

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PRODUCT MEASURABILITY, PARAMETER INTEGRALS, AND A FUBINI COUNTEREXAMPLE
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Kapitel: 3. Measurability
DOI: <https://doi.org/10.5169/seals-64449>

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2.2 A FUBINI COUNTEREXAMPLE

In this section, we give an example of (2). Let \mathcal{A} be as in (5) and define $\mu|_{\mathcal{A}}$ by

$$(11) \quad \mu(A) := \begin{cases} 0 & \text{if } A \text{ is meager,} \\ 1 & \text{if } A \text{ is comeager.} \end{cases}$$

This is possible, since no set $A \subset \mathbf{R}$ is simultaneously meager and comeager, for otherwise $\emptyset = A \cap A^c$ would be comeager, in contradiction to Baire's theorem. It is easy to check that μ is a probability measure on $(\mathbf{R}, \mathcal{A})$. Let again $\nu := \lambda :=$ Lebesgue measure on $\mathcal{B} := \mathcal{B}(\mathbf{R})$, and choose $A \in \mathcal{A}$ meager with $\lambda(A^c) = 0$. Then $1_A(\cdot + y)$ is \mathcal{A} -measurable with

$$\int_{\mathbf{R}} 1_A(x + y) d\mu(x) = \mu(A - y) = 0 \quad (y \in \mathbf{R}).$$

On the other hand, we have

$$\int_{\mathbf{R}} 1_A(x + y) d\nu(y) = \lambda(A - x) = \infty \quad (x \in \mathbf{R}).$$

Hence (2) is obviously true in this case.

3. MEASURABILITY

Here is a positive result, having a certain measurability property of F from (1) among its conclusions. An application of this occurs in Mattner (1999).

3.1. THEOREM. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be σ -finite measure spaces, let $f: \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty]$ be a function measurable with respect to the product σ -algebra $\mathcal{A} \otimes \mathcal{B}$, and put*

$$\mathcal{A}_0 := \sigma(\{f(\cdot, y) : y \in \mathcal{Y}\}),$$

$$\mathcal{B}_0 := \sigma(\{f(x, \cdot) : x \in \mathcal{X}\}),$$

$$\overline{\mathcal{A}}_0 := \{A \in \mathcal{A} : \exists A_0 \in \mathcal{A}_0 \text{ with } A = A_0 \quad [\mu]\},$$

$$\overline{\mathcal{B}}_0 := \{B \in \mathcal{B} : \exists B_0 \in \mathcal{B}_0 \text{ with } B = B_0 \quad [\nu]\},$$

$$\overline{\mathcal{A}_0 \otimes \mathcal{B}_0} := \{C \in \mathcal{A} \otimes \mathcal{B} : \exists C_0 \in \mathcal{A}_0 \otimes \mathcal{B}_0 \text{ with } C = C_0 \quad [\mu \otimes \nu]\}.$$

Then f is $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$ -measurable, $\int_{\mathcal{Y}} f(\cdot, y) d\nu(y)$ is $\overline{\mathcal{A}}_0$ -measurable, and $\int_{\mathcal{X}} f(x, \cdot) d\mu(x)$ is $\overline{\mathcal{B}}_0$ -measurable.

Here and in what follows, we write $A = A_0 \ [\mu]$ for $\mu(A \Delta A_0) = 0$. Below we also use the corresponding notation $f = g \ [\mu]$ for functions, meaning $\mu(\{x : f(x) \neq g(x)\}) = 0$.

3.2 REMARKS

Let us retain the notation and assumptions of Theorem 3.1.

