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4.4. COROLLARY. *The τ_l -spherical functions are exactly the functions $\{\zeta_{l,s} : s \in \mathbf{C}\}$ given by Formulas (3.24) and (4.39). Further, $\zeta_{l,s}$ satisfies $\zeta_{l,s}(g) = \zeta_{l,s}(g^{-1})$ for all $g \in G$. Moreover, $\zeta_{l,s} = \zeta_{l,s'}$ if and only if $s = \pm s'$.*

The functional equation (3.15) with $g_1 = a_t$ and $g_2 = a_\tau$ becomes (cf. [T2], Théorème 1, p. 227)

$$(4.41) \quad \zeta_{l,s}(t)\zeta_{l,s}(\tau) = \int_0^\infty K_l(t, \tau, u)\zeta_{l,s}(u)\Delta(u) du$$

where Δ is as in (1.7) and the kernel $K_l(t, \tau, u)$ is defined as follows. Set

$$B := \frac{\cosh^2 t + \cosh^2 \tau + \cosh^2 u - 1}{2 \cosh t \cosh \tau \cosh u}.$$

Then

$$(4.42) \quad K_l(t, \tau, u) := \frac{2^{-2\rho}\Gamma(2n)}{\sqrt{\pi}\Gamma(2n - \frac{1}{2})} \frac{(\cosh t \cosh \tau \cosh u)^{2n-3}}{(\sinh t \sinh \tau \sinh u)^{4n-2}} (1 - B^2)^{2n-\frac{3}{2}} \\ \times F\left(2n + 2l, 2n - 2l - 2; 2n - \frac{1}{2}; \frac{1}{2}(1 - B)\right)$$

if $B < 1$, and $K_l(t, \tau, u) := 0$ if $B \geq 1$. Using (4.39) and Formula (7.11) in [K2], one can prove that (4.41) holds also outside our group-theoretical setting for all $l \in \mathbf{R}$ satisfying $2n - 1 > 2l \geq 0$.

5. THE POSITIVE DEFINITE τ_l -SPHERICAL FUNCTIONS

A continuous function ζ on a locally compact group G is said to be *positive definite* if for every $f \in C_c(G)$

$$\int_G \int_G \zeta(x^{-1}y) f(x) \overline{f(y)} dx dy \geq 0.$$

In this section we establish which among the $\zeta_{l,s}$ are positive definite.

Let us first introduce some notation and recall some definitions. Let G be a semisimple Lie group with finite center, and let K be a maximal compact subgroup of G . \mathfrak{g} and \mathfrak{k} ($\subset \mathfrak{g}$) are the Lie algebras of G and K , respectively. A (strongly continuous) representation T of G on a Banach space \mathcal{H} is denoted by (T, \mathcal{H}) . We may simply speak of the representation T if \mathcal{H} is understood. Irreducibility for T always means topological irreducibility (= no closed proper invariant subspaces). Let \widehat{K} denote the set of equivalence classes

of finite dimensional irreducible representations of K . We say that $\tau \in \widehat{K}$ occurs in $T|_K$ if there exists a finite dimensional $T|_K$ -invariant subspace V of \mathcal{H} so that $(T|_K, V) \in \tau$. The linear span of all these subspaces V is the K -isotypic subspace of \mathcal{H} of type τ , denoted $\mathcal{H}(\tau)$. If d_τ is the dimension of τ and χ_τ is its character, then

$$E_T(\tau) = d_\tau \int_K T(k^{-1}) \chi_\tau(k) dk$$

is a continuous projection of \mathcal{H} onto $\mathcal{H}(\tau)$. We set $\mathcal{H}_K = \sum_{\tau \in \widehat{K}} \mathcal{H}(\tau)$. T is said to be K -finite if $\dim \mathcal{H}(\tau) < \infty$ for all $\tau \in \widehat{K}$. A Hilbert representation (T, \mathcal{H}) is said to be admissible if it is K -finite and if $T|_K$ acts on \mathcal{H} by unitary operators.

A representation U of an (associative or Lie) algebra \mathcal{A} on a \mathbf{C} -vector space E is denoted (U, E) . The term \mathcal{A} -module is also used. Irreducibility for U always means algebraic irreducibility (= no proper invariant subspaces). Let $\widehat{\mathfrak{k}}_{\mathbf{C}}$ denote the set of equivalence classes of finite dimensional simple $\mathfrak{k}_{\mathbf{C}}$ -modules. The sum of all simple $\mathfrak{k}_{\mathbf{C}}$ -submodules of E which are in the class $\delta \in \widehat{\mathfrak{k}}_{\mathbf{C}}$ is denoted by $E(\delta)$. (U, E) is said \mathfrak{k} -finite if $\dim E(\delta) < \infty$ for all $\delta \in \widehat{\mathfrak{k}}_{\mathbf{C}}$ and if $E = \sum_{\delta \in \widehat{\mathfrak{k}}_{\mathbf{C}}} E(\delta)$.

