5.5. COROLLARY. *Let $E(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto the K-isotypic subspace of type τ_l. Then*

$$\zeta_{l,s}(g) = \frac{1}{d_l}\,\mathrm{tr}[E(\tau_l)T_{l,s}(g)]\,. \tag{5.43}$$

$(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation if and only if the corresponding irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ is unitarizable. The following theorem is thus a consequence of Theorems 5.1 and 5.3 and of Proposition 5.4.

5.6. THEOREM. *$\zeta_{l,s} = \zeta_{l,-s}$ is positive definite if and only if one of the following cases occurs:*

1. $s = i\nu, \ \nu \in \mathbf{R}$.
2. *If* $2l \geq 2n-1$: $\pm s = s_j := 2(l-n-j)+1$ *for integers* $j \geq 0$ *so that* $s_j > 0$. (*discrete series*)
3. *If* $2l < 2n-1$: $s \in (2l-\rho+2, -2l+\rho-2)$. (*complementary series*)

The situation for s real and nonnegative is represented in Figure 6.1.

6. THE τ_l-ABEL TRANSFORM

Proposition 3.2 proves that the τ_l-Abel transform is a *-homomorphism of $\mathcal{D}(G;\chi_l)$ into the convolution algebra $\mathcal{D}_+(\mathbf{R})$ consisting of the even C^∞ functions on $\mathbf{R}$ with compact support. The main theorem of this section states that the τ_l-Abel transform is also a bijection of $\mathcal{D}(G;\chi_l)$ onto $\mathcal{D}_+(\mathbf{R})$, and gives a formula for its inverse.

Identify A with $\mathbf{R}$ under the map $t \mapsto a_t$. Restriction to A then identifies $\mathcal{D}(G;\chi_l)$ with $\mathcal{D}_+(\mathbf{R})$. Let $\mathcal{D}([1,\infty))$ denote the set of the compactly supported C^∞ functions on $[1,\infty)$ (right differentiability at 1 is considered). Define a map H by