- a) The parameter integral $\int_{\mathcal{Y}} f(\cdot, y) d\nu(y)$ need not be \mathcal{A}_0 -measurable and f need not be $\mathcal{A}_0 \otimes \mathcal{B}_0$ -measurable, as the example in 2.1 shows.
- b) The function f need not be $\bar{\mathcal{A}}_0 \otimes \bar{\mathcal{B}}_0$ -measurable. As an example proving this remark, we may take $(\mathcal{X}, \mathcal{A}, \mu) := (\mathcal{Y}, \mathcal{B}, \nu) := ([0, 1], \mathcal{B}([0, 1]), \lambda^1)$, $D := \{(x, x) : x \in [0, 1]\}$, and $f := 1_D$. [We now write λ^d for d -dimensional Lebesgue measure.] Then

$$\mathcal{A}_0 = \mathcal{B}_0 = \{A \in \mathcal{B}([0, 1]) : A \text{ countable or cocountable}\},$$

$$\bar{\mathcal{A}}_0 = \bar{\mathcal{B}}_0 = \left\{A \in \mathcal{B}([0, 1]) : \lambda^1(A) \in \{0, 1\}\right\},$$

and we claim that f is not $\bar{\mathcal{A}}_0 \otimes \bar{\mathcal{B}}_0$ -measurable. To prove this, put

$$\mathcal{C} := \left\{C \in \mathcal{B}([0, 1]^2) : \left(\lambda^2(C), \int_0^1 1_C(x, x) d\lambda^1(x)\right) \in \{(0, 0), (1, 1)\}\right\}.$$

Then \mathcal{C} is a σ -algebra containing $\{A \times B : A \in \bar{\mathcal{A}}_0, B \in \bar{\mathcal{B}}_0\}$, and hence satisfies $\bar{\mathcal{A}}_0 \otimes \bar{\mathcal{B}}_0 \subset \mathcal{C}$. But $D \notin \mathcal{C}$, so that $D \notin \bar{\mathcal{A}}_0 \otimes \bar{\mathcal{B}}_0$.

- c) Let us write more explicitly $\bar{\mathcal{A}}_0(\mu)$ in place of $\bar{\mathcal{A}}_0$. From Theorem 3.1, we may deduce the measurability of $F := \int f(\cdot, y) d\nu(y)$ with respect to $\bigcap_{\mu} \bar{\mathcal{A}}_0(\mu)$, the intersection being over all \mathcal{A} and μ as in the theorem. This, however, must not be confused with the more restrictive property of universal \mathcal{A}_0 -measurability of F [see Cohn (1980), pages 280–283, for the definition and for illuminating facts]. Indeed, our measures μ are supposed to be defined on some \mathcal{A} rendering f $\mathcal{A} \otimes \mathcal{B}$ -measurable, and not merely on \mathcal{A}_0 or its μ -completion. For example, in the situation of 2.1, one can use the measure μ from (11) to deduce that the σ -algebra of all universally \mathcal{A}_0 -measurable sets is contained in $\widetilde{\mathcal{A}}_0 := \{A \subset \mathbf{R} : A \text{ meager or comeager}\}$. Since $\widetilde{\mathcal{A}}_0$ differs from \mathcal{A}_0 only by non-Borel sets, we see that F from (3), (7), (8) is not universally \mathcal{A}_0 -measurable. By the way, the known fact that μ from (11) can not be extended to a measure on $\mathcal{B}(\mathbf{R})$ [see Oxtoby (1980), page 86] follows from our present considerations, since otherwise we would have $\bar{\mathcal{A}}_0(\mu) = \widetilde{\mathcal{A}}_0 \cap \mathcal{B}(\mathbf{R}) = \mathcal{A}_0$, and Theorem 3.1 would yield \mathcal{A}_0 -measurability of F .

3.3 PROOF OF THEOREM 3.1

Obvious arguments show that we may assume in addition that

$$(12) \quad \mu, \nu \text{ are finite and } f \text{ is bounded.}$$

The proof of the theorem splits into two parts as follows.

CLAIM 1. *Under the assumptions of the theorem and (12),*

$$(13) \quad F := \int f(\cdot, y) d\nu(y)$$

is $\bar{\mathcal{A}}_0$ -measurable.

Proof. Let us first recall the “mean value theorem” for vector valued integration: Let E be a topological vector space, $(\Omega, \mathcal{A}, \mu)$ be a measure space, and $g: \Omega \rightarrow E$ be a function. Then an $x \in E$ is called the *weak* (or *Pettis*) integral of g , and we write $\int g d\mu := x$, if

(i) the dual space E' of E separates points on E ,

(ii) the scalar function $\langle y, g(\cdot) \rangle$ belongs to $\mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ for every $y \in E'$, and

(iii) $\int \langle y, g(\omega) \rangle d\mu(\omega) = \langle y, x \rangle$ for every $y \in E'$.

