5. The positive definite τ_l-spherical functions

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4.4. COROLLARY. *The τ_l-spherical functions are exactly the functions $\{\zeta_{l,s} : s \in \mathbf{C}\}$ given by Formulas (3.24) and (4.39). Further, $\zeta_{l,s}$ satisfies $\zeta_{l,s}(g) = \zeta_{l,s}(g^{-1})$ for all $g \in G$. Moreover, $\zeta_{l,s} = \zeta_{l,s'}$ if and only if $s = \pm s'$.*

The functional equation (3.15) with $g_1 = a_t$ and $g_2 = a_\tau$ becomes (cf. [T2], Théorème 1, p. 227)

$$\zeta_{l,s}(t)\zeta_{l,s}(\tau) = \int_0^\infty K_l(t,\tau,u)\zeta_{l,s}(u)\Delta(u)\,du \tag{4.41}$$

where Δ is as in (1.7) and the kernel $K_l(t,\tau,u)$ is defined as follows. Set

$$B := \frac{\cosh^2 t + \cosh^2 \tau + \cosh^2 u - 1}{2\cosh t \cosh \tau \cosh u}\,.$$

Then

$$K_l(t,\tau,u) := \frac{2^{-2\rho}\Gamma(2n)}{\sqrt{\pi}\Gamma(2n-\frac{1}{2})}\,\frac{(\cosh t\cosh \tau \cosh u)^{2n-3}}{(\sinh t \sinh \tau \sinh u)^{4n-2}}\,(1-B^2)^{2n-\frac{3}{2}} \times F\Big(2n+2l, 2n-2l-2; 2n-\frac{1}{2}; \frac{1}{2}(1-B)\Big) \tag{4.42}$$

if $B < 1$, and $K_l(t,\tau,u) := 0$ if $B \geq 1$. Using (4.39) and Formula (7.11) in [K2], one can prove that (4.41) holds also outside our group-theoretical setting for all $l \in \mathbf{R}$ satisfying $2n-1 > 2l \geq 0$.

5. THE POSITIVE DEFINITE τ_l-SPHERICAL FUNCTIONS

A continuous function ζ on a locally compact group G is said to be *positive definite* if for every $f \in C_c(G)$

$$\int_G \int_G \zeta(x^{-1}y)f(x)\overline{f(y)}\,dx\,dy \geq 0\,.$$

In this section we establish which among the $\zeta_{l,s}$ are positive definite.

Let us first introduce some notation and recall some definitions. Let G be a semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. $\mathfrak{g}$ and $\mathfrak{k}$ ($\subset \mathfrak{g}$) are the Lie algebras of G and K, respectively. A (strongly continuous) representation T of G on a Banach space $\mathcal{H}$ is denoted by $(T, \mathcal{H})$. We may simply speak of the representation T if $\mathcal{H}$ is understood. Irreducibility for T always means topological irreducibility (= no closed proper invariant subspaces). Let $\widehat{K}$ denote the set of equivalence classes of finite dimensional irreducible representations of K. We say that $\tau\in\widehat{K}$ occurs in $T|_K$ if there exists a finite dimensional $T|_K$-invariant subspace V of $\mathcal{H}$ so that $(T|_K, V)\in\tau$. The linear span of all these subspaces V is the K-isotypic subspace of $\mathcal{H}$ of type τ, denoted $\mathcal{H}(\tau)$. If d_τ is the dimension of τ and χ_τ is its character, then

$$E_T(\tau)=d_\tau\int_K T(k^{-1})\chi_\tau(k)\,dk$$

is a continuous projection of $\mathcal{H}$ onto $\mathcal{H}(\tau)$. We set $\mathcal{H}_K=\sum_{\tau\in K}\mathcal{H}(\tau)$. T is said to be K-finite if $\dim\mathcal{H}(\tau)<\infty$ for all $\tau\in\widehat{K}$. A Hilbert representation $(T,\mathcal{H})$ is said to be admissible if it is K-finite and if $T|_K$ acts on $\mathcal{H}$ by unitary operators.

