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Autor: Fu, Lei

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immersions. Given  $v \in V'$ , we need to show there exists  $x \in V'$  such that (x, v) and (v, x) are in Z'. This is true if  $v \in V$  by the property of Z. If  $v \in V_s$ , then  $v = a_s$  for some  $a \in V$ . We leave it to the reader to show that  $(x, a_s) \in Z_1$  and  $(a_s, x) \in Z_2$  for generic x in V. This completes the proof of the lemma.

The above lemma allows us to replace V by V', hence to expand V whenever there exists a point s in V such that vs is not defined for all  $v \in V$ , and we can expand V' if there exists a point  $s' \in V'$  such that v's' is not defined for all  $v' \in V'$ . Denote the result of finitely many such expansions also by V', and let  $U \subset V \times V \times V'$  be the closure of  $\Gamma$ . By Lemma 4.3 applied to V', the projection  $p_{12} \colon U \to V \times V$  is an open immersion. Its image is the set of points (a,b) such that  $m \colon V \times V \to V'$  is defined at (a,b). If  $V \times s \not\subset p_{12}(U)$  for some point s in V, then replacing V' by  $V' \cup V_s'$  increases both V' and  $p_{12}(U)$ . Using noetherian induction on open subschemes of  $V \times V$ , we may assume that after finitely many expansions,  $V \times s \subset p_{12}(U)$  for all points  $s \in V$ . Then we have  $p_{12}(U) = V \times V$ .

PROPOSITION 4.5. Let V, V', and U be as above. If  $p_{12}(U) = V \times V$ , then the operation  $m: V' \times V' \to V'$  is everywhere defined on V' and makes V' an algebraic group.

*Proof.* Take (a',b') in  $V' \times V'$ . Choose a point x so that a'x and  $x^{-1}b'$  are both defined and lie in V. Then we can define  $m(a',b')=(a'x)(x^{-1}b')$ . Similarly one can define  $a'^{-1}b'$  and  $b'a'^{-1}$ . In this way we extend m,  $\Phi$ ,  $\Psi$ ,  $\Phi^{-1}$  and  $\Psi^{-1}$  to  $V' \times V'$ . The verification of the group axioms is routine and is omitted.

# 5. FUNDAMENTAL PROPERTIES OF GENERALIZED JACOBIANS

Keep the notations in §3. We have proved that there is a birational group structure on  $(X-S)^{(\pi)}$ . The algebraic group associated to this birational group is called the *generalized jacobian* of  $X_{\mathfrak{m}}$  and is denoted by  $J_{\mathfrak{m}}$ . It is a commutative algebraic group.

Let  $D_0$  be a divisor on X prime to S of degree 0. By Lemma 3.3, the set

$$V_{D_0} = \{ D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D + D_0) = 1, \quad l(D + D_0 - \mathfrak{m}) = 0 \}$$

is a non-empty open subset of  $(X-S)^{(\pi)}$ . We have the following

# LEMMA 5.1. There exists a unique morphism of varieties

$$\alpha_{D_0}\colon V_{D_0}\to (X-S)^{(\pi)}$$

such that  $\alpha_{D_0}(D)$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $D+D_0$  for any  $D \in V_{D_0}$ . Moreover  $\alpha_{D_0}$  is birational.

*Proof.* Consider the Cartesian squares

$$X_{\mathfrak{m}} \times V_{D_0} \subset X_{\mathfrak{m}} \times (X - S)^{(\pi)} \stackrel{p}{\longrightarrow} X_{\mathfrak{m}}$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow$$

