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5.5. COROLLARY. Let  $E(\tau_l)$  denote the projection of  $\mathcal{H}_{l,s}$  onto the  $K$ -isotypic subspace of type  $\tau_l$ . Then

$$(5.43) \quad \zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)].$$

$(T_{l,s}, \mathcal{H}_{l,s})$  is infinitesimally equivalent to a unitary representation if and only if the corresponding irreducible subquotient of  $(\mathcal{H}'_{l,s})_K$  is unitarizable. The following theorem is thus a consequence of Theorems 5.1 and 5.3 and of Proposition 5.4.

5.6. THEOREM.  $\zeta_{l,s} = \zeta_{l,-s}$  is positive definite if and only if one of the following cases occurs:

1.  $s = i\nu$ ,  $\nu \in \mathbf{R}$ .
2. If  $2l \geq 2n - 1$ :  $\pm s = s_j := 2(l - n - j) + 1$  for integers  $j \geq 0$  so that  $s_j > 0$ . (discrete series)
3. If  $2l < 2n - 1$ :  $s \in (2l - \rho + 2, -2l + \rho - 2)$ . (complementary series)

The situation for  $s$  real and nonnegative is represented in Figure 6.1.

## 6. THE $\tau_l$ -ABEL TRANSFORM

Proposition 3.2 proves that the  $\tau_l$ -Abel transform is a  $*$ -homomorphism of  $\mathcal{D}(G; \chi_l)$  into the convolution algebra  $\mathcal{D}_+(\mathbf{R})$  consisting of the even  $C^\infty$  functions on  $\mathbf{R}$  with compact support. The main theorem of this section states that the  $\tau_l$ -Abel transform is also a bijection of  $\mathcal{D}(G; \chi_l)$  onto  $\mathcal{D}_+(\mathbf{R})$ , and gives a formula for its inverse.

Identify  $A$  with  $\mathbf{R}$  under the map  $t \mapsto a_t$ . Restriction to  $A$  then identifies  $\mathcal{D}(G; \chi_l)$  with  $\mathcal{D}_+(\mathbf{R})$ . Let  $\mathcal{D}([1, \infty))$  denote the set of the compactly supported  $C^\infty$  functions on  $[1, \infty)$  (right differentiability at 1 is considered). Define a map  $H$  by

