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The map  $D \mapsto (\mathcal{L}(D), s_D)$  defines a one-to-one correspondence between the set of relative effective Cartier divisors on  $X/T$  and the isomorphism classes of pairs  $(\mathcal{L}, s)$ , where  $\mathcal{L}$  is an invertible sheaf on  $X$  and  $s$  is a global section of  $\mathcal{L}$  such that the map  $s: \mathcal{O}_X \rightarrow \mathcal{L}$  induced by the section  $s$  is injective and  $\mathcal{L}/s\mathcal{O}_X$  is  $\mathcal{O}_T$ -flat.

The proof of the following lemma is straightforward and is left to the reader:

LEMMA 2.2.

(a) If  $D_1$  and  $D_2$  are relative effective Cartier divisors on  $X/T$ , then so is  $D_1 + D_2$ .

(b) Let  $D_1$  and  $D_2$  be two relative effective Cartier divisors on  $X/T$  and let  $\mathcal{I}(D_1)$  and  $\mathcal{I}(D_2)$  be their ideal sheaves. If  $\mathcal{I}(D_1) \subset \mathcal{I}(D_2)$ , then  $D_1 - D_2$  is also a relative effective Cartier divisor on  $X/T$ .

(c) Let  $T' \rightarrow T$  be a base extension and let  $X' = X \times_T T'$ . If  $D$  is a relative effective Cartier divisor on  $X/T$ , then its pull-back to a closed subscheme  $D'$  of  $X'$  is a relative effective Cartier divisor on  $X'/T'$ .

LEMMA 2.3. Assume  $q: X \rightarrow T$  is flat. Let  $\mathcal{I}$  be a coherent sheaf of ideals of  $\mathcal{O}_X$  and let  $D$  be the closed subscheme of  $X$  defined by  $\mathcal{I}$ . If for every point  $x \in D$ , the ideal  $\mathcal{I}_x$  of  $\mathcal{O}_{X,x}$  is generated by one element  $g_x$  whose image in  $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{T,q(x)}} k(q(x))$  is not a zero divisor, then  $D$  is a relative effective Cartier divisor.

*Proof.* It suffices to show that  $g_x$  is not a zero divisor in  $\mathcal{O}_{X,x}$  and that  $\mathcal{O}_{X,x}/(g_x)$  is flat over  $\mathcal{O}_{T,q(x)}$ . This follows from [EGA] §0.10.2.4 by taking  $A = \mathcal{O}_{T,q(x)}$ ,  $B = \mathcal{O}_{X,x}$ ,  $M = N = \mathcal{O}_{X,x}$ , and  $u: M \rightarrow N$  to be the homomorphism  $g_x: \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$  defined by the multiplication by  $g_x$ .

### 3. THE CONSTRUCTION OF A BIRATIONAL GROUP

Let  $X$  be a nonsingular irreducible projective curve over an algebraically closed field  $k$ . A *modulus*  $\mathfrak{m}$  supported on a finite subset  $S$  of  $X$  is a divisor of the form  $\mathfrak{m} = \sum_{P \in S} n_P P$  with each  $n_P > 0$ . For any rational function  $f$  on  $X$ , we write  $f \equiv 0 \pmod{\mathfrak{m}}$  if  $v_P(f) \geq n_P$  for every  $P \in S$ , where  $v_P$  is the valuation defined by  $P$ . Two divisors  $D_1$  and  $D_2$  on  $X$  prime to  $S$  are called *m-equivalent* if there exists a rational function  $f$  satisfying  $f - 1 \equiv 0 \pmod{\mathfrak{m}}$  such that  $D_1 - D_2 = (f)$ . If this holds, we write  $D_1 \sim_{\mathfrak{m}} D_2$ . Define a ringed

space  $(X_m, \mathcal{O}_{X_m})$  as follows: The underlying set of  $X_m$  is  $(X - S) \cup \{Q\}$ . Define

$$\mathcal{O}_{X_m, Q} = k + \{f \mid f \equiv 0 \pmod{m}\}$$

and for every  $x \in X - S$ , define  $\mathcal{O}_{X_m, x} = \mathcal{O}_{X, x}$ . One can show that when  $\deg(m) \geq 2$ , the ringed space  $X_m$  is a singular curve with a unique singular point  $Q$  and its normalization is  $X$ . (It is easy to see that when  $\deg(m) < 2$ , the ringed space  $X_m$  is identified with  $X$  itself.) For a divisor  $D$  of  $X$  prime to  $S$ , we put

$$L_m(D) = H^0(X_m, \mathcal{L}_m), \quad I_m(D) = H^1(X_m, \mathcal{L}_m),$$

where  $\mathcal{L}_m$  is the invertible sheaf on  $X_m$  corresponding to  $D$ . Denote the dimensions of  $L_m(D)$  and  $I_m(D)$  by  $l_m(D)$  and  $i_m(D)$ , respectively. The Riemann-Roch theorem states that

$$l_m(D) - i_m(D) = \deg(D) + 1 - \pi.$$

In this formula,  $\pi$  is the sum  $\pi = g + \delta$ , where  $g$  is the genus of  $X$  and  $\delta = \deg(m) - 1$ . All these results are proved in [S], Chapter IV.