[This is the definition adopted by Edwards (1965), p.566, and by Rudin (1991), p.77.] If now E is in particular locally convex Hausdorff and μ is bounded, then the weak integral, if it exists, necessarily satisfies

$$(14) \quad \int g d\mu \in \mu(\Omega) \cdot \overline{\text{conv}} g(\Omega),$$

with $\overline{\text{conv}}$ indicating convex closure. This “mean value theorem” is surely well known. It follows easily from the Hahn-Banach theorem: Apply Theorem 3.4 (b) of Rudin (1991) to $A := \left\{ \int g d\mu \right\}$ and $B := \mu(\Omega) \cdot \overline{\text{conv}} g(\Omega)$.

We now start with the proof proper. The functions $f(\cdot, y): \mathcal{X} \rightarrow \mathbf{R}$, as well as F from (13), are \mathcal{A} -measurable [by $\mathcal{A} \otimes \mathcal{B}$ -measurability of f and by Fubini] and bounded, and hence belong to $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Let $[f(\cdot, y)], [F] \in L^1(\mathcal{X}, \mathcal{A}, \mu)$ denote their corresponding equivalence classes. We claim that

$$(15) \quad [F] = \int_{\mathcal{Y}} [f(\cdot, y)] d\nu(y),$$

in the weak sense recalled above, applied to the Banach space $E = L^1(\mathcal{X}, \mathcal{A}, \mu)$ with dual space $L^\infty(\mathcal{X}, \mathcal{A}, \mu)$. To prove this, let $h \in [h] \in L^\infty(\mathcal{X}, \mathcal{A}, \mu)$. An obvious Fubini calculation, using the definition of F and the $\mathcal{A} \otimes \mathcal{B}$ -measurability of f , yields

$$\langle [h], [F] \rangle = \int_{\mathcal{X}} h(x)F(x) d\mu(x) = \int_{\mathcal{Y}} \langle [h], [f(\cdot, y)] \rangle d\nu(y),$$

which confirms (15). [Actually, (15) is even true with the right hand side read as a Bochner integral, but we do not need this fact here.] We now use that each $f(\cdot, y)$ is \mathcal{A}_0 -measurable, where of course $\mathcal{A}_0 \subset \mathcal{A}$. This implies that the function $y \mapsto [f(\cdot, y)]$ takes its values in

$$S := \{ \Phi \in L^1(\mathcal{X}, \mathcal{A}, \mu) : \exists \mathcal{A}_0\text{-measurable } \varphi \in \Phi \} ,$$

which is easily seen to be a closed subspace of $L^1(\mathcal{X}, \mathcal{A}, \mu)$. The mean value theorem (14) now yields $[F] \in S$, which is the desired conclusion. \square

CLAIM 2. *Under the assumptions of the theorem and (12), and assuming the truth of Claim 1, f is $\overline{\mathcal{A}_0} \otimes \overline{\mathcal{B}_0}$ -measurable.*

Proof. We consider the restrictions

$$\overline{\mu}_0 := \mu|_{\overline{\mathcal{A}_0}}, \quad \overline{\nu}_0 = \nu|_{\overline{\mathcal{A}_0}},$$

and define a function $\tau: \overline{\mathcal{A}_0} \otimes \overline{\mathcal{B}_0} \rightarrow [0, \infty]$ by

$$(16) \quad \tau(C) := \int_{\mathcal{X}} \int_{\mathcal{Y}} f(x, y) 1_C(x, y) d\overline{\nu}_0(y) d\overline{\mu}_0(x) \quad (C \in \overline{\mathcal{A}_0} \otimes \overline{\mathcal{B}_0}),$$

