2. The convolution algebra $D(G; \lambda_l)$

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for $f\in C_c(G)$. We adopt the usual notation $C_c(G)$ for the space of continuous functions on G with compact support. In the above formulas, $dn = dw\,dz$ ($n=n(w,z)$) and dk is the normalized Haar measure on K.

Let

$$K_1=\left\{\begin{bmatrix}u&0\\0&I\end{bmatrix}:u\in\mathrm{Sp}(1)\right\},\qquad K_2=\left\{\begin{bmatrix}1&0\\0&U\end{bmatrix}:U\in\mathrm{Sp}(n)\right\}.$$

Then every $g\in G$ can be written as $g=k_1k_2a_tk_2'$ for uniquely determined $k_1\in K_1$, $t\geq 0$ and for some $k_2,k_2'\in K_2$. Writing $g=[g_{ij}]_{i,j=0}^n$, we have

$$k_1=\frac{g_{00}}{|g_{00}|}\quad\text{and}\quad\cosh t=|g_{00}|. \tag{1.5}$$

If $g\notin K$, then $t>0$ and k_2,k_2' are uniquely determined modulo the subgroup

$$\left\{\begin{bmatrix}1&0&0\\0&V&0\\0&0&1\end{bmatrix}:V\in\mathrm{Sp}(n-1)\right\}.$$

After dg is normalized according to (1.3), the corresponding integral formula is

$$\int_G f(g)\,dg=\frac{1}{2}\left(\frac{\pi}{4}\right)^{2n}\frac{1}{\Gamma(2n)}\int_{K_1}\int_{K_2}\int_0^\infty\int_{K_2}f(k_1k_2a_tk_2')\Delta(t)\,dk_1\,dk_2\,dt\,dk_2' \tag{1.6}$$

where

$$\Delta(t):=2^{2\rho}(\sinh t)^{4n-1}(\cosh t)^3. \tag{1.7}$$

2. THE CONVOLUTION ALGEBRA $\mathcal{D}(G;\chi_l)$

Let $\mathbf{N}/2$ be the set of nonnegative half-integers $\{0,1/2,1,3/2,\dots\}$. Since $K_1\equiv\mathrm{Sp}(1)$ is isomorphic to $\mathrm{SU}(2)$, $\mathbf{N}/2$ parametrizes the set of equivalence classes of unitary irreducible representations of K_1. We denote with the same symbol τ_l either the equivalence class corresponding to the parameter l or a fixed representative for it. Thus τ_l is a unitary irreducible representation of K_1 in a Hilbert space V_l of dimension $d_l=2l+1$. We extend τ_l to a representation of K by setting $\tau_l\equiv\mathbf{1}$ on K_2. Each τ_l is self-dual, i.e. unitarily equivalent to its contragredient representation. It follows in particular that the character $\chi_l=\operatorname{tr}\tau_l$ of τ_l satisfies $\chi_l(k^{-1})=\chi_l(k)$, $k\in K$.

We denote by $\mathcal{D}(G;\tau_l)$ the convolution algebra of the compactly supported C^∞ maps $F\colon G\to\mathrm{End}(V_l)$ satisfying

$$F(kxk') = \tau_l(k)F(x)\tau_l(k') \qquad (k, k' \in K, x \in G). \tag{2.8}$$

Let $\mathcal{D}(G)$ be the convolution algebra of the C^∞ compactly supported complex valued functions on G. Then $\mathcal{D}(G;\tau_l)$ is isomorphic to the subalgebra $\mathcal{D}(G;\chi_l)$ of $\mathcal{D}(G)$ consisting of the functions $f \in \mathcal{D}(G)$ satisfying

$$f^0 = f \tag{2.9}$$

and

$$f * d_l\chi_l = f\,, \tag{2.10}$$

where

$$f^0(x) := \int_K f(kxk^{-1})\,dk\,. \tag{2.11}$$

The isomorphism is given by $F \longmapsto d_l \operatorname{tr} F$ (see e.g. [Dij], Theorem 1.1).

The commutativity of the algebra $\mathcal{D}(G;\chi_l)$ can be deduced from the fact that the restriction $\tau_l|_M$ of τ_l to M is multiplicity free (according to the general criterion by Deitmar, cf. [Dei] Theorem 3, the commutativity of $\mathcal{D}(G;\chi_l)$ and the multiplicity freeness of $\tau_l|_M$ are in fact equivalent). An elementary direct argument by Takahashi proves the commutativity of a convolution algebra which is slightly bigger than $\mathcal{D}(G;\chi_l)$. Let $\mathcal{D}_1(G)$ be the subalgebra of $\mathcal{D}(G)$ consisting of the functions $f \in \mathcal{D}(G)$ satisfying

$$f(k_2k_1gk_1^{-1}k_2') = f(g)\,, \qquad g \in G, k_1 \in K_1, k_2 \in K_2\,. \tag{2.12}$$

Clearly $\mathcal{D}(G;\chi_l) \subset \mathcal{D}_1(G)$. Moreover $\mathcal{D}_1(G) = \oplus_l \mathcal{D}(G;\chi_l)$. Showing that

$$f(g^{-1}) = f(g) \qquad \text{for all } f \in \mathcal{D}_1(G) \text{ and } g \in G\,, \tag{2.13}$$

one proves the following lemma.

2.1. LEMMA ([T2], Proposition 1). *The algebra $\mathcal{D}_1(G)$ is commutative.*

2.2. LEMMA (cf. [T2], Lemma 2). *For every function $f \in \mathcal{D}(G;\chi_l)$,*

$$f(g) = f(k_1a_t) = \frac{1}{d_l}\chi_l(k_1)f(a_t) \tag{2.14}$$

if $g = k_1k_2a_tk_2'$ $(k_1 \in K_1; k_2, k_2' \in K_2\,;\ t \in \mathbf{R})$.

Proof. For $F \in \mathcal{D}(G;\tau_l)$, $F(a_t)$ is a scalar multiple of the identity, since it commutes with $\tau_l(k_1)$ for all $k_1 \in K_1$. □

2.3. REMARK. Formula (2.13) and Lemma 2.2 remain true for all continuous functions f such that $f = f^0$, $f * d_l\chi_l = f$.