A representation U of an (associative or Lie) algebra $\mathcal{A}$ on a $\mathbf{C}$-vector space E is denoted (U,E). The term $\mathcal{A}$-module is also used. Irreducibility for U always means algebraic irreducibility (= no proper invariant subspaces). Let $\widehat{\mathfrak{k}_{\mathbf{C}}}$ denote the set of equivalence classes of finite dimensional simple $\mathfrak{k}_{\mathbf{C}}$-modules. The sum of all simple $\mathfrak{k}_{\mathbf{C}}$-submodules of E which are in the class $\delta\in\widehat{\mathfrak{k}_{\mathbf{C}}}$ is denoted by $E(\delta)$. (U,E) is said $\mathfrak{k}$-finite if $\dim E(\delta)<\infty$ for all $\delta\in\widehat{\mathfrak{k}_{\mathbf{C}}}$ and if $E=\sum_{\delta\in\widehat{\mathfrak{k}_{\mathbf{C}}}}E(\delta)$.

Every K-finite irreducible representation $(T,\mathcal{H})$ of G induces a $\mathfrak{k}$-finite irreducible representation $(T_K,\mathcal{H}_K)$ of $\mathfrak{U}(\mathfrak{g})$ by differentiation. If, moreover, $\mathcal{H}$ is Hilbert and T is unitary, then $\mathfrak{g}$ acts on $\mathcal{H}_K$ by skew-adjoint operators: $\langle T_K(X)\varphi,\psi\rangle=-\langle\varphi,T_K(X)\psi\rangle$ for all $X\in\mathfrak{g}$ and all $\varphi,\psi\in\mathcal{H}_K$. Two K-finite representations $(T,\mathcal{H})$, $(T',\mathcal{H}')$ of G are said to be *infinitesimally equivalent* if the representations $(T_K,\mathcal{H}_K)$, $(T'_K,\mathcal{H}'_K)$ of $\mathfrak{U}(\mathfrak{g})$ are equivalent.

Assume G is simply connected (which is the case for $G=\mathrm{Sp}(1,n)$). It is a result of Harish-Chandra ([HC1], Theorem 9; see also [W1], pp. 330–331) that if (U,S) is an algebraically irreducible $\mathfrak{k}$-finite representation of $\mathfrak{U}(\mathfrak{g})$ and if S can be endowed with a positive definite Hermitian form $\langle\cdot,\cdot\rangle$ for which $\mathfrak{g}$ acts on $(S,\langle\cdot,\cdot\rangle)$ via skew-adjoint operators, then there is a unique unitary irreducible representation $\widetilde{T}$ of G on the Hilbert completion $\widetilde{\mathcal{H}}$ of S with respect to $\langle\cdot,\cdot\rangle$ so that $\widetilde{\mathcal{H}}_K=S$ and $\widetilde{T}_K=U$. We say in this case that (U,S) – or simply S if U is understood – is *unitarizable*. If, in particular, $(U,S)=(T_K,\mathcal{H}_K)$ for a K-finite irreducible representation $(T,\mathcal{H})$ of G, then $(T,\mathcal{H})$ and $(\widetilde{T},\widetilde{\mathcal{H}})$ are infinitesimally equivalent. The converse is also obvious: if $(T,\mathcal{H})$ is an irreducible K-finite representation of G which is infinitesimally equivalent to a unitary Hilbert representation $(\widetilde{T},\widetilde{\mathcal{H}})$ of G, then $(T_K,\mathcal{H}_K)$ is unitarizable.

