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### 3. THE $\tau_l$ -SPHERICAL FUNCTIONS

Let  $\mathfrak{U}(\mathfrak{g})$  be the universal enveloping algebra of the complexification  $\mathfrak{g}_{\mathbb{C}}$  of  $\mathfrak{g} = \mathfrak{sp}(1, n)$ . The elements of  $\mathfrak{U}(\mathfrak{g})$  are considered as left-invariant differential operators on  $G = \mathrm{Sp}(1, n)$  acting on  $C^\infty$  functions on the right:

$$f(g; X) := \left. \frac{d}{dt} f(g \exp tX) \right|_{t=0} \quad (f \in C^\infty(G), X \in \mathfrak{g}, g \in G).$$

We adopt Harish-Chandra's notation  $f(g; D)$  for the image of  $f \in C^\infty(G)$  under the right action of  $D \in \mathfrak{U}(\mathfrak{g})$ . The set of  $K$ -invariant elements of  $\mathfrak{U}(\mathfrak{g})$  is denoted by  $\mathfrak{U}(\mathfrak{g})^K$ .

3.1. THEOREM ([Go], Theorems 8, 10 and 14; [GaV], Theorems 1.3.14, 1.4.5 and Proposition 1.4.4). *Let  $l \in \mathbf{N}/2$  be fixed. Let  $\zeta$  be a complex-valued continuous function on  $G$  satisfying (2.9), (2.10) and  $\zeta(e) = 1$  ( $e$  is the unit element of  $G$ ). The following statements are mutually equivalent.*

1. *The mapping  $f \mapsto \int_G f(g) \zeta(g) dg$  is an algebra homomorphism of  $\mathcal{D}(G; \chi_l)$  into  $\mathbb{C}$ .*

2.  *$\zeta$  satisfies the functional equation*

$$(3.15) \quad \int_K \zeta(kg_1k^{-1}g_2) dk = \zeta(g_1)\zeta(g_2)$$

*for all  $g_1, g_2 \in G$ .*

3.  *$\zeta$  is a common eigenfunction of the elements of  $\mathfrak{U}(\mathfrak{g})^K$ .*

A function  $\zeta$  satisfying the equivalent conditions of Theorem 3.1 is called a *spherical function of type  $\tau_l$  (and height 1)* or briefly a  *$\tau_l$ -spherical function*<sup>1)</sup>.

Observe that Condition 3 implies in particular that every  $\tau_l$ -spherical function is analytic on  $G$  because  $\mathfrak{U}(\mathfrak{g})^K$  contains an elliptic differential operator.

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<sup>1)</sup> Our definition of  $\tau_l$ -spherical function  $\zeta$  is formally less restrictive than Godement's, which also requires  $\zeta$  to be quasi-bounded with respect to some seminorm on  $G$  (cf. [Go], p. 519). It is possible to show (e.g. [GaV], Theorem 1.3.14) that each of the conditions for  $\zeta$  given by Theorem 3.1 is equivalent to the existence of an irreducible Fréchet representation  $(T, \mathcal{H})$  – with a  $d_l$ -dimensional  $K$ -isotypic subspace  $\mathcal{H}(\tau_l)$  of type  $\tau_l$  – for which

$$(*) \quad \zeta = \frac{1}{d_l} \mathrm{tr}[E(\tau_l)TE(\tau_l)],$$

$E(\tau_l)$  being the projection of  $\mathcal{H}$  onto  $\mathcal{H}(\tau_l)$ .  $(T, \mathcal{H})$  can be chosen to be a Banach representation of  $G$  if and only if  $\zeta$  is quasi-bounded with respect to some seminorm on  $G$ . In Section 5 we will determine, for each  $\tau_l$ -spherical function  $\zeta$ , an irreducible *Hilbert* representation  $(T, \mathcal{H})$  for which  $(*)$  holds. It follows, in particular, that the condition of quasi-boundedness is, in our case, automatically satisfied.

For complex-valued functions  $f$  on  $G$  and  $F$  on  $\mathbf{R}$ , we set

$$\begin{aligned} f^*(g) &= \overline{f(g^{-1})} & (g \in G) \\ F^*(t) &= \overline{F(-t)} & (t \in \mathbf{R}). \end{aligned}$$

The  $\tau_l$ -Abel transform of  $f \in \mathcal{D}(G; \chi_l)$  is the function  $\mathcal{A}_l f$  on  $\mathbf{R}$  defined by

$$(3.16) \quad \mathcal{A}_l f(t) = \frac{1}{d_l^2} e^{\rho t} \int_N f(a_t n) dn.$$

Its properties are summarized in the following proposition.

