Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 45 (1999)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: PRODUCT MEASURABILITY, PARAMETER INTEGRALS, AND A

FUBINI COUNTEREXAMPLE

Autor: Mattner, Lutz

Kapitel: 2. NONMEASURABILITY AND A FUBINI COUNTEREXAMPLE

DOI: https://doi.org/10.5169/seals-64449

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 01.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

in paragraph 2.1 a simple example of a nonmeasurable F, with the right hand side of (1) being of convolution type. On the other hand, Theorem 3.1 contains a positive result, almost yielding \mathcal{A}_0 -measurability of F under additional assumptions.

Now assume that we are also given a σ -finite measure μ on $(\mathcal{X}, \mathcal{A}_0)$. Let us further assume that the function F from (1) is A_0 -measurable and, similarly, that $G := \int_{\mathcal{X}} f(x,\cdot) d\mu(x)$ is \mathcal{B} -measurable. Does it then follow that the "Fubini identity" $\int_{\mathcal{V}} G(y) d\nu(y) = \int_{\mathcal{X}} F(x) d\mu(x)$ holds? Again, the answer is no, as Sierpiński (1920) remarked, essentially by specializing his construction mentioned above. This counterexample has found its way into a number of books, for example Rudin (1987) and Royden (1988), as showing that the assumption of measurability of f with respect to the product σ -algebra $\mathcal{A}_0 \otimes \mathcal{B}$ in the Fubini theorem is not superfluous. In its construction the axiom of choice is still used. The continuum hypothesis is needed only if one insists on specifying the measure spaces, for example as Lebesgue measure. That something beyond the axiom of choice is really needed in the latter case has been proved by Friedman (1980). Below we give, without using the axiom of choice or the continuum hypothesis, a simple construction of a Borel set $A \subset \mathbf{R}$ and of two σ -finite measures μ and ν , defined on suitable σ -algebras on R, such that

(2)
$$\int_{\mathbf{R}} \left[\int_{\mathbf{R}} 1_A(x+y) \, d\mu(x) \right] \, d\nu(y) \neq \int_{\mathbf{R}} \left[\int_{\mathbf{R}} 1_A(x+y) \, d\nu(y) \right] \, d\mu(x) \,,$$

with both iterated integrals existing.

2. Nonmeasurability and a Fubini counterexample

2.1 A NONMEASURABLE CONVOLUTION

In this section, we show that a convolution

(3)
$$F := \int_{\mathbf{R}} g(\cdot - y)h(y) \, dy,$$

with g being a nonnegative bounded Borel function and h nonnegative continuous with compact support, need not be measurable with respect to

(4)
$$\mathcal{A}_0 := \sigma(\{g(\cdot - y) : y \in \mathbf{R}\}),$$

the σ -algebra generated by the translates of g. This yields in particular a counterexample to the measurability of F from (1), with f(x,y) = g(x-y)h(y),

 $\mathcal{X} = \mathcal{Y} = \mathbf{R}$, $\mathcal{A} = \mathcal{A}_0$ from (4), $\mathcal{B} = \mathcal{B}(\mathbf{R}) :=$ Borel σ -algebra on \mathbf{R} , and $\nu = \lambda :=$ Lebesgue measure on $\mathcal{B}(\mathbf{R})$.

The construction becomes clearer if we first drop the nonnegativity and boundedness conditions imposed on g, for which the necessary modifications are indicated afterwards.

Remember that a set $A \subset \mathbf{R}$ is called *meager* [or of the first category] if there is a sequence of closed and nowhere dense sets $F_n \subset \mathbf{R}$ with $A \subset \bigcup_{n \in \mathbf{N}} F_n$. Correspondingly, a set $B \subset \mathbf{R}$ is called *comeager* if its complement B^c is meager, which is equivalent to the existence of a sequence of dense open sets $G_n \subset \mathbf{R}$ with $B \supset \bigcap_{n \in \mathbf{N}} G_n$. It is easily checked that

(5)
$$\mathcal{A} := \{ A \in \mathcal{B}(\mathbf{R}) : A \text{ meager or comeager} \}$$

is a σ -algebra on \mathbf{R} . By Baire's theorem, every comeager set is dense in \mathbf{R} . [We have claimed in the introduction not to use the axiom of choice in constructing this example and the one in 2.2. So we have to note here that we are applying Baire's theorem only in \mathbf{R} , a *separable* complete metric space, where no form of the axiom of choice is needed in its proof. Compare Oxtoby (1980), page 95.] It follows that, for example, the set $[0, \infty[$ is neither meager nor comeager. Hence we surely have the strict inclusion

$$\mathcal{A} \subsetneq \mathcal{B}(\mathbf{R}).$$

Now choose $A \in \mathcal{A}$ meager with $\lambda(A^c) = 0$, for example as in Oxtoby (1980), pages 4–5. Put