Let  $\mathcal{L}$  be the restriction to  $X_{\mathfrak{m}} \times V_{D_0}$  of the invertible sheaf on  $X_{\mathfrak{m}} \times (X-S)^{(\pi)}$  that corresponds to the divisor  $\mathcal{D} + p^*(D_0)$ , where  $\mathcal{D}$  is the universal relative effective Cartier divisor. By Theorem 1.1(c) the sheaf  $q_*\mathcal{L}$  is invertible. The canonical map  $q^*q_*\mathcal{L} \to \mathcal{L}$  induces a homorphism  $s\colon \mathcal{O}_{X_{\mathfrak{m}}\times V_{D_0}} \to \mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$ . Using Remark 2.1, one can show that the pair  $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$  induces a relative effective Cartier divisor on  $(X_{\mathfrak{m}} \times V_{D_0})/V_{D_0}$ . Applying Proposition 3.1 to this divisor, one gets the existence of  $\alpha_{D_0}$ . For any  $D \in V_{D_0}$ , we have  $l_{\mathfrak{m}}(D+D_0)=1$  and  $l(D+D_0-\mathfrak{m})=0$ . So there is one and only one effective divisor  $\mathfrak{m}$ -equivalent to  $D+D_0$ , and this effective divisor is simply  $\alpha_{D_0}(D)$ .

We claim that  $\alpha_{-D_0}$  is the birational inverse of  $\alpha_{D_0}$ . We have

$$\alpha_{D_0}^{-1}(V_{-D_0}) = \{ D \mid D \in V_{D_0}, \ \alpha_{D_0}(D) \in V_{-D_0} \}$$

$$= \{ D \mid D \in V_{D_0}, \ l_{\mathfrak{m}}(\alpha_{D_0}(D) - D_0) = 1, \ l(\alpha_{D_0}(D) - D_0 - \mathfrak{m}) = 0 \}$$

$$= V_{D_0} \cap \{ D \mid l_{\mathfrak{m}}(D) = 1, \ l(D - \mathfrak{m}) = 0 \}$$

$$= V_{D_0} \cap V_0.$$

By Lemma 3.3 both  $V_{D_0}$  and  $V_0$  are open and non-empty. Since  $(X-S)^{(\pi)}$  is irreducible, the set  $V_{D_0} \cap V_0$  is also open and non-empty, that is,  $\alpha_{D_0}^{-1}(V_{-D_0})$  is open and non-empty. One can easily show that on this open set  $\alpha_{-D_0} \circ \alpha_{D_0}$  is defined and is the identity. Similarly one can show  $\alpha_{-D_0}^{-1}(V_{D_0})$  is open and non-empty, and on it  $\alpha_{D_0} \circ \alpha_{-D_0}$  is defined and is the identity. So  $\alpha_{D_0}$  is birational.

We have a birational map  $\varphi \colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$  by the construction of  $J_{\mathfrak{m}}$ . Let  $\operatorname{dom}(\varphi)$  be an open subset of  $(X-S)^{(\pi)}$  such that  $\varphi|_{\operatorname{dom}(\varphi)}$  is an open immersion, Moreover we may assume that for any  $a \in \operatorname{dom}(\varphi)$ , both (a,x)

and (x, a) lie in the set U defined in Lemma 3.4(a) if x is generic, i.e., lies in some open set. In particular, m(a, x) and m(x, a) are defined for generic x.

Let

$$U_{D_0} = V_{D_0} \cap \operatorname{dom}(\varphi) \cap \alpha_{D_0}^{-1}(\operatorname{dom}(\varphi)).$$

Note that  $U_{D_0}$  is open and non-empty since  $(X-S)^{(\pi)}$  is irreducible and  $\alpha_{D_0}$  is birational. Moreover  $\varphi(D)$  and  $\varphi(\alpha_{D_0}(D))$  are defined for any  $D \in U_{D_0}$ . Define

$$\theta_0(D_0) = \varphi(\alpha_{D_0}(D)) - \varphi(D)$$
.

LEMMA 5.2.  $\theta_0(D_0)$  does not depend on the choice of D.