$$(Hf)(\cosh t) := f(a_t) \equiv f(t)$$

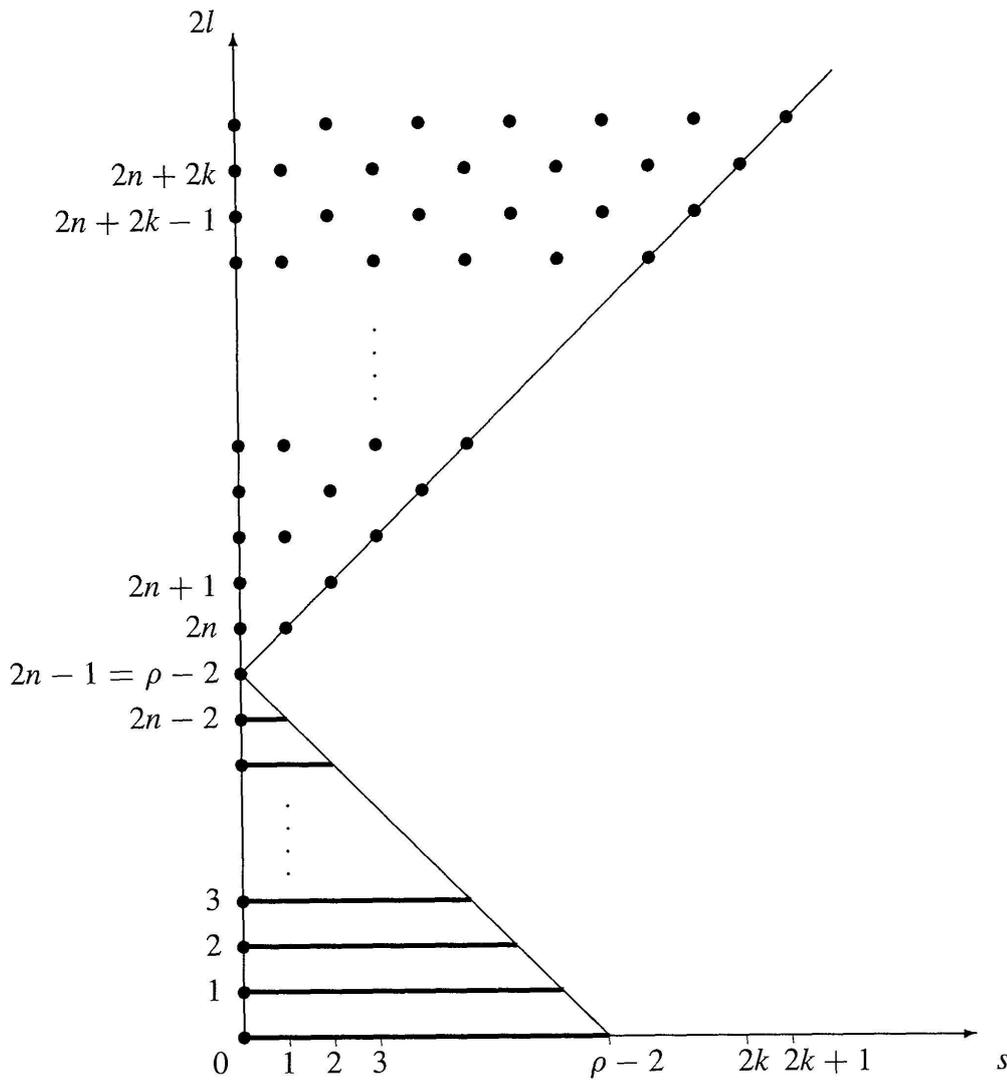


FIGURE 6.1

Positive definite  $\zeta_{l,s}$  for real  $s \geq 0$

for  $f \in \mathcal{D}(G; \chi_l)$ . Lemma 2 and its corollary in [Rou] imply

6.1. LEMMA.  $H$  is a bijection of  $\mathcal{D}(G; \chi_l)$  onto  $\mathcal{D}([1, \infty))$ .

For every  $\mu \in \mathbf{C}$  with  $\Re \mu > 0$ , the Weyl fractional integral transform of  $\varphi \in \mathcal{D}([1, \infty))$  is defined by

$$(6.44) \quad \mathcal{W}_\mu \varphi(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty \varphi(u)(u-x)^{\mu-1} du, \quad x \in [1, \infty).$$

Analytic continuation of  $\mathcal{W}_\mu$  to  $\Re \mu \leq 0$  is obtained via repeated integration by parts of (6.44): for every integer  $m \geq 0$

$$\mathcal{W}_\mu \varphi(u) = \frac{(-1)^m}{\Gamma(\mu + m)} \int_u^\infty \frac{d^m \varphi}{dx^m}(x) (x-u)^{\mu+m-1} dx.$$

For every integer  $m \geq 0$ , the Gegenbauer transform (of dimension 4) of  $\varphi \in \mathcal{D}([1, \infty))$  is defined by

$$(6.45) \quad \mathcal{G}_m \varphi(u) = \frac{4\pi}{m+1} \int_u^\infty \varphi(x) C_m^1\left(\frac{u}{x}\right) (x^2 - u^2)^{\frac{1}{2}} x \, dx, \quad u \in [1, \infty),$$

where

$$(6.46) \quad C_m^1(y) = (m+1)F\left(-m, m+2; \frac{3}{2}; \frac{1-y}{2}\right)$$

is the Gegenbauer polynomial of indices  $(1, m)$  (cf. e.g. [E<sup>+</sup>], 3.15 (3)).

6.2. LEMMA ([K1], Theorem 3.2; [Dea], Formulas (28) and (29)).

1. For every  $\mu \in \mathbf{C}$ ,  $\mathcal{W}_\mu$  is a bijection of  $\mathcal{D}([1, \infty))$  onto itself. The inverse mapping of  $\mathcal{W}_\mu$  is  $\mathcal{W}_{-\mu}$ .

2. For every integer  $m \geq 0$ ,  $\mathcal{G}_m$  is a bijection of  $\mathcal{D}([1, \infty))$  onto itself. The inverse mapping of  $\mathcal{G}_m$  is given by

$$(6.47) \quad \mathcal{G}_m^{-1} \psi(x) = -\frac{1}{2\pi^2(m+1)} \frac{1}{x^2} \int_x^\infty \frac{d^3 \psi}{du^3}(u) C_m^1\left(\frac{u}{x}\right) (u^2 - x^2)^{\frac{1}{2}} \, du$$

for all  $\psi \in \mathcal{D}([1, \infty))$  and all  $x \in [1, \infty)$ .

