

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HARMONIC ANALYSIS ON VECTOR BUNDLES OVER $Sp(1,n)/Sp(1) \times Sp(n)$
Autor: van Dijk, G. / PASQUALE, A.
Kapitel: 5. The positive definite τ_1 -spherical functions
DOI: <https://doi.org/10.5169/seals-64447>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 26.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

4.4. COROLLARY. *The τ_l -spherical functions are exactly the functions $\{\zeta_{l,s} : s \in \mathbf{C}\}$ given by Formulas (3.24) and (4.39). Further, $\zeta_{l,s}$ satisfies $\zeta_{l,s}(g) = \zeta_{l,s}(g^{-1})$ for all $g \in G$. Moreover, $\zeta_{l,s} = \zeta_{l,s'}$ if and only if $s = \pm s'$.*

The functional equation (3.15) with $g_1 = a_t$ and $g_2 = a_\tau$ becomes (cf. [T2], Théorème 1, p.227)

$$(4.41) \quad \zeta_{l,s}(t)\zeta_{l,s}(\tau) = \int_0^\infty K_l(t, \tau, u)\zeta_{l,s}(u)\Delta(u) du$$

where Δ is as in (1.7) and the kernel $K_l(t, \tau, u)$ is defined as follows. Set

$$B := \frac{\cosh^2 t + \cosh^2 \tau + \cosh^2 u - 1}{2 \cosh t \cosh \tau \cosh u}.$$

Then

$$(4.42) \quad K_l(t, \tau, u) := \frac{2^{-2\rho}\Gamma(2n)}{\sqrt{\pi}\Gamma(2n - \frac{1}{2})} \frac{(\cosh t \cosh \tau \cosh u)^{2n-3}}{(\sinh t \sinh \tau \sinh u)^{4n-2}} (1 - B^2)^{2n-\frac{3}{2}} \\ \times F\left(2n + 2l, 2n - 2l - 2; 2n - \frac{1}{2}; \frac{1}{2}(1 - B)\right)$$

if $B < 1$, and $K_l(t, \tau, u) := 0$ if $B \geq 1$. Using (4.39) and Formula (7.11) in [K2], one can prove that (4.41) holds also outside our group-theoretical setting for all $l \in \mathbf{R}$ satisfying $2n - 1 > 2l \geq 0$.

5. THE POSITIVE DEFINITE τ_l -SPHERICAL FUNCTIONS

A continuous function ζ on a locally compact group G is said to be *positive definite* if for every $f \in C_c(G)$

$$\int_G \int_G \zeta(x^{-1}y)f(x)\overline{f(y)} dx dy \geq 0.$$

In this section we establish which among the $\zeta_{l,s}$ are positive definite.

Let us first introduce some notation and recall some definitions. Let G be a semisimple Lie group with finite center, and let K be a maximal compact subgroup of G . \mathfrak{g} and \mathfrak{k} ($\subset \mathfrak{g}$) are the Lie algebras of G and K , respectively. A (strongly continuous) representation T of G on a Banach space \mathcal{H} is denoted by (T, \mathcal{H}) . We may simply speak of the representation T if \mathcal{H} is understood. Irreducibility for T always means topological irreducibility (= no closed proper invariant subspaces). Let \widehat{K} denote the set of equivalence classes

of finite dimensional irreducible representations of K . We say that $\tau \in \widehat{K}$ occurs in $T|_K$ if there exists a finite dimensional $T|_K$ -invariant subspace V of \mathcal{H} so that $(T|_K, V) \in \tau$. The linear span of all these subspaces V is the K -isotypic subspace of \mathcal{H} of type τ , denoted $\mathcal{H}(\tau)$. If d_τ is the dimension of τ and χ_τ is its character, then

$$E_T(\tau) = d_\tau \int_K T(k^{-1})\chi_\tau(k) dk$$

is a continuous projection of \mathcal{H} onto $\mathcal{H}(\tau)$. We set $\mathcal{H}_K = \sum_{\tau \in \widehat{K}} \mathcal{H}(\tau)$. T is said to be K -finite if $\dim \mathcal{H}(\tau) < \infty$ for all $\tau \in \widehat{K}$. A Hilbert representation (T, \mathcal{H}) is said to be admissible if it is K -finite and if $T|_K$ acts on \mathcal{H} by unitary operators.

A representation U of an (associative or Lie) algebra \mathcal{A} on a \mathbf{C} -vector space E is denoted (U, E) . The term \mathcal{A} -module is also used. Irreducibility for U always means algebraic irreducibility (= no proper invariant subspaces). Let $\widehat{\mathfrak{k}}_{\mathbf{C}}$ denote the set of equivalence classes of finite dimensional simple $\mathfrak{k}_{\mathbf{C}}$ -modules. The sum of all simple $\mathfrak{k}_{\mathbf{C}}$ -submodules of E which are in the class $\delta \in \widehat{\mathfrak{k}}_{\mathbf{C}}$ is denoted by $E(\delta)$. (U, E) is said \mathfrak{k} -finite if $\dim E(\delta) < \infty$ for all $\delta \in \widehat{\mathfrak{k}}_{\mathbf{C}}$ and if $E = \sum_{\delta \in \widehat{\mathfrak{k}}_{\mathbf{C}}} E(\delta)$.