For convenience, a closed point on a scheme is just called a point.

Let  $T$  be a connected  $k$ -scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_m \times T & \longrightarrow & X_m \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k). \end{array}$$

Since  $X_m$  is proper and flat over  $\text{spec}(k)$ , the morphism  $q$  is also proper and flat. Let  $D$  be a relative effective Cartier divisor on  $(X_m \times T)/T$  supported on  $(X_m - Q) \times T$  and let  $\mathcal{L}$  be the invertible sheaf corresponding to  $D$ . Applying Theorem 1.1 (a) to the morphism  $q$  and the invertible sheaf  $\mathcal{L}$ , we conclude that  $t \mapsto \chi(\mathcal{L}_t)$  is a constant function on  $T$ . By the Riemann-Roch theorem, we have  $\chi(\mathcal{L}_t) = \deg D_t + 1 - \pi$ . So  $\deg(D_t)$  is also a constant. This constant is called the *degree* of  $D$ . Denote by  $\text{Div}^{(n)}(T)$  the set of all relative effective Cartier divisors of degree  $n$  on  $(X_m \times T)/T$  supported on  $(X_m - Q) \times T$ .

Let  $(X - S)^{(n)}$  be the  $n$ -th symmetric power of  $X - S$ , i.e., the quotient of  $(X - S)^n$  by the action of the  $n$ -th symmetric group  $\mathfrak{S}_n$ , where  $\mathfrak{S}_n$  acts on  $(X - S)^n$  by permuting the factors. In the Appendix we show that there exists a relative effective Cartier divisor  $\mathcal{D} \in \text{Div}^{(n)}((X - S)^{(n)})$ , called the *universal relative effective Cartier divisor*, whose restriction to the fiber of the projection  $X_m \times (X - S)^{(n)} \rightarrow (X - S)^{(n)}$  at  $P_1 + \cdots + P_n \in (X - S)^{(n)}$  is the divisor  $P_1 + \cdots + P_n$  of  $X_m$ . Moreover, we have

PROPOSITION 3.1. *The functor  $T \mapsto \text{Div}^{(n)}(T)$  from the category of  $k$ -schemes to the category of sets is represented by the symmetric power  $(X-S)^{(n)}$ . More precisely, for any relative effective Cartier divisor  $D$  of degree  $n$  on  $(X_m \times T)/T$  supported on  $(X_m - Q) \times T$ , there exists a unique morphism  $f: T \rightarrow (X-S)^{(n)}$  such that the pull-back of  $\mathcal{D}$  by  $\text{id} \times f$  is  $D$ .*

The proof of this proposition is given in the Appendix. The morphism  $T \rightarrow (X-S)^{(n)}$  can be described as follows: For every  $t \in T$ , identifying the fiber of  $q: X_m \times T \rightarrow T$  at  $t$  with  $X_m$ , we may regard the restriction  $D_t$  of  $D$  to the fiber at  $t$  as an effective divisor of degree  $n$  on  $X_m$  supported on  $X_m - Q$ . But this kind of divisor can be thought of as a point in  $(X-S)^{(n)}$ . The morphism  $T \rightarrow (X-S)^{(n)}$  is just  $t \mapsto D_t$ .

LEMMA 3.2. *Let  $D$  be a divisor of  $X$  prime to  $S$  such that  $i_m(D) \geq 1$ . Then there exists an open subset  $U$  of  $X-S$  such that for every  $P \in U$ , we have  $i_m(D+P) = i_m(D) - 1$ .*

*Proof.* If  $P \notin \text{Supp}(D) \cup S$ , then the dual vector space  $I_m(D+P)^*$  of  $I_m(D+P)$  is identified with the subspace of  $I_m(D)^*$  formed by differential forms  $\omega \in I_m(D)^*$  vanishing at  $P$ . Let  $\{\omega_1, \dots, \omega_{i_m(D)}\}$  be a basis of  $I_m(D)^*$ . We can then take  $U$  to be the complement of

$$\text{Supp}(D) \cup S \cup \{P \mid \omega_i(P) = 0 \text{ for } i = 1, \dots, i_m(D)\}.$$

LEMMA 3.3. *Let  $D_0$  be a divisor of  $X$  prime to  $S$  of degree 0. Then the set*

$$V_{D_0} = \{D \in (X-S)^{(\pi)} \mid l_m(D+D_0) = 1 \text{ and } l(D+D_0-m) = 0\}$$

*is non-empty and open in  $(X-S)^{(\pi)}$ .*

*Proof.* Consider the Cartesian square

$$\begin{array}{ccc} X_m \times (X-S)^{(\pi)} & \xrightarrow{p} & X_m \\ q \downarrow & & \downarrow \\ (X-S)^{(\pi)} & \longrightarrow & \text{spec}(k) . \end{array}$$