$$(Hf)(\cosh t) := f(a_t) \equiv f(t)$$

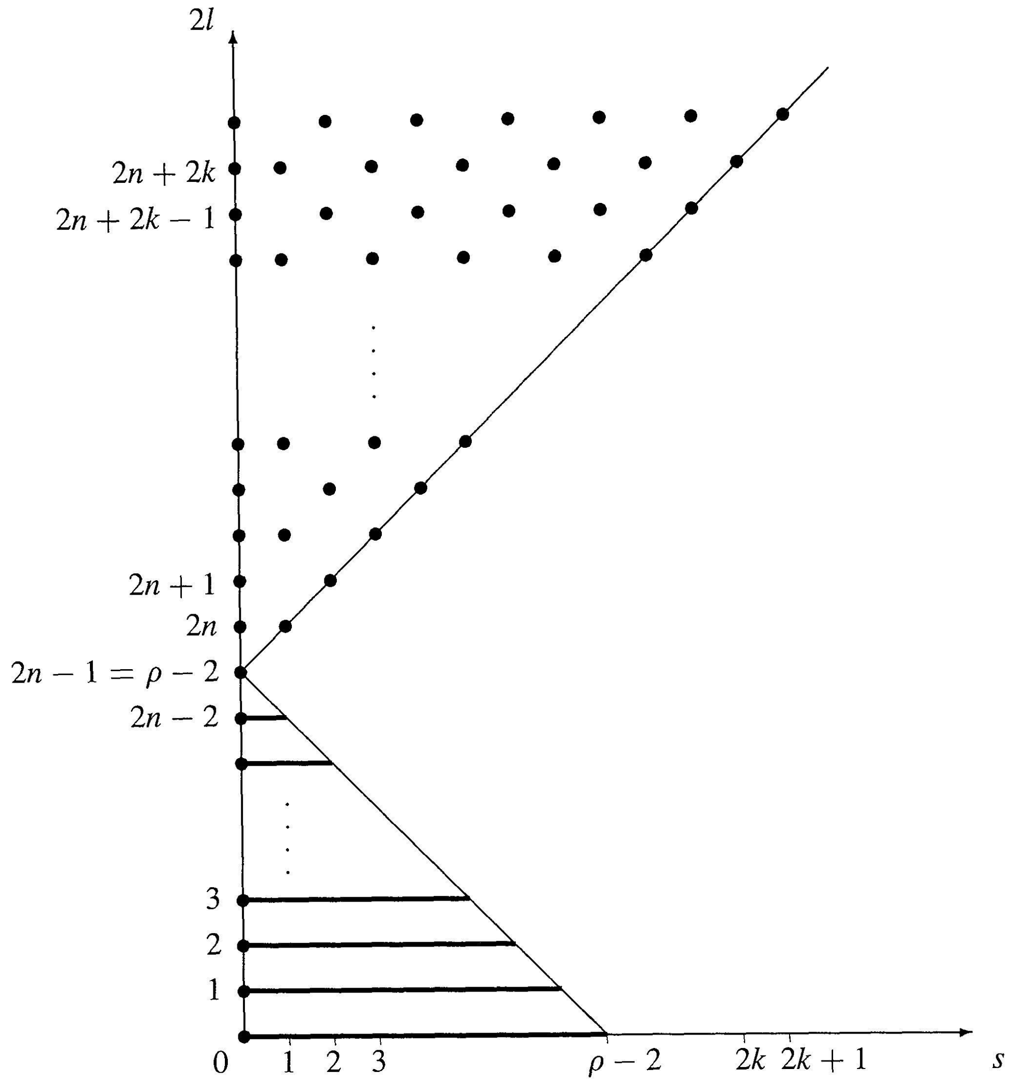

FIGURE 6.1
Positive definite $\zeta_{l,s}$ for real $s \geq 0$

for $f \in \mathcal{D}(G;\chi_l)$. Lemma 2 and its corollary in [Rou] imply

6.1. LEMMA. *H is a bijection of $\mathcal{D}(G;\chi_l)$ onto $\mathcal{D}([1,\infty))$.*

For every $\mu \in \mathbf{C}$ with $\Re\mu > 0$, the Weyl fractional integral transform of $\varphi \in \mathcal{D}([1,\infty))$ is defined by

$$\mathcal{W}_\mu\varphi(x) = \frac{1}{\Gamma(\mu)}\int_x^\infty \varphi(u)(u-x)^{\mu-1}\,du\,, \qquad x\in[1,\infty)\,. \tag{6.44}$$

Analytic continuation of $\mathcal{W}_\mu$ to $\Re\mu \leq 0$ is obtained via repeated integration by parts of (6.44): for every integer $m \geq 0$

$$\mathcal{W}_\mu\varphi(u) = \frac{(-1)^m}{\Gamma(\mu+m)}\int_u^\infty \frac{d^m\varphi}{dx^m}(x)\,(x-u)^{\mu+m-1}\,dx\,.$$

For every integer $m \geq 0$, the Gegenbauer transform (of dimension 4) of $\varphi \in \mathcal{D}([1,\infty))$ is defined by

$$\text{(6.45)} \qquad \mathcal{G}_m\varphi(u) = \frac{4\pi}{m+1}\int_u^\infty \varphi(x)\, C_m^1\left(\frac{u}{x}\right)(x^2-u^2)^{\frac{1}{2}}x\,dx\,, \quad u \in [1,\infty)\,,$$

where

$$\text{(6.46)} \qquad C_m^1(y) = (m+1)F\left(-m, m+2; \frac{3}{2}; \frac{1-y}{2}\right)$$

is the Gegenbauer polynomial of indices $(1,m)$ (cf. e.g. [E$^+$], 3.15 (3)).

6.2. LEMMA ([K1], Theorem 3.2; [Dea], Formulas (28) and (29)).

1. *For every $\mu \in \mathbf{C}$, $\mathcal{W}_\mu$ is a bijection of $\mathcal{D}([1,\infty))$ onto itself. The inverse mapping of $\mathcal{W}_\mu$ is $\mathcal{W}_{-\mu}$.*

2. *For every integer $m \geq 0$, $\mathcal{G}_m$ is a bijection of $\mathcal{D}([1,\infty))$ onto itself. The inverse mapping of $\mathcal{G}_m$ is given by*

$$\text{(6.47)} \qquad \mathcal{G}_m^{-1}\psi(x) = -\frac{1}{2\pi^2(m+1)}\frac{1}{x^2}\int_x^\infty \frac{d^3\psi}{du^3}(u)\, C_m^1\left(\frac{u}{x}\right)(u^2-x^2)^{\frac{1}{2}}\,du$$

for all $\psi \in \mathcal{D}([1,\infty))$ and all $x \in [1,\infty)$.