*Proof.* Let  $D_1$  and  $D_2$  be two elements in  $U_{D_0}$ . We need to show that

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_2)) - \varphi(D_2).$$

Choose  $D_3 \in U_{D_0}$  so that  $(\alpha_{D_0}(D_1), D_3)$ ,  $(D_1, \alpha_{D_0}(D_3))$ ,  $(\alpha_{D_0}(D_2), D_3)$  and  $(D_2, \alpha_{D_0}(D_3))$  all lie in the set U defined in Lemma 3.4 (a). Such a  $D_3$  exists. Indeed, if  $(\alpha_{D_0}(D_1), x)$ ,  $(D_1, x)$ ,  $(\alpha_{D_0}(D_2), x)$  and  $(D_2, x)$  all lie in U for x lying in an open set O, then we may choose  $D_3$  to be any element in  $U_{D_0} \cap O \cap \alpha_{D_0}^{-1}(O)$ . Note that  $U_{D_0} \cap O \cap \alpha_{D_0}^{-1}(O)$  is not empty since  $\alpha_{D_0}$  is birational and  $(X - S)^{(\pi)}$  is irreducible.

We have

$$\varphi(\alpha_{D_0}(D_1)) + \varphi(D_3) = \varphi(m(\alpha_{D_0}(D_1), D_3)),$$
  
$$\varphi(D_1) + \varphi(\alpha_{D_0}(D_3)) = \varphi(m(D_1, \alpha_{D_0}(D_3)).$$

Since

$$m(\alpha_{D_0}(D_1), D_3) \sim_{\mathfrak{m}} \alpha_{D_0}(D_1) + D_3 - \pi P_0 \sim_{\mathfrak{m}} D_1 + D_0 + D_3 - \pi P_0,$$
  
 $m(D_1, \alpha_{D_0}(D_3)) \sim_{\mathfrak{m}} D_1 + \alpha_{D_0}(D_3) - \pi P_0 \sim_{\mathfrak{m}} D_1 + D_3 + D_0 - \pi P_0,$ 

we have

$$m(\alpha_{D_0}(D_1), D_3) = m(D_1, \alpha_{D_0}(D_3)).$$

Hence

$$\varphi(\alpha_{D_0}(D_1)) + \varphi(D_3) = \varphi(D_1) + \varphi(\alpha_{D_0}(D_3)),$$

that is,

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_3)) - \varphi(D_3).$$

Similarly we have

$$\varphi(\alpha_{D_0}(D_2)) - \varphi(D_2) = \varphi(\alpha_{D_0}(D_3)) - \varphi(D_3).$$

Therefore

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_2)) - \varphi(D_2).$$

This proves the lemma.

Thus we have a well-defined map  $\theta_0$ :  $\mathrm{Div}^{(0)} \to J_{\mathfrak{m}}$  from the set of divisors of degree 0 on X prime to S to  $J_{\mathfrak{m}}$ .

LEMMA 5.3.  $\theta_0$  is a homomorphism.

*Proof.* Let  $D_0, E_0 \in \operatorname{Div}^{(0)}$  and let  $F_0 = D_0 + E_0$ . Choose  $D \in U_{D_0}$ ,  $E \in U_{E_0}$  and  $F \in U_{F_0}$  so that

$$(\alpha_{D_0}(D), \alpha_{E_0}(E)), (D, E), (m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F) \text{ and } (m(D, E), \alpha_{F_0}(F))$$

all lie in the set U defined in Lemma 3.4(a). We have

$$\alpha_{D_0}(D) + \alpha_{E_0}(E) + F \sim_{\mathfrak{m}} D + D_0 + E + E_0 + F = D + E + F + D_0 + E_0,$$

$$D + E + \alpha_{F_0}(F) \sim_{\mathfrak{m}} D + E + F + F_0 = D + E + F + D_0 + E_0.$$

So

$$m(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F) = m(m(D, E), \alpha_{F_0}(F)).$$

Hence

$$\varphi(m(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F)) = \varphi(m(m(D, E), \alpha_{F_0}(F)).$$

Therefore

$$\varphi(\alpha_{D_0}(D)) + \varphi(\alpha_{E_0}(E)) + \varphi(F) = \varphi(D) + \varphi(E) + \varphi(\alpha_{F_0}(F)),$$

or equivalently,

$$(\varphi(\alpha_{D_0}(D)) - \varphi(D)) + (\varphi(\alpha_{E_0}(E)) - \varphi(E)) = \varphi(\alpha_{F_0}(F)) - \varphi(F).$$

This last equality is exactly

$$\theta_0(D_0) + \theta_0(E_0) = \theta_0(D_0 + E_0)$$
.

