

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 45 (1999)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** HARMONIC ANALYSIS ON VECTOR BUNDLES OVER  
Sp(1,n)/Sp(1)xSp(n)  
**Autor:** van Dijk, G. / PASQUALE, A.  
**Kapitel:** 1. The fine structure of Sp(1,n)  
**DOI:** <https://doi.org/10.5169/seals-64447>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

In Section 6 we prove that the  $\tau$ -Abel transform is an isomorphism of  $\mathcal{D}(G; \chi_\tau)$  onto the convolution algebra  $\mathcal{D}_+(\mathbf{R})$  of the even  $C^\infty$  compactly supported functions on  $\mathbf{R}$ . The inversion formula is explicitly written. The Paley-Wiener Theorem for the  $\tau$ -spherical transform is an immediate consequence. The final Section 7 contains the inversion formula and the Plancherel Theorem for the  $\tau$ -spherical transform.

Similar results for  $SU(n, 1)$  have been obtained as a specialization of the Hermitian symmetric case by Shimeno [Shi] and Heckman [HS, Part 1].

**ACKNOWLEDGMENT.** During the preparation of this paper, the second author has been financially supported by the Dutch Organization for Scientific Research (N.W.O.).

## 1. THE FINE STRUCTURE OF $\mathrm{Sp}(1, n)$

Let  $\mathbf{H}$  be the skew-field of the quaternions. Consider on the right  $\mathbf{H}$ -vector space  $\mathbf{H}^{n+1}$  the Hermitian form

$$(1.1) \quad [x, y] = \bar{y}_0 x_0 - \bar{y}_1 x_1 - \cdots - \bar{y}_n x_n,$$

the bar sign denoting quaternionic conjugation: if  $1, i, j, k$  are the quaternionic units and  $q = a + ib + jc + kd \in \mathbf{H}$  (with  $a, b, c, d \in \mathbf{R}$ ), then  $\bar{q} = a - ib - jc - kd$ . Let  $G = \mathrm{Sp}(1, n)$  be the group  $\mathrm{U}(1, n; \mathbf{H})$  of  $(n+1) \times (n+1)$  matrices with coefficients in  $\mathbf{H}$  which preserve this form. For  $n = 1$ ,  $G$  is called the De Sitter group. Let  $\mathrm{Sp}(m)$  indicate the group  $\mathrm{U}(m; \mathbf{H})$  of  $m \times m$  matrices with coefficients in  $\mathbf{H}$  which preserve the inner product  $(x, y) = \bar{y}_1 x_1 + \cdots + \bar{y}_m x_m$  of  $\mathbf{H}^m$ . In particular,  $\mathrm{Sp}(1)$  consists of the quaternions  $q = a + ib + jc + kd$  with norm  $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$  equal to 1.  $\mathrm{Sp}(1)$  is canonically isomorphic to  $SU(2)$ . The group  $G$  acts on the projective space  $P_n(\mathbf{H})$ . Let  $\Omega$  denote the image of the open set  $\{x \in \mathbf{H}^{n+1} : [x, x] > 0\}$  under the canonical map  $\mathbf{H}^{n+1} \setminus \{0\} \rightarrow P_n(\mathbf{H})$ . Then  $G$  acts transitively on  $\Omega$ , and the stabilizer of the quaternionic line generated by the vector  $(1, 0, \dots, 0)$  is the group

$$K = \left\{ \begin{bmatrix} u & 0 \\ 0 & U \end{bmatrix} : u \in \mathrm{Sp}(1), U \in \mathrm{Sp}(n) \right\} \equiv \mathrm{Sp}(1) \times \mathrm{Sp}(n).$$

The homogeneous space  $G/K$  is called the hyperbolic quaternionic space.  $K$  is a maximally compact subgroup of  $G$ .  $G$  is connected and simply connected.

To study the fine structure of  $G$ , we consider its Lie algebra  $\mathfrak{g} = \mathfrak{sp}(1, n)$ . Let  $J$  be the  $(n + 1) \times (n + 1)$  matrix  $\mathrm{diag}(-1, 1, \dots, 1)$ . For any matrix  $X$  of type  $(n + 1, n + 1)$  with coefficients in  $\mathbf{H}$  we set  $X^* = J\bar{X}^t J$ , the symbol  $t$  denoting transposition.