6.3. THEOREM. *The τ_l-Abel transform is a bijection of $\mathcal{D}(G;\chi_l)$ onto $\mathcal{D}_+(\mathbf{R})$. It can be written as the composition*

$$\mathcal{A}_l = \frac{(2\pi)^{2(n-1)}}{d_l^2}\, H^{-1}\circ \mathcal{W}_{2n-2}\circ \mathcal{G}_{2l}\circ H\,,$$

and its inverse is given by

$$\mathcal{A}_l^{-1} = \frac{d_l^2}{(2\pi)^{2(n-1)}}\, H^{-1}\circ \mathcal{G}_{2l}^{-1}\circ \mathcal{W}_{2-2n}\circ H\,.$$

Moreover, the support of the restriction to $A \equiv \mathbf{R}$ of $f \in \mathcal{D}(G;\chi_l)$ is contained in $[-R,R]$ if and only if the support of $\mathcal{A}_l f$ is contained in $[-R,R]$.

Proof. Identify the set of pure quaternions $w = i\,b + j\,c + k\,d \in \mathbf{H}$ with $\mathbf{R}^3$, and $\mathbf{H}^{n-1}$ with $\mathbf{R}^{4(n-1)}$. If $z \in \mathbf{H}^{n-1}$, then $-[z,z] \equiv |z|^2$ is the square of the Euclidean norm of z in $\mathbf{R}^{4(n-1)}$. For $a_t \in A$ and $n = n(w,z) \in N$ we have

$$(a_t n)_{00} = \cosh t + e^t(w + \tfrac{1}{2}|z|^2)\,.$$

Let $f \in \mathcal{D}(G;\chi_l)$. Applying Lemma 3.3 and Formulas (1.5), we obtain

$$\mathcal{A}_l f(t) = \frac{1}{d_l^2} e^{\rho t} \int_N f(a_t n)\, dn$$

$$= \frac{1}{d_l^3} e^{\rho t} \int_N \chi_l\left(\frac{(a_t n)_{00}}{|(a_t n)_{00}|}\right) Hf(|(a_t n)_{00}|)\, dn$$

$$= \frac{1}{d_l^3} e^{\rho t} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} \chi_l\left(\frac{\cosh t + e^t(w + \frac{1}{2}|z|^2)}{|\cosh t + e^t(w + \frac{1}{2}|z|^2)|}\right)$$

$$\times Hf(|\cosh t + e^t(w + \tfrac{1}{2}|z|^2)|)\, dz\, dw$$

$$= \frac{1}{d_l^3} e^{\rho t} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} C_{2l}^1\left(\frac{\cosh t + \frac{1}{2}e^t|z|^2}{[(\cosh t + \frac{1}{2}e^t|z|^2)^2 + e^{2t}|w|^2]^{\frac{1}{2}}}\right)$$

$$\times Hf([(\cosh t + \tfrac{1}{2}e^t|z|^2)^2 + e^{2t}|w|^2]^{\frac{1}{2}})\, dz\, dw$$

(by Formula (4.40))

$$= \frac{4^{n-1}}{d_l^3} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} C_{2l}^1\left(\frac{\cosh t + |X|^2}{[(\cosh t + |X|^2)^2 + |Y|^2]^{\frac{1}{2}}}\right)$$

$$\times Hf([(\cosh t + |X|^2)^2 + |Y|^2]^{\frac{1}{2}})\, dX\, dY$$

(by substituting $X = \frac{1}{\sqrt{2}} e^{\frac{t}{2}} z,\ Y = e^t w$)

$$= \frac{2^\rho}{d_l^3} \frac{\pi^{2n-1}}{\Gamma(2n-2)} \int_0^\infty \int_0^\infty C_{2l}^1\left(\frac{\cosh t + r^2}{[(\cosh t + r^2)^2 + s^2]^{\frac{1}{2}}}\right)$$

$$\times Hf([(\cosh t + r^2)^2 + s^2]^{\frac{1}{2}})\, r^{4n-5} s^2\, ds\, dr$$

(by passing to spherical coordinates in $\mathbf{R}^{4(n-1)}$ and in $\mathbf{R}^3$)

$$= \frac{2^{2n}}{d_l^3} \frac{\pi^{2n-1}}{\Gamma(2n-2)} \int_{\cosh t}^{\infty} \left[\int_0^{\infty} C_{2l}^1 \left(\frac{u}{[u^2+s^2]^{\frac{1}{2}}} \right) Hf([u^2+s^2]^{\frac{1}{2}})\, s^2\, ds \right] \times (u-\cosh t)^{2n-3}\, du$$