As we are going to show, the τ_l-spherical functions can be written as

$$\zeta_{l,s}(g) = \frac{1}{d_l}\,\mathrm{tr}[E(\tau_l)T_{l,s}(g)E(\tau_l)] = \frac{1}{d_l}\,\mathrm{tr}[E(\tau_l)T_{l,s}(g)]$$

for certain admissible irreducible Hilbert representations $(T_{l,s}, \mathcal{H}_{l,s})$ of $G = \mathrm{Sp}(1,n)$ satisfying $\dim \mathcal{H}_{l,s}(\tau_l) = d_l$ (for the second equality see e.g. [HC2], Lemma 1). The positive definite $\zeta_{l,s}$ can then be selected by applying the following theorem.

5.1. THEOREM ([Sak], Theorem 3; [B], I.4.8, p. 44). *$\zeta_{l,s}$ is positive definite if and only if $(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation.*

Realize τ_l as a unitary representation on a $(2l+1)$-dimensional Hilbert space V_l with inner product $\langle\cdot,\cdot\rangle_l$. For all $s \in \mathbf{C}$, define a representation $\theta_{l,s}$ of $P = MAN$ on V_l by

$$\theta_{l,s}(ma_tn) = e^{-(s-\rho)t}\tau_l(m)\,.$$

Consider the representation $T'_{l,s} = \mathrm{Ind}_P^G(\theta_{l,s})$ of $G = \mathrm{Sp}(1,n)$: the representation space is the Hilbert completion $\mathcal{H}'_{l,s}$ of the set of the C^∞ functions $F\colon G \to V_l$ satisfying

$$F(gp) = \theta_{l,s}(p^{-1})F(g) = e^{(s-\rho)t}\tau_l(m^{-1})F(g)\,, \qquad g \in G,\ p = ma_tn \in P\,,$$

with respect to the inner product

$$(F_1, F_2)_l = \int_K \langle F_1(k), F_2(k)\rangle_l\, dk\,.$$

G acts according to

$$(T'_{l,s}(g)F)(g') = F(g^{-1}g'), \qquad g, g' \in G\,.$$

$T'_{l,s}$ is admissible, but need not be irreducible.

The following lemma is a straightforward generalization of the result in Section 16, pp. 526–528, of [Go]. We therefore omit its proof.

5.2. LEMMA. *For all $l \in \mathbf{N}/2$ and $s \in \mathbf{C}$, let $E'(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto its K-isotypic subspace of type τ_l. Then*

$$\zeta_{l,s}(g) = \frac{1}{d_l}\,\mathrm{tr}[E'(\tau_l)T'_{l,s}(g)]\,.$$

The composition series structure and unitarity for the $T'_{l,s}$ have been determined by Howe and Tan with infinitesimal methods. In [HT], the results about the $T'_{l,s}$ are deduced from those obtained for a certain family of representations of $\mathrm{Sp}(1,n) \times \mathbf{H}^\times$ which are equivalent to $T'_{l,s} \otimes \tau_{l,s}$. Here $\mathbf{H}^\times = \mathbf{R}_+^\times \cdot \mathrm{Sp}(1)$ denotes the group of quaternionic dilations, acting on the space V_l of τ_l according to

$$\tau_{l,s}(h) = |h|^{s-\rho}\tau_l\left(h/|h|\right), \qquad h \in \mathbf{H}^\times .$$

5.3. THEOREM ([HT], Theorem 5.6 and p. 58).
1. *$(\mathcal{H}'_{l,s})_K$ is equivalent as a $\mathfrak{U}(\mathfrak{g})$-module to $(\mathcal{H}'_{l,-s})_K$.*

2. *$(\mathcal{H}'_{l,s})_K$ is a reducible $\mathfrak{U}(\mathfrak{g})$-module if and only if $s \in \mathbf{Z}$, $s \equiv 2(l-n)+1$ (mod 2) and $s \notin (2l-\rho+2, -2l+\rho-2)$.*

3. *Suppose $(\mathcal{H}'_{l,s})_K$ irreducible. Then $(\mathcal{H}'_{l,s})_K$ is unitarizable if and only if one of the following two cases occurs:*

(a) $s = i\nu$, $\nu \in \mathbf{R}$.