3.2. PROPOSITION. For all  $f \in \mathcal{D}(G; \chi_l)$ ,  $\mathcal{A}_l f$  is a  $C^\infty$  function on  $\mathbf{R}$  with compact support. If  $f, f_1, f_2 \in \mathcal{D}(G; \chi_l)$  and  $a_1, a_2 \in \mathbf{C}$ , then

$$(3.17) \quad (\mathcal{A}_l f)^* = \mathcal{A}_l(f^*),$$

$$(3.18) \quad \mathcal{A}_l(a_1 f_1 + a_2 f_2) = a_1 \mathcal{A}_l f_1 + a_2 \mathcal{A}_l f_2,$$

$$(3.19) \quad \mathcal{A}_l(f_1 * f_2) = \mathcal{A}_l f_1 * \mathcal{A}_l f_2.$$

Formula (3.17) is equivalent to the fact that  $\mathcal{A}_l f$  is an even function.

*Proof.* Formulas (3.17)–(3.19) are immediately proven by passing to  $\mathcal{D}(G; \tau_l)$ . For the last statement, recall that  $f(g^{-1}) = f(g)$  for  $f \in \mathcal{D}(G; \chi_l)$ .  $\square$

The following lemma relates our definition of  $\mathcal{A}_l$  to the definition often found in the literature (cf. e.g. [W2], p. 34).

3.3. LEMMA. For  $f \in \mathcal{D}(G; \chi_l)$  one has

$$\int_N f(ka_t n) dn = \frac{1}{d_l} \chi_l(k) \int_N f(a_t n) dn.$$

*Proof.* Let  $F \in \mathcal{D}(G; \tau_l)$ . Then  $\int_N F(a_t n) dn$  commutes with  $\tau(m)$  ( $m \in M$ ) so with  $\tau(k_1)$  ( $k_1 \in K_1$ ), hence is a scalar multiple of the identity. The lemma follows by taking traces.  $\square$

We now use the  $\tau_l$ -Abel transform to construct  $\tau_l$ -spherical functions. Because of Proposition 3.2, for any complex number  $s$ , the map

$$(3.20) \quad \lambda_s : f \longmapsto \int_{-\infty}^{\infty} \mathcal{A}_l f(t) e^{-st} dt$$

is an algebra homomorphism of  $\mathcal{D}(G; \chi_l)$  into  $\mathbf{C}$ .

Set

$$(3.21) \quad \alpha_{l,s}(ka_t n) = \frac{1}{d_l} \chi_l(k) e^{-(s+\rho)t}.$$

Since  $f = f * d_l \chi_l$  and  $\chi_l(k^{-1}) = \chi_l(k)$  for  $k \in K$ , for every  $f \in \mathcal{D}(G; \chi_l)$

$$(3.22) \quad \begin{aligned} \lambda_s(f) &= \frac{1}{d_l} \int_K \int_{-\infty}^{\infty} \int_N f(ka_t n) \chi_l(k) e^{(-s+\rho)t} dk dt dn \\ &= \int_G f(g) \alpha_{l,s}(g) dg \\ &= \int_G f(g) \int_K \alpha_{l,s}(kgk^{-1}) dk dg \\ &= \int_G f(g) \zeta_{l,s}(g) dg \end{aligned}$$

with

$$(3.23) \quad \zeta_{l,s} := \int_K \alpha_{l,s}(kgk^{-1}) dk.$$

One easily checks that  $\zeta_{l,s}$  satisfies  $\zeta_{l,s} = \zeta_{l,s}^0$ ,  $\zeta_{l,s} * d_l \chi_l = \zeta_{l,s}$  and  $\zeta_{l,s}(e) = 1$ . Thus  $\zeta_{l,s}$  is a  $\tau_l$ -spherical function. It will be shown in the next section that any  $\tau_l$ -spherical function is of the form (3.24).

By Remark 2.3, we have

$$(3.24) \quad \zeta_{l,s}(g) = \frac{1}{d_l} \chi_l(k_1) \zeta_{l,s}(a_t) \quad \text{for } g = k_1 k_2 a_t k_2',$$

so  $\zeta_{l,s}$  is uniquely determined by its restriction to  $A$ .

#### 4. THE DIFFERENTIAL EQUATION FOR THE $\tau_l$ -SPHERICAL FUNCTIONS

For a subalgebra  $\mathfrak{u}$  of  $\mathfrak{g}$ , let  $\mathfrak{u}_{\mathbf{C}}$  denote the complex subalgebra of  $\mathfrak{g}_{\mathbf{C}}$  generated by  $\mathfrak{u}$ . The universal enveloping algebra  $\mathfrak{U}(\mathfrak{u})$  of  $\mathfrak{u}_{\mathbf{C}}$  is considered as a subalgebra of  $\mathfrak{U}(\mathfrak{g})$ .

The representation  $\tau_l$  of  $K_1$  induces differentiated representations of the Lie algebra  $\mathfrak{k}_1$  of  $K_1$  and of the universal enveloping algebra  $\mathfrak{U}(\mathfrak{k}_1)$  of  $(\mathfrak{k}_1)_{\mathbf{C}}$ . We indicate these representations with the same letter  $\tau_l$ . Let  $\mathfrak{k}_2$  be the Lie algebra of  $K_2$ . Every element  $Y \in \mathfrak{k}_{\mathbf{C}}$  can be uniquely decomposed as