(by setting $u = \cosh t + r^2$)

(6.48)

$$= \frac{2^{2n}\pi^{2n-1}}{d_l^3 \Gamma(2n-2)} \int_{\cosh t}^{\infty} \left[\int_u^{\infty} C_{2l}^1 \left(\frac{u}{x} \right) Hf(x)\,(x^2-u^2)^{\frac{1}{2}} x\, dx \right] (u-\cosh t)^{2n-3} du$$

(by setting $x = [u^2+s^2]^{\frac{1}{2}}$)

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2 \Gamma(2n-1)} \int_{\cosh t}^{\infty} (\mathcal{G}_{2l} Hf)(u)\,(u-\cosh t)^{2n-3}\, du$$

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2} \mathcal{W}_{2n-2}\, \mathcal{G}_{2l}\, Hf(\cosh t)$$

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2} (H^{-1}\, \mathcal{W}_{2n-2}\, \mathcal{G}_{2l}\, Hf)(t),$$

i.e.

$$\mathcal{A}_l = \frac{(2\pi)^{2(n-1)}}{d_l^2} (H^{-1} \circ \mathcal{W}_{2n-2} \circ \mathcal{G}_{2l} \circ H).$$

The inversion formula immediately follows from Lemma 6.2.

The restriction to $A \equiv \mathbf{R}$ of $f \in \mathcal{D}(G;\chi_l)$ has its support $\operatorname{supp} f$ contained in $[-R,R]$ if and only if $\operatorname{supp} Hf \subset [1,\cosh R]$. Moreover, if $\operatorname{supp}\varphi \subset [1,\cosh R]$, then $\operatorname{supp}\mathcal{W}_\mu\varphi$, $\operatorname{supp}\mathcal{G}_m\varphi$ and $\operatorname{supp}\mathcal{G}_m^{-1}\varphi$ are also contained in $[1,\cosh R]$. The last statement then follows from the formulas for $\mathcal{A}_l$ and $\mathcal{A}_l^{-1}$. □

The τ_l*-spherical transform* of $f \in \mathcal{D}(G;\chi_l)$ is the function $\widehat{f}_l$ on $\mathbf{C}$ defined by

$$\widehat{f}_l(s) = \int_G f(g)\zeta_{l,s}(g)\,dg, \qquad s \in \mathbf{C}.$$

Let $\mathcal{S}_l\colon f \mapsto \widehat{f}_l$ denote the τ_l-spherical transform, and let $\mathcal{F}$ denote the Fourier-Laplace transform on $\mathbf{R}$. Formulas (3.20) and (3.22) yield

$$\mathcal{S}_l = \mathcal{F}\circ\mathcal{A}_l\,. \tag{6.49}$$

Let $\mathcal{H}_+^R(\mathbf{R})$ denote the set of even functions h on $\mathbf{C}$ which are entire rapidly decreasing functions of exponential type R: for every integer $N \geq 0$ there is a constant $C_N > 0$ so that

$$|h(s)| \leq C_N(1+|s|)^{-N}e^{R|\Re s|} \quad \text{for all } s \in \mathbf{C}\,.$$

Set $\mathcal{H}_+(\mathbf{R}) := \bigcup_{R>0}\mathcal{H}_+^R(\mathbf{R})$. Theorem 6.3 and the Paley-Wiener Theorem for the Fourier-Laplace transform of even functions on $\mathbf{R}$ prove the following theorem.