The Lie algebra  $\mathfrak{g}$  consists of the matrices  $X$  which verify the relation

$$X + X^* = 0.$$

These are the matrices of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ \bar{Z}_2^t & Z_3 \end{bmatrix}$$

with  $Z_1$  and  $Z_3$  anti-Hermitian of type  $(1, 1)$  and  $(n, n)$ , respectively, and  $Z_2$  arbitrary. Let  $\theta$  be the anti-involutive automorphism of  $\mathfrak{g}$  defined by

$$\theta X = JXJ.$$

Then  $\theta$  is a Cartan involution with the usual decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ . Here  $\mathfrak{k}$  is the Lie algebra of  $K$ . Let  $L$  be the following element of  $\mathfrak{g}$ :

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathbf{0} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then  $L \in \mathfrak{p}$  and  $\mathfrak{a} = RL$  is a maximal Abelian subspace of  $\mathfrak{p}$ . We are going to diagonalize  $\mathrm{ad} L$ . The centralizer of  $L$  in  $\mathfrak{k}$  is the subalgebra  $\mathfrak{m}$  of  $\mathfrak{g}$  of the matrices

$$\begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with  $u \in \mathbf{H}$ ,  $u + \bar{u} = 0$  and  $V$  a matrix of type  $(n - 1, n - 1)$  satisfying  $V + \bar{V}^t = \mathbf{0}$ . The non-zero eigenvalues of  $\mathrm{ad} L$  are  $\alpha = 1, -\alpha, \pm 2\alpha$ . The space  $\mathfrak{g}_\alpha$  consists of the matrices

$$X = \begin{bmatrix} 0 & z^* & 0 \\ z & \mathbf{0} & -z \\ 0 & z^* & 0 \end{bmatrix}$$

where  $z$  is a matrix of type  $(n - 1, 1)$  with coefficients in  $\mathbf{H}$ , and  $z^* := \bar{z}^t$ . The real dimension of  $\mathfrak{g}_\alpha$  is  $m_\alpha = 4(n - 1)$ . The space  $\mathfrak{g}_{2\alpha}$  consists of the matrices of the form

$$X = \begin{bmatrix} w & 0 & -w \\ 0 & \mathbf{0} & 0 \\ w & 0 & -w \end{bmatrix}$$

with  $w \in \mathbf{H}$ ,  $w + \bar{w} = 0$ . The dimension of  $\mathfrak{g}_{2\alpha}$  is equal to  $m_{2\alpha} = 3$ . We have  $\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{m} + \mathfrak{a} + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ .

Let  $A$  be the subgroup  $\exp \mathfrak{a}$ . This is the subgroup of the matrices

$$a_t = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}$$

where  $t$  is a real number. The centralizer of  $A$  in  $K$  is the subgroup  $M$  of the matrices

$$m(u, V) = \begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with  $u \in \mathrm{Sp}(1)$  and  $V \in \mathrm{Sp}(n-1)$ . The Lie algebra of  $M$  is  $\mathfrak{m}$ . The subspace  $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$  is a (real) nilpotent subalgebra. Set  $N = \exp \mathfrak{n}$ . This is the subgroup of the matrices

$$n(w, z) = \begin{bmatrix} 1 + w - \frac{1}{2}[z, z] & z^* & -w + \frac{1}{2}[z, z] \\ z & I & -z \\ w - \frac{1}{2}[z, z] & z^* & 1 - w + \frac{1}{2}[z, z] \end{bmatrix}$$

where  $w \in \mathbf{H}$  satisfies  $w + \bar{w} = 0$  and  $z = [z_1, \dots, z_{n-1}]^t$  is a matrix of type  $(n-1, 1)$  with coefficients in  $\mathbf{H}$ . We have set  $z^* = \bar{z}^t$  and  $[z, z] = -\bar{z}_1 z_1 - \cdots - \bar{z}_{n-1} z_{n-1}$ .

