

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	45 (1999)
Heft:	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	PRODUCT MEASURABILITY, PARAMETER INTEGRALS, AND A FUBINI COUNTEREXAMPLE
Autor:	Mattner, Lutz
Kapitel:	3. Measurability
DOI:	https://doi.org/10.5169/seals-64449

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 16.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

2.2 A FUBINI COUNTEREXAMPLE

In this section, we give an example of (2). Let \mathcal{A} be as in (5) and define $\mu|_{\mathcal{A}}$ by

$$(11) \quad \mu(A) := \begin{cases} 0 & \text{if } A \text{ is meager,} \\ 1 & \text{if } A \text{ is comeager.} \end{cases}$$

This is possible, since no set $A \subset \mathbf{R}$ is simultaneously meager and comeager, for otherwise $\emptyset = A \cap A^c$ would be comeager, in contradiction to Baire's theorem. It is easy to check that μ is a probability measure on $(\mathbf{R}, \mathcal{A})$. Let again $\nu := \lambda :=$ Lebesgue measure on $\mathcal{B} := \mathcal{B}(\mathbf{R})$, and choose $A \in \mathcal{A}$ meager with $\lambda(A^c) = 0$. Then $1_A(\cdot + y)$ is \mathcal{A} -measurable with

$$\int_{\mathbf{R}} 1_A(x + y) d\mu(x) = \mu(A - y) = 0 \quad (y \in \mathbf{R}).$$

On the other hand, we have

$$\int_{\mathbf{R}} 1_A(x + y) d\nu(y) = \lambda(A - x) = \infty \quad (x \in \mathbf{R}).$$

Hence (2) is obviously true in this case.

3. MEASURABILITY

Here is a positive result, having a certain measurability property of F from (1) among its conclusions. An application of this occurs in Mattner (1999).

3.1. **THEOREM.** *Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be σ -finite measure spaces, let $f: \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty]$ be a function measurable with respect to the product σ -algebra $\mathcal{A} \otimes \mathcal{B}$, and put*

$$\mathcal{A}_0 := \sigma(\{f(\cdot, y) : y \in \mathcal{Y}\}),$$

$$\mathcal{B}_0 := \sigma(\{f(x, \cdot) : x \in \mathcal{X}\}),$$

$$\bar{\mathcal{A}}_0 := \{A \in \mathcal{A} : \exists A_0 \in \mathcal{A}_0 \text{ with } A = A_0 \text{ } [\mu]\},$$

$$\bar{\mathcal{B}}_0 := \{B \in \mathcal{B} : \exists B_0 \in \mathcal{B}_0 \text{ with } B = B_0 \text{ } [\nu]\},$$

$$\overline{\mathcal{A}_0 \otimes \mathcal{B}_0} := \{C \in \mathcal{A} \otimes \mathcal{B} : \exists C_0 \in \mathcal{A}_0 \otimes \mathcal{B}_0 \text{ with } C = C_0 \text{ } [\mu \otimes \nu]\}.$$

Then f is $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$ -measurable, $\int_{\mathcal{Y}} f(\cdot, y) d\nu(y)$ is $\bar{\mathcal{A}}_0$ -measurable, and $\int_{\mathcal{X}} f(x, \cdot) d\mu(x)$ is $\bar{\mathcal{B}}_0$ -measurable.

Here and in what follows, we write $A = A_0$ $[\mu]$ for $\mu(A \Delta A_0) = 0$. Below we also use the corresponding notation $f = g$ $[\mu]$ for functions, meaning $\mu(\{x : f(x) \neq g(x)\}) = 0$.

3.2 REMARKS

Let us retain the notation and assumptions of Theorem 3.1.

