by s . We have

$$\begin{aligned} \text{Hom}(\mathcal{L}_1, \mathcal{L}_1) &\cong \text{Hom}(\mathcal{O}_{X_m \times T}, \mathcal{L}_1 \otimes \mathcal{L}_1^{-1}) \\ &\cong \text{Hom}(\mathcal{O}_{X_m \times T}, \mathcal{O}_{X_m \times T}) \cong \mathcal{O}_{X_m \times T}(X_m \times T). \end{aligned}$$

Hence the left vertical arrow in the above diagram is bijective. Similarly the right vertical arrow is also bijective. Since $q_*\mathcal{O}_{X_m \times T} = \mathcal{O}_T$, we have $\mathcal{O}_{X_m \times T}(X_m \times T) \cong \mathcal{O}(T)$, and the upper horizontal arrow is bijective. Hence $\text{Hom}(\mathcal{L}_1, \mathcal{L}_1) \cong \text{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$ by the commutativity of the above diagram.

LEMMA 6.2. *Let $\{U_i\}$ be an open covering of T and let \mathcal{L}_i be invertible sheaves on $X_m \times U_i$. Assume $s^*\mathcal{L}_i \cong \mathcal{O}_{U_i}$ and $\mathcal{L}_i|_{X_m \times (U_i \cap U_j)} \cong \mathcal{L}_j|_{X_m \times (U_i \cap U_j)}$. Then there exists an invertible sheaf \mathcal{L} on $X_m \times T$ such that $\mathcal{L}|_{X_m \times U_i} \cong \mathcal{L}_i$ and $s^*\mathcal{L} \cong \mathcal{O}_T$. Moreover \mathcal{L} is unique up to isomorphism.*

Proof. Fix an isomorphism $\alpha_i: s^*\mathcal{L}_i \rightarrow \mathcal{O}_{U_i}$ for each i . Let

$$\alpha_{ij}: s^*\mathcal{L}_i|_{U_i \cap U_j} \rightarrow s^*\mathcal{L}_j|_{U_i \cap U_j}$$

be the isomorphism $(\alpha_j|_{U_i \cap U_j})^{-1} \circ (\alpha_i|_{U_i \cap U_j})$. By Lemma 6.1 the canonical map

$$\text{Hom}(\mathcal{L}_i|_{X_m \times (U_i \cap U_j)}, \mathcal{L}_j|_{X_m \times (U_i \cap U_j)}) \rightarrow \text{Hom}(s^*\mathcal{L}_i|_{U_i \cap U_j}, s^*\mathcal{L}_j|_{U_i \cap U_j})$$

is bijective. So α_{ij} can be lifted uniquely to an isomorphism

$$A_{ij}: \mathcal{L}_i|_{X_m \times (U_i \cap U_j)} \rightarrow \mathcal{L}_j|_{X_m \times (U_i \cap U_j)}.$$

By the uniqueness of the lifting and the fact that $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$ on $U_i \cap U_j \cap U_k$, we have $A_{jk}A_{ij} = A_{ik}$ on $X_m \times (U_i \cap U_j \cap U_k)$. So A_{ij} defines glueing data and we can glue the \mathcal{L}_i together to get an invertible sheaf \mathcal{L} on $X_m \times T$. By the construction of \mathcal{L} we have $s^*\mathcal{L} \cong \mathcal{O}_T$. This proves the existence of \mathcal{L} . Similarly using Lemma 6.1 one can prove \mathcal{L} is unique up to isomorphism.

LEMMA 6.3. Assume T is integral. Let \mathcal{L}_1 and \mathcal{L}_2 be two invertible sheaves on $X_m \times T$ satisfying $\mathcal{L}_{1,t} \cong \mathcal{L}_{2,t}$ for all $t \in T$. Then there is an invertible sheaf \mathcal{M} on T such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes q^* \mathcal{M}$.