(b) $s \in (2l-\rho+2, -2l+\rho-2)$.

Case (b) corresponds to the complementary series for Sp(1, n). *They exist if and only if $2l < 2n-1$.*

The fact that τ_l occurs exactly once in $T'_{l,s}|_K$ for the irreducible $T'_{l,s}$ is known a priori ([Go], Corollary to Theorem 8, p. 522; [Dei], Theorem 3). The explicit K-module decomposition of $(\mathcal{H}'_{l,s})_K$ in [HT], pp. 53–54, shows that this is actually true for all the $T'_{l,s}$. The K-submodule of $(\mathcal{H}'_{l,s})_K$ equivalent to τ_l is the only element in the "fiber of K-types" over the point $(0, 2l)$ in Diagrams 5.10 and 5.14 of [HT]. It is contained in a unique subquotient of $T'_{l,s}$, which can then be located in the diagrams used to determine the unitarizability of the various subquotients ([HT], pp. 25 and 30). We therefore obtain the following proposition.

5.4. PROPOSITION. *Suppose $(\mathcal{H}'_{l,s})_K$ is a reducible $\mathfrak{U}(\mathfrak{g})$-module and assume $s \geq 0$. The irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ in which τ_l occurs is unitarizable if and only if $s \equiv 2(l-n)+1$ (mod 2) and $2l > s-\rho+4n-2$. That is, if and only if $2l \geq 2n-1$ and $s \in \{s_j = 2(l-n-j)+1 : j = 0, 1, \ldots; s_j \geq 0\}$.*

Let $(T_{l,s}, \mathcal{H}_{l,s})$ denote the subquotient representation of $T'_{l,s}$ corresponding to the irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ in which τ_l occurs. Then $T_{l,s}$ is an admissible Hilbert representation of Sp(1, n), and $T_{l,s}(g)v = T'_{l,s}(g)v$ for all $v \in \mathcal{H}'_{l,s}(\tau_l)$. Lemma 5.2 yields

5.5. COROLLARY. *Let $E(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto the K-isotypic subspace of type τ_l. Then*

$$\zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)]\,. \tag{5.43}$$

$(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation if and only if the corresponding irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ is unitarizable. The following theorem is thus a consequence of Theorems 5.1 and 5.3 and of Proposition 5.4.

5.6. THEOREM. *$\zeta_{l,s} = \zeta_{l,-s}$ is positive definite if and only if one of the following cases occurs:*

1. $s = i\nu, \;\; \nu \in \mathbf{R}$.
2. *If* $2l \geq 2n-1$: $\;\pm s = s_j := 2(l-n-j)+1$ *for integers* $j \geq 0$ *so that* $s_j > 0$. (*discrete series*)
3. *If* $2l < 2n-1$: $\;s \in (2l-\rho+2, -2l+\rho-2)$. (*complementary series*)

The situation for s real and nonnegative is represented in Figure 6.1.

6. THE τ_l-ABEL TRANSFORM

Proposition 3.2 proves that the τ_l-Abel transform is a *-homomorphism of $\mathcal{D}(G;\chi_l)$ into the convolution algebra $\mathcal{D}_+(\mathbf{R})$ consisting of the even C^∞ functions on $\mathbf{R}$ with compact support. The main theorem of this section states that the τ_l-Abel transform is also a bijection of $\mathcal{D}(G;\chi_l)$ onto $\mathcal{D}_+(\mathbf{R})$, and gives a formula for its inverse.

Identify A with $\mathbf{R}$ under the map $t \mapsto a_t$. Restriction to A then identifies $\mathcal{D}(G;\chi_l)$ with $\mathcal{D}_+(\mathbf{R})$. Let $\mathcal{D}([1,\infty))$ denote the set of the compactly supported C^∞ functions on $[1,\infty)$ (right differentiability at 1 is considered). Define a map H by

$$(Hf)(\cosh t) := f(a_t) \equiv f(t)$$