So  $\theta_0$  is a homomorphism.

We define  $\theta \colon \operatorname{Div} \to J_{\mathfrak{m}}$  from the group of divisors on X prime to S to  $J_{\mathfrak{m}}$  by

$$\theta(D) = \theta_0(D - \deg(D)P_0).$$

Obviously  $\theta$  is a homomorphism.

PROPOSITION 5.4. The homomorphism  $\theta$  is surjective and  $\ker(\theta)$  consists of divisors  $\mathfrak{m}$ -equivalent to integral multiples of  $P_0$ .

*Proof.* Assume  $\sum_{i=1}^{\pi} P_i$  is in dom $(\varphi)$ . We have

$$\theta\left(\sum_{i=1}^{\pi} P_i\right) = \theta_0\left(\sum_{i=1}^{\pi} P_i - \pi P_0\right) = \varphi(\alpha_{D_0}(D)) - \varphi(D),$$

where  $D_0 = \sum_{i=1}^{\pi} P_i - \pi P_0$  and  $D \in U_{D_0}$ . We may choose D so that  $m(\sum_{i=1}^{\pi} P_i, D)$  is defined and is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $\sum_{i=1}^{\pi} P_i + D - \pi P_0$ . Since  $\alpha_{D_0}(D)$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $D + D_0 = D + \sum_{i=1}^{\pi} P_i - \pi P_0$ , we have  $m(\sum_{i=1}^{\pi} P_i, D) = \alpha_{D_0}(D)$ . Hence  $\varphi(m(\sum_{i=1}^{\pi} P_i, D)) = \varphi(\alpha_{D_0}(D))$ . So  $\varphi(\sum_{i=1}^{\pi} P_i) + \varphi(D) = \varphi(\alpha_{D_0}(D))$ . Therefore  $\varphi(\alpha_{D_0}(D)) - \varphi(D) = \varphi(\sum_{i=1}^{\pi} P_i)$ , that is,

$$\theta\left(\sum_{i=1}^{\pi} P_i\right) = \varphi\left(\sum_{i=1}^{\pi} P_i\right).$$

This is true whenever  $\sum_{i=1}^{\pi} P_i$  is in dom $(\varphi)$ .

Since  $\varphi|_{\text{dom}(\varphi)}$  is an open immersion,  $\varphi(\text{dom}(\varphi))$  is an open subset of  $J_{\mathfrak{m}}$ . The image of  $\theta$  contains this open subset. But  $J_{\mathfrak{m}}$  is generated by any open subset. So we must have  $\text{Im}(\theta) = J_{\mathfrak{m}}$  and  $\theta$  is surjective.

Assume  $E \in \ker(\theta)$ . Then  $\theta_0(E - \deg(E)P_0) = 0$ . Put  $E_0 = E - \deg(E)P_0$ . Then for any  $F \in U_{E_0}$ , we have

$$\varphi(\alpha_{E_0}(F)) - \varphi(F) = \theta_0(E - \deg(E)P_0) = 0.$$

Hence  $\varphi(\alpha_{E_0}(F)) = \varphi(F)$ . But  $\varphi$  is an open immersion on  $\operatorname{dom}(\varphi)$ . So we have  $\alpha_{E_0}(F) = F$ . Since  $\alpha_{E_0}(F) \sim_{\mathfrak{m}} F + E_0$ , we have  $F \sim_{\mathfrak{m}} F + E_0$ . Hence  $E_0 \sim_{\mathfrak{m}} 0$ , that is,  $E \sim_{\mathfrak{m}} \operatorname{deg}(E)P_0$ . So E is  $\mathfrak{m}$ -equivalent to an integral multiple of  $P_0$ .