Every K -finite irreducible representation (T, \mathcal{H}) of G induces a \mathfrak{k} -finite irreducible representation (T_K, \mathcal{H}_K) of $\mathfrak{U}(\mathfrak{g})$ by differentiation. If, moreover, \mathcal{H} is Hilbert and T is unitary, then \mathfrak{g} acts on \mathcal{H}_K by skew-adjoint operators: $\langle T_K(X)\varphi, \psi \rangle = -\langle \varphi, T_K(X)\psi \rangle$ for all $X \in \mathfrak{g}$ and all $\varphi, \psi \in \mathcal{H}_K$. Two K -finite representations $(T, \mathcal{H}), (T', \mathcal{H}')$ of G are said to be *infinitesimally equivalent* if the representations $(T_K, \mathcal{H}_K), (T'_K, \mathcal{H}'_K)$ of $\mathfrak{U}(\mathfrak{g})$ are equivalent.

Assume G is simply connected (which is the case for $G = \mathrm{Sp}(1, n)$). It is a result of Harish-Chandra ([HC1], Theorem 9; see also [W1], pp. 330–331) that if (U, S) is an algebraically irreducible \mathfrak{k} -finite representation of $\mathfrak{U}(\mathfrak{g})$ and if S can be endowed with a positive definite Hermitian form $\langle \cdot, \cdot \rangle$ for which \mathfrak{g} acts on $(S, \langle \cdot, \cdot \rangle)$ via skew-adjoint operators, then there is a unique unitary irreducible representation \widetilde{T} of G on the Hilbert completion $\widetilde{\mathcal{H}}$ of S with respect to $\langle \cdot, \cdot \rangle$ so that $\widetilde{\mathcal{H}}_K = S$ and $\widetilde{T}_K = U$. We say in this case that (U, S) – or simply S if U is understood – is *unitarizable*. If, in particular, $(U, S) = (T_K, \mathcal{H}_K)$ for a K -finite irreducible representation (T, \mathcal{H}) of G , then (T, \mathcal{H}) and $(\widetilde{T}, \widetilde{\mathcal{H}})$ are infinitesimally equivalent. The converse is also obvious: if (T, \mathcal{H}) is an irreducible K -finite representation of G which is infinitesimally equivalent to a unitary Hilbert representation $(\widetilde{T}, \widetilde{\mathcal{H}})$ of G , then (T_K, \mathcal{H}_K) is unitarizable.

As we are going to show, the τ_l -spherical functions can be written as

$$\zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)E(\tau_l)] = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)]$$

for certain admissible irreducible Hilbert representations $(T_{l,s}, \mathcal{H}_{l,s})$ of $G = \operatorname{Sp}(1, n)$ satisfying $\dim \mathcal{H}_{l,s}(\tau_l) = d_l$ (for the second equality see e.g. [HC2], Lemma 1). The positive definite $\zeta_{l,s}$ can then be selected by applying the following theorem.

5.1. THEOREM ([Sak], Theorem 3; [B], I.4.8, p.44). *$\zeta_{l,s}$ is positive definite if and only if $(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation.*

Realize τ_l as a unitary representation on a $(2l + 1)$ -dimensional Hilbert space V_l with inner product $\langle \cdot, \cdot \rangle_l$. For all $s \in \mathbf{C}$, define a representation $\theta_{l,s}$ of $P = MAN$ on V_l by

$$\theta_{l,s}(ma_t n) = e^{-(s-\rho)t} \tau_l(m).$$

Consider the representation $T'_{l,s} = \operatorname{Ind}_P^G(\theta_{l,s})$ of $G = \operatorname{Sp}(1, n)$: the representation space is the Hilbert completion $\mathcal{H}'_{l,s}$ of the set of the C^∞ functions $F: G \rightarrow V_l$ satisfying

$$F(gp) = \theta_{l,s}(p^{-1})F(g) = e^{(s-\rho)t} \tau_l(m^{-1})F(g), \quad g \in G, p = ma_t n \in P,$$

with respect to the inner product

$$(F_1, F_2)_l = \int_K \langle F_1(k), F_2(k) \rangle_l dk.$$

G acts according to

$$(T'_{l,s}(g)F)(g') = F(g^{-1}g'), \quad g, g' \in G.$$

$T'_{l,s}$ is admissible, but need not be irreducible.

The following lemma is a straightforward generalization of the result in Section 16, pp. 526–528, of [Go]. We therefore omit its proof.