Every K -finite irreducible representation (T, \mathcal{H}) of G induces a \mathfrak{k} -finite irreducible representation (T_K, \mathcal{H}_K) of $\mathfrak{U}(\mathfrak{g})$ by differentiation. If, moreover, \mathcal{H} is Hilbert and T is unitary, then \mathfrak{g} acts on \mathcal{H}_K by skew-adjoint operators: $\langle T_K(X)\varphi, \psi \rangle = -\langle \varphi, T_K(X)\psi \rangle$ for all $X \in \mathfrak{g}$ and all $\varphi, \psi \in \mathcal{H}_K$. Two K -finite representations (T, \mathcal{H}) , (T', \mathcal{H}') of G are said to be *infinitesimally equivalent* if the representations (T_K, \mathcal{H}_K) , (T'_K, \mathcal{H}'_K) of $\mathfrak{U}(\mathfrak{g})$ are equivalent.

Assume G is simply connected (which is the case for $G = \mathrm{Sp}(1, n)$). It is a result of Harish-Chandra ([HC1], Theorem 9; see also [W1], pp. 330–331) that if (U, S) is an algebraically irreducible \mathfrak{k} -finite representation of $\mathfrak{U}(\mathfrak{g})$ and if S can be endowed with a positive definite Hermitian form $\langle \cdot, \cdot \rangle$ for which \mathfrak{g} acts on $(S, \langle \cdot, \cdot \rangle)$ via skew-adjoint operators, then there is a unique unitary irreducible representation \widetilde{T} of G on the Hilbert completion $\widetilde{\mathcal{H}}$ of S with respect to $\langle \cdot, \cdot \rangle$ so that $\widetilde{\mathcal{H}}_K = S$ and $\widetilde{T}_K = U$. We say in this case that (U, S) – or simply S if U is understood – is *unitarizable*. If, in particular, $(U, S) = (T_K, \mathcal{H}_K)$ for a K -finite irreducible representation (T, \mathcal{H}) of G , then (T, \mathcal{H}) and $(\widetilde{T}, \widetilde{\mathcal{H}})$ are infinitesimally equivalent. The converse is also obvious: if (T, \mathcal{H}) is an irreducible K -finite representation of G which is infinitesimally equivalent to a unitary Hilbert representation $(\widetilde{T}, \widetilde{\mathcal{H}})$ of G , then (T_K, \mathcal{H}_K) is unitarizable.

As we are going to show, the τ_l -spherical functions can be written as

$$\zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)E(\tau_l)] = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)]$$

for certain admissible irreducible Hilbert representations $(T_{l,s}, \mathcal{H}_{l,s})$ of $G = \operatorname{Sp}(1, n)$ satisfying $\dim \mathcal{H}_{l,s}(\tau_l) = d_l$ (for the second equality see e.g. [HC2], Lemma 1). The positive definite $\zeta_{l,s}$ can then be selected by applying the following theorem.

5.1. THEOREM ([Sak], Theorem 3; [B], I.4.8, p.44). *$\zeta_{l,s}$ is positive definite if and only if $(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation.*

Realize τ_l as a unitary representation on a $(2l+1)$ -dimensional Hilbert space V_l with inner product $\langle \cdot, \cdot \rangle_l$. For all $s \in \mathbf{C}$, define a representation $\theta_{l,s}$ of $P = MAN$ on V_l by

$$\theta_{l,s}(ma_t n) = e^{-(s-\rho)t} \tau_l(m).$$

Consider the representation $T'_{l,s} = \operatorname{Ind}_P^G(\theta_{l,s})$ of $G = \operatorname{Sp}(1, n)$: the representation space is the Hilbert completion $\mathcal{H}'_{l,s}$ of the set of the C^∞ functions $F: G \rightarrow V_l$ satisfying

$$F(gp) = \theta_{l,s}(p^{-1})F(g) = e^{(s-\rho)t} \tau_l(m^{-1})F(g), \quad g \in G, p = ma_t n \in P,$$

with respect to the inner product

$$(F_1, F_2)_l = \int_K \langle F_1(k), F_2(k) \rangle_l dk.$$

G acts according to

$$(T'_{l,s}(g)F)(g') = F(g^{-1}g'), \quad g, g' \in G.$$

$T'_{l,s}$ is admissible, but need not be irreducible.

The following lemma is a straightforward generalization of the result in Section 16, pp.526–528, of [Go]. We therefore omit its proof.