Proof. Let $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$. Then $\mathcal{L}_t \cong \mathcal{O}_{X_m}$. It suffices to show that $\mathcal{L} \cong q^* \mathcal{M}$ for some invertible sheaf \mathcal{M} on T . We have $H^0(X_m, \mathcal{L}_t) = H^0(X_m, \mathcal{O}_{X_m}) = k$. By Theorem 1.1(c), the sheaf $q_* \mathcal{L}$ is invertible and $q_* \mathcal{L} \otimes k(t) = H^0(X_m, \mathcal{L}_t)$. So the restriction $(q^* q_* \mathcal{L})_t \rightarrow \mathcal{L}_t$ of the canonical map $q^* q_* \mathcal{L} \rightarrow \mathcal{L}$ to the fiber of q at $t \in T$ is $H^0(X_m, \mathcal{L}_t) \otimes \mathcal{O}_{X_m} \rightarrow \mathcal{L}_t$, which is an isomorphism since $\mathcal{L}_t \cong \mathcal{O}_{X_m}$. By Nakayama's Lemma, the canonical map $q^* q_* \mathcal{L} \rightarrow \mathcal{L}$ is surjective. But since it is a homomorphism of invertible sheaves, it must be bijective. Hence $\mathcal{L} \cong q^* q_* \mathcal{L}$.

Now we use the above lemmas to construct a canonical invertible sheaf on $X_m \times J_m$.

On $X_m \times (X - S)^{(\pi)}$ we have the invertible sheaf corresponding to the divisor $\mathcal{D} - p^*(\pi P_0)$, where \mathcal{D} is the universal relative effective Cartier divisor and $p: X_m \times (X - S)^{(\pi)} \rightarrow X_m$ is the projection. Since $\theta: (X - S)^{(\pi)} \rightarrow J_m$ is birational, there exist open subsets U in $(X - S)^{(\pi)}$ and V in J_m such that θ induces an isomorphism $U \cong V$. Hence we can push-forward the above invertible sheaf on $X_m \times (X - S)^{(\pi)}$ to get an invertible sheaf \mathcal{L}_V on $X_m \times V$. For each $t \in J_m$, denote by $\mathcal{L}(t)$ the invertible sheaf on X_m corresponding to the divisor class in C_m^0 that is mapped to $t \in J_m$ under the canonical isomorphism $C_m^0 \cong J_m$. Obviously the restriction $\mathcal{L}_{V,t}$ of \mathcal{L}_V to the fiber of the projection $q: X_m \times J_m \rightarrow J_m$ at $t \in V$ is isomorphic to $\mathcal{L}(t)$. The invertible sheaf $\mathcal{L}_V \otimes (q^* s^* \mathcal{L}_V)^{-1}$ has the same property, where $s: J_m \rightarrow X_m \times J_m$ is the section $t \mapsto (P_0, t)$. Thus replacing \mathcal{L}_V by $\mathcal{L}_V \otimes (q^* s^* \mathcal{L}_V)^{-1}$ if necessary, we may assume that $s^* \mathcal{L}_V \cong \mathcal{O}_V$.

For each $a \in J_m$, let $T_{-a}: J_m \rightarrow J_m$ be the translation $t \mapsto t - a$. Consider the invertible sheaf $\mathcal{L}_{a+V} = (\text{id} \times T_{-a})^* \mathcal{L}_V \otimes p^* \mathcal{L}(a)$ on $X_m \otimes (a + V)$, where $p: X_m \times J_m \rightarrow X_m$ is the projection. The restriction $\mathcal{L}_{a+V,a+t}$ of \mathcal{L}_{a+V} to the fiber of q at $a + t \in a + V$ is

$$((\text{id} \times T_{-a})^* \mathcal{L}_V \otimes p^* \mathcal{L}(a))_{a+t} = \mathcal{L}_{V,t} \otimes \mathcal{L}(a) = \mathcal{L}(t) \otimes \mathcal{L}(a) = \mathcal{L}(a + t),$$