6.4. THEOREM (Paley-Wiener Theorem). *The τ_l-spherical transform is a bijection of $\mathcal{D}(G;\chi_l)$ onto $\mathcal{H}_+(\mathbf{R})$. Moreover, the restriction of $f \in \mathcal{D}(G;\chi_l)$ to $A \equiv \mathbf{R}$ has support in $[-R,R]$ if and only if $\widehat{f}_l \in \mathcal{H}_+^R(\mathbf{R})$.*

We conclude this section by observing that the τ_l-Abel transform is related, as one should expect, to the Abel transform of [K2], §5.

Reversing the order of integration and substituting $x = \cosh\tau$ and $u = \cosh w$, we obtain from (6.48)

$$\mathcal{A}_l f(t) = \int_t^\infty A_l(t,\tau)f(\tau)\,d\tau \tag{6.50}$$

where

$$A_l(t,\tau) := \frac{(2\pi)^{2n-1}}{d_l^3\Gamma(2n-2)}\sinh(2\tau)\int_t^s C_{2l}^1\left(\frac{\cosh w}{\cosh\tau}\right)(\cosh^2\tau-\cosh^2 w)^{\frac{1}{2}}$$
$$\times(\cosh w-\cosh t)^{2n-3}\sinh w\,dw\,.$$

Substituting also $y = \dfrac{\cosh\tau-\cosh w}{\cosh\tau-\cosh t}$ and setting

$$\gamma(t,\tau) = \frac{\cosh\tau-\cosh t}{2\cosh\tau} \qquad \text{and} \qquad K_l = \frac{(2\pi)^{2n-1}}{d_l^3\Gamma(2n-2)},$$

we get from Formula (6.46)

$$A_l(t,\tau) = \sqrt{2}\,K_l \sinh(2\tau)(\cosh\tau - \cosh t)^{2n-\frac{3}{2}}(\cosh\tau)^{\frac{1}{2}}$$

$$\times \int_0^1 C^1_{2l}(1-2\gamma(t,\tau)y)y^{\frac{1}{2}}(1-y)^{2n-3}(1-\gamma(t,\tau)y)^{\frac{1}{2}}\,dy$$

$$= \sqrt{2}(2l+1)K_l \sinh(2\tau)(\cosh\tau - \cosh t)^{2n-\frac{3}{2}}(\cosh\tau)^{\frac{1}{2}}$$

$$\times \int_0^1 F\Big(\frac{3}{2}+2l, -2l-\frac{1}{2}; \frac{3}{2}; \gamma(t,\tau)y\Big)y^{\frac{1}{2}}(1-y)^{2n-3}(1-\gamma(t,\tau)y)^{\frac{1}{2}}dy\,.$$

If we now apply the relation ([E+], 2.9(2))

(6.51) $$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a, c-b; c; z)$$

and Bateman's Formula ([E+], 2.4(2))

(6.52) $$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(c-s)}\int_0^1 x^{s-1}(1-x)^{c-s-1}F(a,b;s;xz)\,dx$$

for $\Re c > \Re s > 0, z \neq 1, |\arg(1-z)| < \pi$,

we finally obtain

(6.53) $$A_l(t,\tau) = \frac{(2\pi)^{2n-\frac{1}{2}}}{2\Gamma(2n-\frac{1}{2})}\frac{1}{d_l^2}\sinh(2\tau)(\cosh\tau - \cosh t)^{2n-\frac{3}{2}}(\cosh\tau)^{\frac{1}{2}}$$

$$\times F\left(\frac{3}{2}+2l, -2l-\frac{1}{2}, 2n-\frac{1}{2}, \gamma(t,\tau)\right).$$

The comparison of Formula (6.53) with the kernel $A_{2n-1,2l+1}(t,\tau)$ in [K2], Formula (5.60), gives

(6.54)

$$A_l(t,\tau) = \frac{1}{2}\frac{1}{{d_l}^2}\left(\frac{\pi}{4}\right)^{2n}\frac{1}{\Gamma(2n)}2^{-4l}(\cosh\tau)^{-2l}A_{2n-1,2l+1}(t,\tau)$$

$$= 2^{-4l}C_l(\cosh\tau)^{-2l}A_{2n-1,2l+1}(t,\tau)$$

where we have set

(6.55) $$C_l := \frac{1}{2}\frac{1}{(2l+1)^2}\left(\frac{\pi}{4}\right)^{2n}\frac{1}{\Gamma(2n)}\,.$$