6.3. THEOREM. The  $\tau_l$ -Abel transform is a bijection of  $\mathcal{D}(G; \chi_l)$  onto  $\mathcal{D}_+(\mathbf{R})$ . It can be written as the composition

$$\mathcal{A}_l = \frac{(2\pi)^{2(n-1)}}{d_l^2} H^{-1} \circ \mathcal{W}_{2n-2} \circ \mathcal{G}_{2l} \circ H,$$

and its inverse is given by

$$\mathcal{A}_l^{-1} = \frac{d_l^2}{(2\pi)^{2(n-1)}} H^{-1} \circ \mathcal{G}_{2l}^{-1} \circ \mathcal{W}_{2-2n} \circ H.$$

Moreover, the support of the restriction to  $A \equiv \mathbf{R}$  of  $f \in \mathcal{D}(G; \chi_l)$  is contained in  $[-R, R]$  if and only if the support of  $\mathcal{A}_l f$  is contained in  $[-R, R]$ .

*Proof.* Identify the set of pure quaternions  $w = i b + j c + k d \in \mathbf{H}$  with  $\mathbf{R}^3$ , and  $\mathbf{H}^{n-1}$  with  $\mathbf{R}^{4(n-1)}$ . If  $z \in \mathbf{H}^{n-1}$ , then  $-[z, z] \equiv |z|^2$  is the square of the Euclidean norm of  $z$  in  $\mathbf{R}^{4(n-1)}$ . For  $a_t \in A$  and  $n = n(w, z) \in N$  we have

$$(a_t n)_{00} = \cosh t + e^t (w + \frac{1}{2} |z|^2).$$

Let  $f \in \mathcal{D}(G; \chi_l)$ . Applying Lemma 3.3 and Formulas (1.5), we obtain

$$\begin{aligned}
\mathcal{A}f(t) &= \frac{1}{d_l^2} e^{\rho t} \int_N f(a_t n) dn \\
&= \frac{1}{d_l^3} e^{\rho t} \int_N \chi_l \left( \frac{(a_t n)_{00}}{|(a_t n)_{00}|} \right) Hf(|(a_t n)_{00}|) dn \\
&= \frac{1}{d_l^3} e^{\rho t} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} \chi_l \left( \frac{\cosh t + e^t(w + \frac{1}{2}|z|^2)}{|\cosh t + e^t(w + \frac{1}{2}|z|^2)|} \right) \\
&\quad \times Hf(|\cosh t + e^t(w + \frac{1}{2}|z|^2)|) dz dw \\
&= \frac{1}{d_l^3} e^{\rho t} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} C_{2l}^1 \left( \frac{\cosh t + \frac{1}{2}e^t|z|^2}{[(\cosh t + \frac{1}{2}e^t|z|^2)^2 + e^{2t}|w|^2]^{\frac{1}{2}}} \right) \\
&\quad \times Hf([( \cosh t + \frac{1}{2}e^t|z|^2)^2 + e^{2t}|w|^2]^{\frac{1}{2}}) dz dw \\
&\hspace{15em} \text{(by Formula (4.40))}
\end{aligned}$$

$$\begin{aligned}
&= \frac{4^{n-1}}{d_l^3} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} C_{2l}^1 \left( \frac{\cosh t + |X|^2}{[(\cosh t + |X|^2)^2 + |Y|^2]^{\frac{1}{2}}} \right) \\
&\quad \times Hf([( \cosh t + |X|^2)^2 + |Y|^2]^{\frac{1}{2}}) dX dY \\
&\hspace{15em} \text{(by substituting } X = \frac{1}{\sqrt{2}}e^{\frac{t}{2}}z, Y = e^t w)
\end{aligned}$$

$$\begin{aligned}
&= \frac{2^\rho}{d_l^3} \frac{\pi^{2n-1}}{\Gamma(2n-2)} \int_0^\infty \int_0^\infty C_{2l}^1 \left( \frac{\cosh t + r^2}{[(\cosh t + r^2)^2 + s^2]^{\frac{1}{2}}} \right) \\
&\quad \times Hf([( \cosh t + r^2)^2 + s^2]^{\frac{1}{2}}) r^{4n-5} s^2 ds dr
\end{aligned}$$