that is, $\mathcal{L}_{a+V,a+t} = \mathcal{L}(a + t)$. Hence for any $t \in V \cap (a + V)$, we have $\mathcal{L}_{V,t} = \mathcal{L}_{a+V,t}$. By Lemma 6.3, we have

$$\mathcal{L}_V|_{X_m \times (V \cap (a + V))} \cong \mathcal{L}_{a+V}|_{X_m \times (V \cap (a + V))} \otimes q^* \mathcal{M}$$

for some invertible sheaf \mathcal{M} on $V \cap (a + V)$. But since $s^* \mathcal{L}_V \cong \mathcal{O}_V$, we also have $s^* \mathcal{L}_{a+V} = \mathcal{O}_{a+V}$. Hence $\mathcal{M} \cong \mathcal{O}_{V \cap (a + V)}$. Therefore $\mathcal{L}_V|_{X_m \times (V \cap (a + V))} \cong$

$\mathcal{L}_{a+V}|_{X_m \times (V \cap (a+V))}$. By Lemma 6.2, we can glue \mathcal{L}_{a+V} ($a \in J_m$) together to get an invertible sheaf \mathcal{L}_{J_m} on $X_m \times J_m$. It has the property that its restriction to the fiber of q at $t \in J_m$ is isomorphic to $\mathcal{L}(t)$ and $s^* \mathcal{L}_{J_m} \cong \mathcal{O}_{J_m}$.

Define

$$P^0(T) = \{\mathcal{L} \in \text{Pic}(X_m \times T) \mid \deg(\mathcal{L}) = 0\} / q^* \text{Pic}(T),$$

where $\deg(\mathcal{L})$ is defined as the leading coefficient of $\chi(\mathcal{L}_t^{\otimes n})$ as a polynomial in n . Since $s^* q^* = \text{id}$, we may define

$$P^0(T) = \{\mathcal{L} \in \text{Pic}(X_m \times T) \mid \deg(\mathcal{L}) = 0 \text{ and } s^* \mathcal{L} \cong \mathcal{O}_T\}$$

as well. In particular, we have $\mathcal{L}_{J_m} \in P^0(J_m)$. Using the first definition of $P^0(T)$ and Lemma 6.3, one can show that the pull-back of \mathcal{L}_{J_m} by $\text{id} \times \theta: X_m \times (X - S)^{(\pi)} \rightarrow X_m \times J_m$ is the invertible sheaf on $X_m \times (X - S)^{(\pi)}$ corresponding to the divisor $\mathcal{D} - p^*(\pi P_0)$.

The following theorem says that J_m is the Picard scheme of X_m .

THEOREM 3. *The functor $T \rightarrow P^0(T)$ is represented by J_m . More precisely, for any invertible sheaf \mathcal{L} on $X_m \times T$ of degree 0 satisfying $s^* \mathcal{L} \cong \mathcal{O}_T$, there is one and only one morphism of schemes $f: T \rightarrow J_m$ such that \mathcal{L} is the pull-back of \mathcal{L}_{J_m} by $\text{id} \times f: X_m \times T \rightarrow X_m \times J_m$.*

Proof. Let $V_0 = \{D \in (X - S)^{(\pi)} \mid l_m(D) = 1, l(D - m) = 0\}$. By Lemma 3.3, we know V_0 is non-empty and open in $(X - S)^{(\pi)}$. Note that for every $D \in V_0$, there is one and only one effective divisor in X_m that is m -equivalent to D . Hence the restriction $\theta|_{V_0}$ of $\theta: (X - S)^{(\pi)} \rightarrow J_m$ to V_0 is injective. By [EGA] III, §4.4.9, $\theta|_{V_0}$ is an open immersion.