Conversely assume E is m-equivalent to an integral multiple of  $P_0$  and let us prove that  $\theta(E)=0$ . Again let  $E_0=E-\deg(E)P_0$ . Then  $E_0\sim_{\mathfrak{m}} 0$ . Choose  $F\in U_{E_0}\cap U_0$ , where  $U_0$  is the set  $U_{D_0}$  defined before by taking  $D_0=0$ . We have

$$\theta(E) = \theta_0(E_0) = \varphi(\alpha_{E_0}(F)) - \varphi(F),$$
  
$$\theta(0) = \varphi(\alpha_0(F)) - \varphi(F).$$

Note that  $F + E_0 \sim_{\mathfrak{m}} F$  since  $E_0 \sim_{\mathfrak{m}} 0$ . But  $\alpha_{E_0}(F)$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $F + E_0$ , and  $\alpha_0(F)$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to F. So we must have  $\alpha_{E_0}(F) = \alpha_0(F)$ . Therefore  $\theta(E) = \theta(0) = 0$ .

Regarding a point P in X-S as a divisor, we can calculate  $\theta(P)$ . In this way we get a map  $\theta: X-S \to J_{\mathfrak{m}}$ .

PROPOSITION 5.5. The map  $\theta: X - S \to J_{\mathfrak{m}}$  is a morphism of algebraic varieties.

*Proof.* Let  $P \in X - S$  and let  $D_0 = P - P_0$ . Fix a  $D \in U_{D_0}$ . Consider the set  $W_1 = \{R \in X - S \mid l_m(D + R - P_0) = 1\}$ . By the Riemann-Roch theorem, for any R in X - S, we have  $l_m(D + R - P_0) \ge 1$ . Applying Theorem 1.1 (b) to the projection  $q: X_{\mathfrak{m}} \times (X-S) \to X-S$  and the invertible sheaf corresponding to the divisor  $\mathcal{D} + p^*(D - P_0)$ , where  $\mathcal{D}$  is the universal relative effective Cartier divisor on  $X_{\mathfrak{m}} \times (X - S)$  and  $p: X_{\mathfrak{m}} \times (X - S) \to X_{\mathfrak{m}}$ is another projection, we see that  $W_1$  is open in X - S. Similarly one can show  $W_2 = \{R \in X - S \mid l(D + R - P_0 - \mathfrak{m}) = 0\}$  is also open in X - S. So  $W = W_1 \cap W_2 = \{ R \in X - S \mid l_{\mathfrak{m}}(D + R - P_0) = 1, \quad l(D + R - P_0 - \mathfrak{m}) = 0 \}$ is open in X-S. It is non-empty since  $P \in W$  by our choice of D. By Proposition 3.1 we have a morphism  $\gamma \colon W \to (X-S)^{(\pi)}$  of algebraic varieties such that for every  $R \in W$ ,  $\gamma(R)$  is the unique effective divisor that is  $\mathfrak{m}$ equivalent to  $D+R-P_0$ . Since  $\alpha_{R-P_0}(D)$  is the unique effective divisor that is m-equivalent to  $D+R-P_0$ , we have  $\gamma(R)=\alpha_{R-P_0}(D)$ . Replacing W by an open subset containing P, we may assume  $\text{Im}(\gamma) \subset \text{dom}(\varphi)$ . Note that for any  $R \in W$ , we have  $D \in U_{R-P_0}$ , and

$$\theta(R) = \theta_0(R - P_0) = \varphi((\alpha_{R - P_0}(D)) - \varphi(D)) = \varphi(\gamma(R)) - \varphi(D),$$

that is,  $\theta(R) = \varphi(\gamma(R)) - \varphi(D)$ . So  $\theta = \varphi \circ \gamma - \varphi(D)$  on W. This proves  $\theta$  is a morphism of algebraic varieties in an open subset containing P. Since  $P \in X - S$  is arbitrary,  $\theta$  is a morphism of algebraic varieties.

The morphism  $\theta\colon X-S\to J_{\mathfrak{m}}$  induces a morphism of algebraic varieties  $\theta\colon (X-S)^{(\pi)}\to J_{\mathfrak{m}}$ .