The composition law in  $N$  is the following:

$$n(w, z) \cdot n(w', z') = n(w + w' + \Im[z, z'], z + z'),$$

where  $\Im q := \frac{q - \bar{q}}{2}$  for  $q \in \mathbf{H}$ . The subgroups  $A$  and  $M$  normalize  $N$ :

$$\begin{aligned} a_t n(w, z) a_{-t} &= n(e^{2t} w, e^t z), \\ m(u, V) n(w, z) m(u, V)^{-1} &= n(uw\bar{u}, Vz\bar{u}). \end{aligned}$$

Let  $2\rho$  be the trace of the restriction of  $\mathrm{ad} L$  to  $\mathfrak{n}$ :

$$(1.2) \quad \rho = \frac{1}{2}(m_\alpha + 2m_{2\alpha}) = 2n + 1.$$

We have the Iwasawa decomposition  $G = KAN = KNA$  and the corresponding integral formulas:

$$(1.3) \quad \int_G f(g) dg = \int_K \int_{-\infty}^{+\infty} \int_N f(ka_t n) e^{2\rho t} dk dt dn$$

$$(1.4) \quad = \int_K \int_N \int_{-\infty}^{+\infty} f(kna_t) dk dn dt$$

for  $f \in C_c(G)$ . We adopt the usual notation  $C_c(G)$  for the space of continuous functions on  $G$  with compact support. In the above formulas,  $dn = dw dz$  ( $n = n(w, z)$ ) and  $dk$  is the normalized Haar measure on  $K$ .

Let

$$K_1 = \left\{ \begin{bmatrix} u & 0 \\ 0 & I \end{bmatrix} : u \in \mathrm{Sp}(1) \right\}, \quad K_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} : U \in \mathrm{Sp}(n) \right\}.$$

Then every  $g \in G$  can be written as  $g = k_1 k_2 a_t k'_2$  for uniquely determined  $k_1 \in K_1$ ,  $t \geq 0$  and for some  $k_2, k'_2 \in K_2$ . Writing  $g = [g_{ij}]_{i,j=0}^n$ , we have

$$(1.5) \quad k_1 = \frac{g_{00}}{|g_{00}|} \quad \text{and} \quad \cosh t = |g_{00}|.$$

If  $g \notin K$ , then  $t > 0$  and  $k_2, k'_2$  are uniquely determined modulo the subgroup

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 1 \end{bmatrix} : V \in \mathrm{Sp}(n-1) \right\}.$$

After  $dg$  is normalized according to (1.3), the corresponding integral formula is

$$(1.6) \quad \int_G f(g) dg = \frac{1}{2} \left( \frac{\pi}{4} \right)^{2n} \frac{1}{\Gamma(2n)} \int_{K_1} \int_{K_2} \int_0^\infty \int_{K_2} f(k_1 k_2 a_t k'_2) \Delta(t) dk_1 dk_2 dt dk'_2$$

where

$$(1.7) \quad \Delta(t) := 2^{2\rho} (\sinh t)^{4n-1} (\cosh t)^3.$$

## 2. THE CONVOLUTION ALGEBRA $\mathcal{D}(G; \chi_l)$

Let  $\mathbf{N}/2$  be the set of nonnegative half-integers  $\{0, 1/2, 1, 3/2, \dots\}$ . Since  $K_1 \cong \mathrm{Sp}(1)$  is isomorphic to  $\mathrm{SU}(2)$ ,  $\mathbf{N}/2$  parametrizes the set of equivalence classes of unitary irreducible representations of  $K_1$ . We denote with the same symbol  $\tau_l$  either the equivalence class corresponding to the parameter  $l$  or a fixed representative for it. Thus  $\tau_l$  is a unitary irreducible representation of  $K_1$  in a Hilbert space  $V_l$  of dimension  $d_l = 2l + 1$ . We extend  $\tau_l$  to a representation of  $K$  by setting  $\tau_l \equiv \mathbf{1}$  on  $K_2$ . Each  $\tau_l$  is self-dual, i.e. unitarily equivalent to its contragredient representation. It follows in particular that the character  $\chi_l = \mathrm{tr} \tau_l$  of  $\tau_l$  satisfies  $\chi_l(k^{-1}) = \chi_l(k)$ ,  $k \in K$ .

We denote by  $\mathcal{D}(G; \tau_l)$  the convolution algebra of the compactly supported  $C^\infty$  maps  $F: G \rightarrow \mathrm{End}(V_l)$  satisfying