5.2. LEMMA. *For all $l \in \mathbf{N}/2$ and $s \in \mathbf{C}$, let $E'(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto its K -isotypic subspace of type τ_l . Then*

$$\zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E'(\tau_l)T'_{l,s}(g)].$$

The composition series structure and unitarity for the $T'_{l,s}$ have been determined by Howe and Tan with infinitesimal methods. In [HT], the results about the $T'_{l,s}$ are deduced from those obtained for a certain family of representations of $\mathrm{Sp}(1, n) \times \mathbf{H}^\times$ which are equivalent to $T'_{l,s} \otimes \tau_{l,s}$. Here $\mathbf{H}^\times = \mathbf{R}_+^\times \cdot \mathrm{Sp}(1)$ denotes the group of quaternionic dilations, acting on the space V_l of τ_l according to

$$\tau_{l,s}(h) = |h|^{s-\rho} \tau_l(h/|h|), \quad h \in \mathbf{H}^\times.$$

5.3. THEOREM ([HT], Theorem 5.6 and p.58).

1. $(\mathcal{H}'_{l,s})_K$ is equivalent as a $\mathfrak{U}(\mathfrak{g})$ -module to $(\mathcal{H}'_{l,-s})_K$.
2. $(\mathcal{H}'_{l,s})_K$ is a reducible $\mathfrak{U}(\mathfrak{g})$ -module if and only if $s \in \mathbf{Z}$, $s \equiv 2(l-n)+1 \pmod{2}$ and $s \notin (2l - \rho + 2, -2l + \rho - 2)$.
3. Suppose $(\mathcal{H}'_{l,s})_K$ irreducible. Then $(\mathcal{H}'_{l,s})_K$ is unitarizable if and only if one of the following two cases occurs:

- (a) $s = i\nu$, $\nu \in \mathbf{R}$.
- (b) $s \in (2l - \rho + 2, -2l + \rho - 2)$.

Case (b) corresponds to the complementary series for $\mathrm{Sp}(1, n)$. They exist if and only if $2l < 2n - 1$.

The fact that τ_l occurs exactly once in $T'_{l,s}|_K$ for the irreducible $T'_{l,s}$ is known a priori ([Go], Corollary to Theorem 8, p.522; [Dei], Theorem 3). The explicit K -module decomposition of $(\mathcal{H}'_{l,s})_K$ in [HT], pp.53–54, shows that this is actually true for all the $T'_{l,s}$. The K -submodule of $(\mathcal{H}'_{l,s})_K$ equivalent to τ_l is the only element in the “fiber of K -types” over the point $(0, 2l)$ in Diagrams 5.10 and 5.14 of [HT]. It is contained in a unique subquotient of $T'_{l,s}$, which can then be located in the diagrams used to determine the unitarizability of the various subquotients ([HT], pp.25 and 30). We therefore obtain the following proposition.

5.4. PROPOSITION. Suppose $(\mathcal{H}'_{l,s})_K$ is a reducible $\mathfrak{U}(\mathfrak{g})$ -module and assume $s \geq 0$. The irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ in which τ_l occurs is unitarizable if and only if $s \equiv 2(l-n)+1 \pmod{2}$ and $2l > s - \rho + 4n - 2$. That is, if and only if $2l \geq 2n - 1$ and $s \in \{s_j = 2(l-n-j)+1 : j = 0, 1, \dots; s_j \geq 0\}$.

Let $(T_{l,s}, \mathcal{H}_{l,s})$ denote the subquotient representation of $T'_{l,s}$ corresponding to the irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ in which τ_l occurs. Then $T_{l,s}$ is an admissible Hilbert representation of $\mathrm{Sp}(1, n)$, and $T_{l,s}(g)v = T'_{l,s}(g)v$ for all $v \in \mathcal{H}'_{l,s}(\tau_l)$. Lemma 5.2 yields

5.5. COROLLARY. *Let $E(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto the K -isotypic subspace of type τ_l . Then*

$$(5.43) \quad \zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)].$$

$(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation if and only if the corresponding irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ is unitarizable. The following theorem is thus a consequence of Theorems 5.1 and 5.3 and of Proposition 5.4.

5.6. THEOREM. $\zeta_{l,s} = \zeta_{l,-s}$ is positive definite if and only if one of the following cases occurs:

1. $s = i\nu$, $\nu \in \mathbf{R}$.
2. If $2l \geq 2n - 1$: $\pm s = s_j := 2(l - n - j) + 1$ for integers $j \geq 0$ so that $s_j > 0$. (discrete series)
3. If $2l < 2n - 1$: $s \in (2l - \rho + 2, -2l + \rho - 2)$. (complementary series)

The situation for s real and nonnegative is represented in Figure 6.1.

6. THE τ_l -ABEL TRANSFORM

Proposition 3.2 proves that the τ_l -Abel transform is a $*$ -homomorphism of $\mathcal{D}(G; \chi_l)$ into the convolution algebra $\mathcal{D}_+(\mathbf{R})$ consisting of the even C^∞ functions on \mathbf{R} with compact support. The main theorem of this section states that the τ_l -Abel transform is also a bijection of $\mathcal{D}(G; \chi_l)$ onto $\mathcal{D}_+(\mathbf{R})$, and gives a formula for its inverse.

Identify A with \mathbf{R} under the map $t \mapsto a_t$. Restriction to A then identifies $\mathcal{D}(G; \chi_l)$ with $\mathcal{D}_+(\mathbf{R})$. Let $\mathcal{D}([1, \infty))$ denote the set of the compactly supported C^∞ functions on $[1, \infty)$ (right differentiability at 1 is considered). Define a map H by

$$(Hf)(\cosh t) := f(a_t) \equiv f(t)$$