PROPOSITION 5.6.  $\theta: (X-S)^{(\pi)} \to J_{\mathfrak{m}}$  coincides with the birational map  $\varphi: (X-S)^{(\pi)} \to J_{\mathfrak{m}}$ . In particular  $\varphi$  is everywhere defined.

*Proof.* Let  $\sum_{i=1}^{\pi} P_i \in \text{dom}(\varphi)$ . By the proof of Proposition 5.4, we have  $\varphi(\sum_{i=1}^{\pi} P_i) = \theta(\sum_{i=1}^{\pi} P_i)$ . So  $\varphi = \theta$  as rational maps.

Thus there is no difference between  $\varphi$  and  $\theta$ . From now on we denote the map  $\varphi$  also by  $\theta$ . We summarize what we have so far in the following theorem.

THEOREM 1. There is a morphism of algebraic varieties  $\theta: X - S \to J_{\mathfrak{m}}$  satisfying the following properties:

- (a) The extension of  $\theta$  to the group of divisors on X prime to S induces, by passing to quotient, an isomorphism between the group  $C_{\mathfrak{m}}^0$  of classes of divisors of degree zero with respect to  $\mathfrak{m}$ -equivalence and the group  $J_{\mathfrak{m}}$ .
- (b) The extension of  $\theta$  to  $(X-S)^{(\pi)}$  induces a birational map from  $X^{(\pi)}$  to  $J_{\mathfrak{m}}$ .

The following theorem characterizes  $J_{\mathfrak{m}}$  by a universal property:

THEOREM 2. Let  $f: X \to G$  be a rational map from X to a commutative algebraic group G and assume  $\mathfrak{m}$  is a modulus for f. Then there is a unique homomorphism  $F: J_{\mathfrak{m}} \to G$  of algebraic groups such that  $f = F \circ \theta + f(P_0)$ .

Proof. Replacing f by  $f - f(P_0)$ , we may assume  $f(P_0) = 0$ . Since m is a modulus for f, the extension of f to the group of divisors of X prime to S induces a homomorphism  $C_{\mathfrak{m}}^0 \to G$  by passing to quotient. By Theorem 1(a) we have  $J_{\mathfrak{m}} \cong C_{\mathfrak{m}}^0$  as groups. So we have a homomorphism of groups  $F: J_{\mathfrak{m}} \to G$  such that  $f = F\theta$ . It remains to prove F is a morphism of algebraic varieties. By Theorem 1(b) we have a birational map  $\theta: (X-S)^{(\pi)} \to J_{\mathfrak{m}}$ . Denote the extension of f to  $(X-S)^{(\pi)}$  by f'. Then  $F\theta = f'$ . Since  $\theta$  is birational, it induces an isomorphism between an open subvariety of  $(X-S)^{(\pi)}$  and an open subvariety of  $J_{\mathfrak{m}}$ . Moreover f' is a morphism of algebraic varieties. Hence F is a morphism of algebraic varieties when restricted to some open subset of  $J_{\mathfrak{m}}$ . The whole  $J_{\mathfrak{m}}$  can be obtained from this open subset by translation. So F is a morphism of algebraic varieties.

## 6. GENERALIZED JACOBIANS AND PICARD SCHEMES

In this section we prove  $J_{\mathfrak{m}}$  is the Picard scheme of  $X_{\mathfrak{m}}$ . Let T be a k-scheme. Consider the Cartesian square

$$\begin{array}{cccc} X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\ \downarrow & & & \downarrow \\ T & \longrightarrow & \operatorname{spec}(k) \ . \end{array}$$

We have  $q_*\mathcal{O}_{X_{\mathfrak{m}}\times T}=\mathcal{O}_T$  by [EGA] III, §1.4.15, the fact  $H^0(X_{\mathfrak{m}},\mathcal{O}_{X_{\mathfrak{m}}})=k$ , and the fact that  $T\to \operatorname{spec}(k)$  is flat. The morphism q has a section  $s\colon T\to X_{\mathfrak{m}}\times T$ ,  $t\mapsto (P_0,t)$ .