- a) The parameter integral $\int_{\mathcal{Y}} f(\cdot, y) d\nu(y)$ need not be \mathcal{A}_0 -measurable and f need not be $\mathcal{A}_0 \otimes \mathcal{B}_0$ -measurable, as the example in 2.1 shows.
- b) The function f need not be $\bar{\mathcal{A}}_0 \otimes \bar{\mathcal{B}}_0$ -measurable. As an example proving this remark, we may take $(\mathcal{X}, \mathcal{A}, \mu) := (\mathcal{Y}, \mathcal{B}, \nu) := ([0, 1], \mathcal{B}([0, 1]), \lambda^1)$, $D := \{(x, x) : x \in [0, 1]\}$, and $f := 1_D$. [We now write λ^d for d -dimensional Lebesgue measure.] Then

$$\mathcal{A}_0 = \mathcal{B}_0 = \{A \in \mathcal{B}([0, 1]) : A \text{ countable or cocountable}\} ,$$

$$\bar{\mathcal{A}}_0 = \bar{\mathcal{B}}_0 = \left\{A \in \mathcal{B}([0, 1]) : \lambda^1(A) \in \{0, 1\}\right\} ,$$

and we claim that f is not $\bar{\mathcal{A}}_0 \otimes \bar{\mathcal{B}}_0$ -measurable. To prove this, put

$$\mathcal{C} := \left\{C \in \mathcal{B}([0, 1]^2) : \left(\lambda^2(C), \int_0^1 1_C(x, x) d\lambda^1(x)\right) \in \{(0, 0), (1, 1)\}\right\} .$$

Then \mathcal{C} is a σ -algebra containing $\{A \times B : A \in \bar{\mathcal{A}}_0, B \in \bar{\mathcal{B}}_0\}$, and hence satisfies $\bar{\mathcal{A}}_0 \otimes \bar{\mathcal{B}}_0 \subset \mathcal{C}$. But $D \notin \mathcal{C}$, so that $D \notin \bar{\mathcal{A}}_0 \otimes \bar{\mathcal{B}}_0$.

- c) Let us write more explicitly $\bar{\mathcal{A}}_0(\mu)$ in place of $\bar{\mathcal{A}}_0$. From Theorem 3.1, we may deduce the measurability of $F := \int f(\cdot, y) d\nu(y)$ with respect to $\bigcap_{\mu} \bar{\mathcal{A}}_0(\mu)$, the intersection being over all \mathcal{A} and μ as in the theorem. This, however, must not be confused with the more restrictive property of universal \mathcal{A}_0 -measurability of F [see Cohn (1980), pages 280–283, for the definition and for illuminating facts]. Indeed, our measures μ are supposed to be defined on some \mathcal{A} rendering f $\mathcal{A} \otimes \mathcal{B}$ -measurable, and not merely on \mathcal{A}_0 or its μ -completion. For example, in the situation of 2.1, one can use the measure μ from (11) to deduce that the σ -algebra of all universally \mathcal{A}_0 -measurable sets is contained in $\widetilde{\mathcal{A}}_0 := \{A \subset \mathbf{R} : A \text{ meager or comeager}\}$. Since $\widetilde{\mathcal{A}}_0$ differs from \mathcal{A}_0 only by non-Borel sets, we see that F from (3), (7), (8) is not universally \mathcal{A}_0 -measurable. By the way, the known fact that μ from (11) can not be extended to a measure on $\mathcal{B}(\mathbf{R})$ [see Oxtoby (1980), page 86] follows from our present considerations, since otherwise we would have $\bar{\mathcal{A}}_0(\mu) = \widetilde{\mathcal{A}}_0 \cap \mathcal{B}(\mathbf{R}) = \mathcal{A}_0$, and Theorem 3.1 would yield \mathcal{A}_0 -measurability of F .

3.3 PROOF OF THEOREM 3.1

Obvious arguments show that we may assume in addition that

$$(12) \quad \mu, \nu \text{ are finite and } f \text{ is bounded.}$$

The proof of the theorem splits into two parts as follows.

CLAIM 1. *Under the assumptions of the theorem and (12),*

$$(13) \quad F := \int f(\cdot, y) d\nu(y)$$

is $\bar{\mathcal{A}}_0$ -measurable.

Proof. Let us first recall the “mean value theorem” for vector valued integration: Let E be a topological vector space, $(\Omega, \mathcal{A}, \mu)$ be a measure space, and $g: \Omega \rightarrow E$ be a function. Then an $x \in E$ is called the *weak* (or *Pettis*) integral of g , and we write $\int g d\mu := x$, if

- (i) the dual space E' of E separates points on E ,
- (ii) the scalar function $\langle y, g(\cdot) \rangle$ belongs to $\mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ for every $y \in E'$, and
- (iii) $\int \langle y, g(\omega) \rangle d\mu(\omega) = \langle y, x \rangle$ for every $y \in E'$.