(by passing to spherical coordinates in  $\mathbf{R}^{4(n-1)}$  and in  $\mathbf{R}^3$ )

$$= \frac{2^{2n} \pi^{2n-1}}{d_l^3 \Gamma(2n-2)} \int_{\cosh t}^{\infty} \left[ \int_0^{\infty} C_{2l}^1 \left( \frac{u}{[u^2 + s^2]^{\frac{1}{2}}} \right) Hf([u^2 + s^2]^{\frac{1}{2}}) s^2 ds \right] \\ \times (u - \cosh t)^{2n-3} du$$

(by setting  $u = \cosh t + r^2$ )

(6.48)

$$= \frac{2^{2n} \pi^{2n-1}}{d_l^3 \Gamma(2n-2)} \int_{\cosh t}^{\infty} \left[ \int_u^{\infty} C_{2l}^1 \left( \frac{u}{x} \right) Hf(x) (x^2 - u^2)^{\frac{1}{2}} x dx \right] (u - \cosh t)^{2n-3} du$$

(by setting  $x = [u^2 + s^2]^{\frac{1}{2}}$ )

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2 \Gamma(2n-1)} \int_{\cosh t}^{\infty} (\mathcal{G}_{2l} Hf)(u) (u - \cosh t)^{2n-3} du \\ = \frac{(2\pi)^{2(n-1)}}{d_l^2} \mathcal{W}_{2n-2} \mathcal{G}_{2l} Hf(\cosh t) \\ = \frac{(2\pi)^{2(n-1)}}{d_l^2} (H^{-1} \mathcal{W}_{2n-2} \mathcal{G}_{2l} Hf)(t),$$

i.e.

$$\mathcal{A}_l = \frac{(2\pi)^{2(n-1)}}{d_l^2} (H^{-1} \circ \mathcal{W}_{2n-2} \circ \mathcal{G}_{2l} \circ H).$$

The inversion formula immediately follows from Lemma 6.2.

The restriction to  $A \equiv \mathbf{R}$  of  $f \in \mathcal{D}(G; \chi_l)$  has its support  $\text{supp} f$  contained in  $[-R, R]$  if and only if  $\text{supp} Hf \subset [1, \cosh R]$ . Moreover, if  $\text{supp} \varphi \subset [1, \cosh R]$ , then  $\text{supp} \mathcal{W}_\mu \varphi$ ,  $\text{supp} \mathcal{G}_m \varphi$  and  $\text{supp} \mathcal{G}_m^{-1} \varphi$  are also contained in  $[1, \cosh R]$ . The last statement then follows from the formulas for  $\mathcal{A}_l$  and  $\mathcal{A}_l^{-1}$ .  $\square$

The  $\tau_l$ -spherical transform of  $f \in \mathcal{D}(G; \chi_l)$  is the function  $\widehat{f}_l$  on  $\mathbf{C}$  defined by

$$\widehat{f}_l(s) = \int_G f(g) \zeta_{l,s}(g) dg, \quad s \in \mathbf{C}.$$

Let  $S_l: f \mapsto \widehat{f}_l$  denote the  $\tau_l$ -spherical transform, and let  $\mathcal{F}$  denote the Fourier-Laplace transform on  $\mathbf{R}$ . Formulas (3.20) and (3.22) yield

$$(6.49) \quad S_l = \mathcal{F} \circ \mathcal{A}_l.$$

Let  $\mathcal{H}_+^R(\mathbf{R})$  denote the set of even functions  $h$  on  $\mathbf{C}$  which are entire rapidly decreasing functions of exponential type  $R$ : for every integer  $N \geq 0$  there is a constant  $C_N > 0$  so that

$$|h(s)| \leq C_N (1 + |s|)^{-N} e^{R|\Re s|} \quad \text{for all } s \in \mathbf{C}.$$

Set  $\mathcal{H}_+(\mathbf{R}) := \bigcup_{R>0} \mathcal{H}_+^R(\mathbf{R})$ . Theorem 6.3 and the Paley-Wiener Theorem for the Fourier-Laplace transform of even functions on  $\mathbf{R}$  prove the following theorem.