Consider the Cartesian square

$$\begin{array}{ccc} X_m \times T & \xrightarrow{p} & X_m \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k). \end{array}$$

Let $\mathcal{L}' = \mathcal{L} \otimes p^* \mathcal{L}(\pi P_0)$, where $\mathcal{L}(\pi P_0)$ is the invertible sheaf on X_m corresponding to the divisor πP_0 . Let us prove the theorem under the extra assumption that for every $t \in T$, we have $\dim H^0(X_m, \mathcal{L}'_t) = 1$ and $\dim H^0(X, \mathcal{L}'_t \otimes \mathcal{L}(-m)) = 0$, where $\mathcal{L}(-m)$ is the invertible sheaf on X corresponding to the divisor $-m$. By the Riemann-Roch theorem, for every $t \in T$, we have $\dim H^1(X_m, \mathcal{L}'_t) = 0$. By Theorem 1.1 (d) the sheaf $q_* \mathcal{L}'$ is invertible. The canonical map $q^* q_* \mathcal{L}' \rightarrow \mathcal{L}'$ induces

$$s: \mathcal{O}_{X_m \times T} \rightarrow \mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}.$$

Using Remark 2.1, one can show that the pair $(\mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}, s)$ defines a relative effective Cartier divisor on $(X_m \times T)/T$. By Proposition 3.1, there exists a unique morphism of schemes $g: T \rightarrow (X - S)^{(\pi)}$ such that the pull-back by $\text{id} \times g$ of the universal relative effective Cartier divisor \mathcal{D} is the divisor defined by $(\mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}, s)$. Let $f = \theta g$. Then the pull-back of \mathcal{L}_{J_m} by $\text{id} \times f$ is \mathcal{L} . This proves the existence of f . To prove f is unique, assume $f: T \rightarrow J_m$ is a morphism such that the pull-back of \mathcal{L}_{J_m} by $\text{id} \times f$ is \mathcal{L} . By our extra assumption, we must have $\text{Im}(f) \subset \theta(V_0)$. But $\theta|_{V_0}$ is an open immersion. So there exists a morphism $g: T \rightarrow (X - S)^{(\pi)}$ such that $f = \theta g$. We leave it to the reader to prove that the pull-back of the universal relative effective Cartier divisor \mathcal{D} by $\text{id} \times g$ is the divisor defined by the pair $(\mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}, s)$. By Proposition 3.1, such kind of g is unique. So f is also unique.

Now let us prove the theorem. Let t_0 be a point in T . For every point $D \in (X - S)^{(\pi)}$, denote by $\mathcal{L}(D)$ the invertible sheaf on X or on X_m corresponding to the divisor D . By Lemma 3.3, the set

$$\{D \in (X - S)^{(\pi)} \mid \dim H^0(X_m, \mathcal{L}_{t_0} \otimes \mathcal{L}(D)) = 1, \dim H^0(X, \mathcal{L}_{t_0} \otimes \mathcal{L}(D - m)) = 0\}$$

is non-empty (and open). Fix an element D in this set. Consider the set

$$U_{t_0} = \{t \in T \mid \dim H^0(X_m, \mathcal{L}_t \otimes \mathcal{L}(D)) = 1, \dim H^0(X, \mathcal{L}_t \otimes \mathcal{L}(D - m)) = 0\}.$$

This set is open by the Riemann-Roch theorem and Theorem 1.1 (b). Obviously it contains t_0 . So U_{t_0} is an open neighbourhood of t_0 . By the theorem with the extra assumption that we have already proved, there exists a unique morphism $f'_{U_{t_0}}: U_{t_0} \rightarrow J_m$ such that the pull-back of \mathcal{L}_{J_m} by $\text{id} \times f'_{U_{t_0}}$ is $(\mathcal{L} \otimes p^* \mathcal{L}(D - \pi P_0))|_{X_m \times U_{t_0}}$. Put $f_{U_{t_0}} = f'_{U_{t_0}} + a$, where a is the point in J_m corresponding to the divisor class $\pi P_0 - D$ in C_m^0 . Obviously the pull-back of \mathcal{L}_{J_m} by the morphism $\text{id} \times f_{U_{t_0}}$ is $\mathcal{L}|_{X_m \times U_{t_0}}$. Moreover, such an $f_{U_{t_0}}$ is unique. So we can glue $f_{U_{t_0}}$ together to get $f: T \rightarrow J_m$.