and we emphasize that the right hand side has to be read as an iterated integral. In order to show its existence, we have to check that the function $x \mapsto \int_{\mathcal{Y}} f(x, y) 1_C(x, y) d\overline{\nu}_0(y)$ is $\overline{\mathcal{A}_0}$ -measurable. For the special case $C = A \times B$ with $A \in \overline{\mathcal{A}_0}$ and $B \in \overline{\mathcal{B}_0}$, this follows from Claim 1, applied to $\overline{\mathcal{A}_0}$ in place of \mathcal{A}_0 and $f(x, y) 1_B(y)$ in place of $f(x, y)$, and using $\overline{\overline{\mathcal{A}_0}} = \overline{\mathcal{A}_0}$. The general case follows as usual via Sierpiński's lemma [Satz I.6.8 in Elstrodt (1996)]. Thus τ is well-defined. It is easily checked that τ is a measure, and that every set of $\overline{\mu}_0 \otimes \overline{\nu}_0$ -measure zero is of τ -measure zero as well. Hence the Lebesgue-Radon-Nikodym theorem yields the existence of an $\overline{\mathcal{A}_0} \otimes \overline{\mathcal{B}_0}$ -measurable function $\tilde{f}: \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty]$ such that

$$\tau(C) = \int_C \tilde{f} d\overline{\mu}_0 \otimes \overline{\nu}_0 \quad (C \in \overline{\mathcal{A}_0} \otimes \overline{\mathcal{B}_0}).$$

By (16) and Fubini, this implies in particular

$$(17) \quad \int_{A_0} \int_{B_0} f(x, y) d\overline{\nu}_0(y) d\overline{\mu}_0(x) = \int_{A_0} \int_{B_0} \tilde{f}(x, y) d\overline{\nu}_0(y) d\overline{\mu}_0(x)$$

($A_0 \in \overline{\mathcal{A}_0}$, $B_0 \in \overline{\mathcal{B}_0}$). Since, using (12), both sides in (17) are always finite, we may conclude for every $B_0 \in \overline{\mathcal{B}_0}$:

$$\int_{B_0} f(\cdot, y) d\bar{\nu}_0(y) = \int_{B_0} \tilde{f}(\cdot, y) d\bar{\nu}_0(y) \quad [\bar{\mu}_0].$$

Trivially, this remains true if $[\bar{\mu}_0]$ is replaced by $[\mu]$, and an integration yields

$$(18) \quad \int_A \int_{B_0} f(x, y) d\bar{\nu}_0(y) d\mu(x) = \int_A \int_{B_0} \tilde{f}(x, y) d\bar{\nu}_0(y) d\mu(x)$$

($A \in \mathcal{A}$, $B_0 \in \bar{\mathcal{B}}_0$). We now want to interchange the order of integrations. Since \tilde{f} is trivially $\mathcal{A} \otimes \bar{\mathcal{B}}_0$ -measurable, we may obviously do this on the right hand side of (18). To do the same on the left hand side, we rewrite it successively as

$$\int_A \int_{B_0} f(x, y) d\nu(y) d\mu(x) = \int_{B_0} \int_A f(x, y) d\mu(x) d\nu(y) = \int_{B_0} \int_A f(x, y) d\mu(x) d\bar{\nu}_0(y),$$

where the last equality follows from a second application of Claim 1, with the role of the variables interchanged. Thus (18) yields

$$(19) \quad \int_{B_0} \int_A f(x, y) d\mu(x) d\bar{\nu}_0(y) = \int_{B_0} \int_A \tilde{f}(x, y) d\mu(x) d\bar{\nu}_0(y)$$

($A \in \mathcal{A}$, $B_0 \in \bar{\mathcal{B}}_0$). Now the argument leading from (17) to (18) can be repeated to lead from (19) to a corresponding statement with B in place of B_0 , ν in place of $\bar{\nu}_0$, and \mathcal{B} in place of $\bar{\mathcal{B}}_0$, which is equivalent to

$$\int_{A \times B} f d\mu \otimes \nu = \int_{A \times B} \tilde{f} d\mu \otimes \nu \quad (A \in \mathcal{A}, B \in \mathcal{B}).$$

This shows that $f = \tilde{f}$ $[\mu \otimes \nu]$, which yields the desired conclusion. \square

ACKNOWLEDGEMENT. The present work was supported by a Heisenberg grant of the Deutsche Forschungsgemeinschaft. I thank H. von Weizsäcker for helpful remarks on an earlier version.

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