6.4. THEOREM (Paley-Wiener Theorem). *The  $\tau_l$ -spherical transform is a bijection of  $\mathcal{D}(G; \chi_l)$  onto  $\mathcal{H}_+(\mathbf{R})$ . Moreover, the restriction of  $f \in \mathcal{D}(G; \chi_l)$  to  $A \equiv \mathbf{R}$  has support in  $[-R, R]$  if and only if  $\widehat{f}_l \in \mathcal{H}_+^R(\mathbf{R})$ .*

We conclude this section by observing that the  $\tau_l$ -Abel transform is related, as one should expect, to the Abel transform of [K2], §5.

Reversing the order of integration and substituting  $x = \cosh \tau$  and  $u = \cosh w$ , we obtain from (6.48)

$$(6.50) \quad \mathcal{A}_l f(t) = \int_t^\infty A_l(t, \tau) f(\tau) d\tau$$

where

$$A_l(t, \tau) := \frac{(2\pi)^{2n-1}}{d_l^3 \Gamma(2n-2)} \sinh(2\tau) \int_t^s C_{2l}^1 \left( \frac{\cosh w}{\cosh \tau} \right) (\cosh^2 \tau - \cosh^2 w)^{\frac{1}{2}} \\ \times (\cosh w - \cosh t)^{2n-3} \sinh w dw.$$

Substituting also  $y = \frac{\cosh \tau - \cosh w}{\cosh \tau - \cosh t}$  and setting

$$\gamma(t, \tau) = \frac{\cosh \tau - \cosh t}{2 \cosh \tau} \quad \text{and} \quad K_l = \frac{(2\pi)^{2n-1}}{d_l^3 \Gamma(2n-2)},$$

we get from Formula (6.46)

$$\begin{aligned} A_l(t, \tau) &= \sqrt{2} K_l \sinh(2\tau) (\cosh \tau - \cosh t)^{2n-\frac{3}{2}} (\cosh \tau)^{\frac{1}{2}} \\ &\quad \times \int_0^1 C_{2l}^1 (1 - 2\gamma(t, \tau)y) y^{\frac{1}{2}} (1-y)^{2n-3} (1 - \gamma(t, \tau)y)^{\frac{1}{2}} dy \\ &= \sqrt{2} (2l+1) K_l \sinh(2\tau) (\cosh \tau - \cosh t)^{2n-\frac{3}{2}} (\cosh \tau)^{\frac{1}{2}} \\ &\quad \times \int_0^1 F\left(\frac{3}{2} + 2l, -2l - \frac{1}{2}; \frac{3}{2}; \gamma(t, \tau)y\right) y^{\frac{1}{2}} (1-y)^{2n-3} (1 - \gamma(t, \tau)y)^{\frac{1}{2}} dy. \end{aligned}$$

If we now apply the relation ([E<sup>+</sup>], 2.9(2))

$$(6.51) \quad F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

and Bateman's Formula ([E<sup>+</sup>], 2.4(2))

$$(6.52) \quad F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(c-s)} \int_0^1 x^{s-1} (1-x)^{c-s-1} F(a, b; s; xz) dx$$

for  $\Re c > \Re s > 0, z \neq 1, |\arg(1-z)| < \pi,$

we finally obtain

$$(6.53) \quad A_l(t, \tau) = \frac{(2\pi)^{2n-\frac{1}{2}}}{2\Gamma(2n-\frac{1}{2})} \frac{1}{d_l^2} \sinh(2\tau) (\cosh \tau - \cosh t)^{2n-\frac{3}{2}} (\cosh \tau)^{\frac{1}{2}} \\ \times F\left(\frac{3}{2} + 2l, -2l - \frac{1}{2}, 2n - \frac{1}{2}, \gamma(t, \tau)\right).$$

The comparison of Formula (6.53) with the kernel  $A_{2n-1, 2l+1}(t, \tau)$  in [K2], Formula (5.60), gives

$$(6.54) \quad \begin{aligned} A_l(t, \tau) &= \frac{1}{2} \frac{1}{d_l^2} \left(\frac{\pi}{4}\right)^{2n} \frac{1}{\Gamma(2n)} 2^{-4l} (\cosh \tau)^{-2l} A_{2n-1, 2l+1}(t, \tau) \\ &= 2^{-4l} C_l (\cosh \tau)^{-2l} A_{2n-1, 2l+1}(t, \tau) \end{aligned}$$

where we have set

$$(6.55) \quad C_l := \frac{1}{2} \frac{1}{(2l+1)^2} \left(\frac{\pi}{4}\right)^{2n} \frac{1}{\Gamma(2n)}.$$