5.2. LEMMA. *For all $l \in \mathbf{N}/2$ and $s \in \mathbf{C}$, let $E'(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto its K -isotypic subspace of type τ_l . Then*

$$\zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E'(\tau_l)T'_{l,s}(g)].$$

The composition series structure and unitarity for the $T'_{l,s}$ have been determined by Howe and Tan with infinitesimal methods. In [HT], the results about the $T'_{l,s}$ are deduced from those obtained for a certain family of representations of $\mathrm{Sp}(1, n) \times \mathbf{H}^\times$ which are equivalent to $T'_{l,s} \otimes \tau_{l,s}$. Here $\mathbf{H}^\times = \mathbf{R}_+^\times \cdot \mathrm{Sp}(1)$ denotes the group of quaternionic dilations, acting on the space V_l of τ_l according to

$$\tau_{l,s}(h) = |h|^{s-\rho} \tau_l(h/|h|), \quad h \in \mathbf{H}^\times.$$

5.3. THEOREM ([HT], Theorem 5.6 and p.58).

1. $(\mathcal{H}'_{l,s})_K$ is equivalent as a $\mathfrak{U}(\mathfrak{g})$ -module to $(\mathcal{H}'_{l,-s})_K$.
2. $(\mathcal{H}'_{l,s})_K$ is a reducible $\mathfrak{U}(\mathfrak{g})$ -module if and only if $s \in \mathbf{Z}$, $s \equiv 2(l-n)+1 \pmod{2}$ and $s \notin (2l-\rho+2, -2l+\rho-2)$.
3. Suppose $(\mathcal{H}'_{l,s})_K$ irreducible. Then $(\mathcal{H}'_{l,s})_K$ is unitarizable if and only if one of the following two cases occurs:

$$(a) \ s = i\nu, \ \nu \in \mathbf{R}.$$

$$(b) \ s \in (2l-\rho+2, -2l+\rho-2).$$

Case (b) corresponds to the complementary series for $\mathrm{Sp}(1, n)$. They exist if and only if $2l < 2n-1$.

The fact that τ_l occurs exactly once in $T'_{l,s}|_K$ for the irreducible $T'_{l,s}$ is known a priori ([Go], Corollary to Theorem 8, p.522; [Dei], Theorem 3). The explicit K -module decomposition of $(\mathcal{H}'_{l,s})_K$ in [HT], pp.53–54, shows that this is actually true for all the $T'_{l,s}$. The K -submodule of $(\mathcal{H}'_{l,s})_K$ equivalent to τ_l is the only element in the “fiber of K -types” over the point $(0, 2l)$ in Diagrams 5.10 and 5.14 of [HT]. It is contained in a unique subquotient of $T'_{l,s}$, which can then be located in the diagrams used to determine the unitarizability of the various subquotients ([HT], pp.25 and 30). We therefore obtain the following proposition.

5.4. PROPOSITION. Suppose $(\mathcal{H}'_{l,s})_K$ is a reducible $\mathfrak{U}(\mathfrak{g})$ -module and assume $s \geq 0$. The irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ in which τ_l occurs is unitarizable if and only if $s \equiv 2(l-n)+1 \pmod{2}$ and $2l > s-\rho+4n-2$. That is, if and only if $2l \geq 2n-1$ and $s \in \{s_j = 2(l-n-j)+1 : j = 0, 1, \dots; s_j \geq 0\}$.

Let $(T_{l,s}, \mathcal{H}_{l,s})$ denote the subquotient representation of $T'_{l,s}$ corresponding to the irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ in which τ_l occurs. Then $T_{l,s}$ is an admissible Hilbert representation of $\mathrm{Sp}(1, n)$, and $T_{l,s}(g)v = T'_{l,s}(g)v$ for all $v \in \mathcal{H}'_{l,s}(\tau_l)$. Lemma 5.2 yields

5.5. COROLLARY. Let $E(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto the K -isotypic subspace of type τ_l . Then

$$(5.43) \quad \zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)].$$

$(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation if and only if the corresponding irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ is unitarizable. The following theorem is thus a consequence of Theorems 5.1 and 5.3 and of Proposition 5.4.

5.6. THEOREM. $\zeta_{l,s} = \zeta_{l,-s}$ is positive definite if and only if one of the following cases occurs:

1. $s = i\nu$, $\nu \in \mathbf{R}$.
2. If $2l \geq 2n - 1$: $\pm s = s_j := 2(l - n - j) + 1$ for integers $j \geq 0$ so that $s_j > 0$. (discrete series)
3. If $2l < 2n - 1$: $s \in (2l - \rho + 2, -2l + \rho - 2)$. (complementary series)

The situation for s real and nonnegative is represented in Figure 6.1.

6. THE τ_l -ABEL TRANSFORM

Proposition 3.2 proves that the τ_l -Abel transform is a $*$ -homomorphism of $\mathcal{D}(G; \chi_l)$ into the convolution algebra $\mathcal{D}_+(\mathbf{R})$ consisting of the even C^∞ functions on \mathbf{R} with compact support. The main theorem of this section states that the τ_l -Abel transform is also a bijection of $\mathcal{D}(G; \chi_l)$ onto $\mathcal{D}_+(\mathbf{R})$, and gives a formula for its inverse.

Identify A with \mathbf{R} under the map $t \mapsto a_t$. Restriction to A then identifies $\mathcal{D}(G; \chi_l)$ with $\mathcal{D}_+(\mathbf{R})$. Let $\mathcal{D}([1, \infty))$ denote the set of the compactly supported C^∞ functions on $[1, \infty)$ (right differentiability at 1 is considered). Define a map H by

$$(Hf)(\cosh t) := f(a_t) \equiv f(t)$$