[This is the definition adopted by Edwards (1965), p. 566, and by Rudin (1991), p. 77.] If now E is in particular locally convex Hausdorff and μ is bounded, then the weak integral, if it exists, necessarily satisfies

$$(14) \quad \int g d\mu \in \mu(\Omega) \cdot \overline{\text{conv}} g(\Omega),$$

with $\overline{\text{conv}}$ indicating convex closure. This “mean value theorem” is surely well known. It follows easily from the Hahn-Banach theorem: Apply Theorem 3.4 (b) of Rudin (1991) to $A := \{\int g d\mu\}$ and $B := \mu(\Omega) \cdot \overline{\text{conv}} g(\Omega)$.

We now start with the proof proper. The functions $f(\cdot, y): \mathcal{X} \rightarrow \mathbf{R}$, as well as F from (13), are \mathcal{A} -measurable [by $\mathcal{A} \otimes \mathcal{B}$ -measurability of f and by Fubini] and bounded, and hence belong to $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Let $[f(\cdot, y)], [F] \in L^1(\mathcal{X}, \mathcal{A}, \mu)$ denote their corresponding equivalence classes. We claim that

$$(15) \quad [F] = \int_{\mathcal{Y}} [f(\cdot, y)] d\nu(y),$$

in the weak sense recalled above, applied to the Banach space $E = L^1(\mathcal{X}, \mathcal{A}, \mu)$ with dual space $L^\infty(\mathcal{X}, \mathcal{A}, \mu)$. To prove this, let $h \in [h] \in L^\infty(\mathcal{X}, \mathcal{A}, \mu)$. An obvious Fubini calculation, using the definition of F and the $\mathcal{A} \otimes \mathcal{B}$ -measurability of f , yields

$$\langle [h], [F] \rangle = \int_{\mathcal{X}} h(x) F(x) d\mu(x) = \int_{\mathcal{Y}} \langle [h], [f(\cdot, y)] \rangle d\nu(y),$$

which confirms (15). [Actually, (15) is even true with the right hand side read as a Bochner integral, but we do not need this fact here.] We now use that each $f(\cdot, y)$ is \mathcal{A}_0 -measurable, where of course $\mathcal{A}_0 \subset \mathcal{A}$. This implies that the function $y \mapsto [f(\cdot, y)]$ takes its values in

$$S := \{ \Phi \in L^1(\mathcal{X}, \mathcal{A}, \mu) : \exists \mathcal{A}_0\text{-measurable } \varphi \in \Phi \},$$

which is easily seen to be a closed subspace of $L^1(\mathcal{X}, \mathcal{A}, \mu)$. The mean value theorem (14) now yields $[F] \in S$, which is the desired conclusion. \square

CLAIM 2. *Under the assumptions of the theorem and (12), and assuming the truth of Claim 1, f is $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$ -measurable.*

Proof. We consider the restrictions

$$\overline{\mu}_0 := \mu|_{\overline{\mathcal{A}_0}}, \quad \overline{\nu}_0 = \nu|_{\overline{\mathcal{A}_0}},$$

and define a function $\tau: \overline{\mathcal{A}_0 \otimes \mathcal{B}_0} \rightarrow [0, \infty]$ by

$$(16) \quad \tau(C) := \int_{\mathcal{X}} \int_{\mathcal{Y}} f(x, y) 1_C(x, y) d\overline{\nu}_0(y) d\overline{\mu}_0(x) \quad (C \in \overline{\mathcal{A}_0 \otimes \mathcal{B}_0}),$$

and we emphasize that the right hand side has to be read as an iterated integral. In order to show its existence, we have to check that the function $x \mapsto \int_{\mathcal{Y}} f(x, y) 1_C(x, y) d\overline{\nu}_0(y)$ is $\overline{\mathcal{A}_0}$ -measurable. For the special case $C = A \times B$ with $A \in \overline{\mathcal{A}_0}$ and $B \in \overline{\mathcal{B}_0}$, this follows from Claim 1, applied to $\overline{\mathcal{A}_0}$ in place of \mathcal{A}_0 and $f(x, y) 1_B(y)$ in place of $f(x, y)$, and using $\overline{\overline{\mathcal{A}_0}} = \overline{\mathcal{A}_0}$. The general case follows as usual via Sierpiński's lemma [Satz I.6.8 in Elstrodt (1996)]. Thus τ is well-defined. It is easily checked that τ is a measure, and that every set of $\overline{\mu}_0 \otimes \overline{\nu}_0$ -measure zero is of τ -measure zero as well. Hence the Lebesgue-Radon-Nikodym theorem yields the existence of an $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$ -measurable function $\tilde{f}: \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty]$ such that