(7)
$$g(x) := x 1_A(x) \qquad (x \in \mathbf{R}),$$

(8)
$$h(x) := (1 - |x|)_{+} \qquad (x \in \mathbf{R}),$$

and define F as in (3) and A_0 as in (4). Then

$$\mathcal{A}_0 \subset \mathcal{A},$$

because every $g(\cdot - y)$ is Borel and vanishes on the comeager set $(y+A)^c$, and is hence A-measurable. On the other hand, since $\lambda(A^c) = 0$ and $\int h \, dy = 1$, $\int yh(y) \, dy = 0$,

(10)
$$F(x) = \int_{\mathbf{R}} (x - y)h(y) dy = x \qquad (x \in \mathbf{R}).$$

Hence $\sigma(F)$, the σ -algebra generated by F, is just $\mathcal{B}(\mathbf{R})$, and by (6), (9) it follows that F is not \mathcal{A}_0 -measurable.

To obtain that same conclusion for a nonnegative and bounded g, we may replace g from (7) by $g(x) := (\pi/2 + \arctan x) 1_A(x)$. Instead of calculating F explicitly, we then argue that F is still strictly increasing, and this suffices to deduce that $\sigma(F) = \mathcal{B}(\mathbf{R})$.

2.2 A FUBINI COUNTEREXAMPLE

In this section, we give an example of (2). Let $\mathcal A$ be as in (5) and define $\mu|_{\mathcal A}$ by

(11)
$$\mu(A) := \begin{cases} 0 & \text{if } A \text{ is meager,} \\ 1 & \text{if } A \text{ is comeager.} \end{cases}$$

This is possible, since no set $A \subset \mathbf{R}$ is simultaneously meager and comeager, for otherwise $\emptyset = A \cap A^c$ would be comeager, in contradiction to Baire's theorem. It is easy to check that μ is a probability measure on $(\mathbf{R}, \mathcal{A})$. Let again $\nu := \lambda :=$ Lebesgue measure on $\mathcal{B} := \mathcal{B}(\mathbf{R})$, and choose $A \in \mathcal{A}$ meager with $\lambda(A^c) = 0$. Then $1_A(\cdot + y)$ is \mathcal{A} -measurable with

$$\int_{\mathbf{R}} 1_A(x+y) \, d\mu(x) = \mu(A-y) \, = \, 0 \qquad (y \in \mathbf{R}) \, .$$

On the other hand, we have

$$\int_{\mathbf{R}} 1_A(x+y) \, d\nu(y) = \lambda(A-x) = \infty \qquad (x \in \mathbf{R}).$$

Hence (2) is obviously true in this case.

3. Measurability

Here is a positive result, having a certain measurability property of F from (1) among its conclusions. An application of this occurs in Mattner (1999).

3.1. THEOREM. Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be σ -finite measure spaces, let $f: \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ be a function measurable with respect to the product σ -algebra $\mathcal{A} \otimes \mathcal{B}$, and put

$$\mathcal{A}_{0} := \sigma(\{f(\cdot, y) : y \in \mathcal{Y}\}),$$

$$\mathcal{B}_{0} := \sigma(\{f(x, \cdot) : x \in \mathcal{X}\}),$$

$$\overline{\mathcal{A}}_{0} := \{A \in \mathcal{A} : \exists A_{0} \in \mathcal{A}_{0} \text{ with } A = A_{0} \quad [\mu]\},$$

$$\overline{\mathcal{B}}_{0} := \{B \in \mathcal{B} : \exists B_{0} \in \mathcal{A}_{0} \text{ with } B = B_{0} \quad [\nu]\},$$

$$\overline{\mathcal{A}_{0} \otimes \mathcal{B}_{0}} := \{C \in \mathcal{A} \otimes \mathcal{B} : \exists C_{0} \in \mathcal{A}_{0} \otimes \mathcal{B}_{0} \text{ with } C = C_{0} \quad [\mu \otimes \nu]\}.$$

Then f is $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$ -measurable, $\int_{\mathcal{Y}} f(\cdot, y) d\nu(y)$ is $\overline{\mathcal{A}}_0$ -measurable, and $\int_{\mathcal{X}} f(x, \cdot) d\mu(x)$ is $\overline{\mathcal{B}}_0$ -measurable.