$$\tau(C) = \int_C \tilde{f} d\overline{\mu}_0 \otimes \overline{\nu}_0 \quad (C \in \overline{\mathcal{A}_0 \otimes \mathcal{B}_0}).$$

By (16) and Fubini, this implies in particular

$$(17) \quad \int_{A_0} \int_{B_0} f(x, y) d\overline{\nu}_0(y) d\overline{\mu}_0(x) = \int_{A_0} \int_{B_0} \tilde{f}(x, y) d\overline{\nu}_0(y) d\overline{\mu}_0(x)$$

($A_0 \in \overline{\mathcal{A}_0}$, $B_0 \in \overline{\mathcal{B}_0}$). Since, using (12), both sides in (17) are always finite, we may conclude for every $B_0 \in \overline{\mathcal{B}_0}$:

$$\int_{B_0} f(\cdot, y) d\bar{\nu}_0(y) = \int_{B_0} \tilde{f}(\cdot, y) d\bar{\nu}_0(y) \quad [\bar{\mu}_0].$$

Trivially, this remains true if $[\bar{\mu}_0]$ is replaced by $[\mu]$, and an integration yields

$$(18) \quad \int_A \int_{B_0} f(x, y) d\bar{\nu}_0(y) d\mu(x) = \int_A \int_{B_0} \tilde{f}(x, y) d\bar{\nu}_0(y) d\mu(x)$$

($A \in \mathcal{A}$, $B_0 \in \bar{\mathcal{B}}_0$). We now want to interchange the order of integrations. Since \tilde{f} is trivially $\mathcal{A} \otimes \bar{\mathcal{B}}_0$ -measurable, we may obviously do this on the right hand side of (18). To do the same on the left hand side, we rewrite it successively as

$$\int_A \int_{B_0} f(x, y) d\nu(y) d\mu(x) = \int_{B_0} \int_A f(x, y) d\mu(x) d\nu(y) = \int_{B_0} \int_A f(x, y) d\mu(x) d\bar{\nu}_0(y),$$

where the last equality follows from a second application of Claim 1, with the role of the variables interchanged. Thus (18) yields

$$(19) \quad \int_{B_0} \int_A f(x, y) d\mu(x) d\bar{\nu}_0(y) = \int_{B_0} \int_A \tilde{f}(x, y) d\mu(x) d\bar{\nu}_0(y)$$

($A \in \mathcal{A}$, $B_0 \in \bar{\mathcal{B}}_0$). Now the argument leading from (17) to (18) can be repeated to lead from (19) to a corresponding statement with B in place of B_0 , ν in place of $\bar{\nu}_0$, and \mathcal{B} in place of $\bar{\mathcal{B}}_0$, which is equivalent to

$$\int_{A \times B} f d\mu \otimes \nu = \int_{A \times B} \tilde{f} d\mu \otimes \nu \quad (A \in \mathcal{A}, B \in \mathcal{B}).$$

This shows that $f = \tilde{f}$ $[\mu \otimes \nu]$, which yields the desired conclusion. \square

ACKNOWLEDGEMENT. The present work was supported by a Heisenberg grant of the Deutsche Forschungsgemeinschaft. I thank H. von Weizsäcker for helpful remarks on an earlier version.

REFERENCES

COHN, D. L. (1980) *Measure Theory*. Birkhäuser.

EDWARDS, R. E. (1965) *Functional Analysis. Theory and Applications*. Holt, Rinehart and Winston.

ELSTRODT, J. (1996) *Maß- und Integrationstheorie*. Springer.

FRIEDMAN, H. (1980) A consistent Fubini-Tonelli theorem for nonmeasurable functions. *Illinois J. Math.* 24, 390–395.