Applying Theorem 1.1 (b) to  $q$  and the invertible sheaf  $\mathcal{L}$  on  $X_m \times (X-S)^{(\pi)}$  corresponding to the divisor  $\mathcal{D} + p^*(D_0)$ , where  $\mathcal{D}$  is the universal relative effective Cartier divisor, we conclude that the set

$$V_1 = \{t \in (X - S)^{(\pi)} \mid \dim H^0(X_m, \mathcal{L}_t) \leq 1\}$$

is open, that is,

$$V_1 = \{D \in (X - S)^{(\pi)} \mid l_m(D + D_0) \leq 1\}$$

is open. By the Riemann-Roch theorem we have, for any  $D \in (X - S)^{(\pi)}$ ,

$$l_m(D + D_0) \geq \deg(D + D_0) + 1 - \pi = 1.$$

So we must have

$$V_1 = \{D \in (X - S)^{(\pi)} \mid l_m(D + D_0) = 1\}.$$

If  $l_m(D_0) \neq 0$ , then there exists a rational function  $f$  on  $X$  such that  $(f) + D_0$  is an effective divisor on  $X$  prime to  $S$ . This effective divisor must be 0 since it is of degree 0. Hence  $l_m(D_0) = l_m((f) + D_0) = l_m(0) = 1$ . So in any case we have  $l_m(D_0) \leq 1$ . By the Riemann-Roch theorem, we have  $i_m(D_0) \leq \pi$ . Applying Lemma 3.2 repeatedly, we can find  $P_1, \dots, P_{i_m(D_0)}$  in  $X - S$  so that  $i_m(D_0 + P_1 + \dots + P_{i_m(D_0)}) = 0$ . Choose  $P_{i_m(D_0)+1}, \dots, P_\pi$  in  $X - S$  arbitrarily. We have

$$i_m(D_0 + P_1 + \dots + P_{i_m(D_0)}) \geq i_m(D_0 + P_1 + \dots + P_{i_m(D_0)} + P_{i_m(D_0)+1} + \dots + P_\pi).$$

(This can be seen by interpreting  $i_m(D)$  as the dimension of the vector space of differential forms  $\omega$  regular at  $Q$  satisfying  $(\omega) \geq D$ .) So we have  $i_m(D_0 + P_1 + \dots + P_\pi) = 0$ . By the Riemann-Roch theorem, we have  $l_m(D_0 + P_1 + \dots + P_\pi) = 1$ . Hence  $P_1 + \dots + P_\pi$  is in the set  $V_1$  and  $V_1$  is not empty.

Similarly by Theorem 1.1 (b) applied to the projection  $q: X \times (X - S)^{(\pi)} \rightarrow (X - S)^{(\pi)}$  and the invertible sheaf on  $X \times (X - S)^{(\pi)}$  corresponding to the divisor  $\mathcal{D} + p^*(D_0 - m)$ , where  $p: X \times (X - S)^{(\pi)} \rightarrow X$  is another projection, we see that the set

$$V_2 = \{D \in (X - S)^{(\pi)} \mid l(D + D_0 - m) = 0\}$$

is open. Since  $\deg(D_0 - m) < 0$ , we have  $l(D_0 - m) = 0$ . By the Riemann-Roch theorem, we have  $i(D_0 - m) = \pi$ . Applying Lemma 3.2 repeatedly (but taking  $m = 0$ ), we can find  $P_1, \dots, P_\pi \in X - S$  such that  $i(D_0 - m + P_1 + \dots + P_\pi) = 0$ . Then by the Riemann-Roch theorem we have  $l(D_0 - m + P_1 + \dots + P_\pi) = 0$ . So  $P_1 + \dots + P_\pi$  is in  $V_2$  and  $V_2$  is not empty.

Since  $(X - S)^{(\pi)}$  is irreducible, the set  $V_{D_0} = V_1 \cap V_2$  is open and non-empty.

LEMMA 3.4. *Fix a point  $P_0$  in  $S$ .*

(a) *The set*

$$U = \{(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_m(D_1 + D_2 - \pi P_0) = 1, \quad l(D_1 + D_2 - \pi P_0 - m) = 0\}$$

*is a non-empty open subset of  $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$ .*

(b) *The set*

$$V = \{(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_m(D_2 - D_1 + \pi P_0) = 1, \quad l(D_2 - D_1 + \pi P_0 - m) = 0\}$$

*is a non-empty open subset of  $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$ .*

*Proof.* (a) Let  $p_1, p_2: (X - S)^{(\pi)} \times (X - S)^{(\pi)} \rightarrow (X - S)^{(\pi)}$  be the projections and let  $E_i$  ( $i = 1, 2$ ) be the pull-backs by  $\text{id} \times p_i$  of the universal relative effective Cartier divisor  $\mathcal{D}$  on  $X_m \times (X - S)^{(\pi)}$ . Put  $E = E_1 + E_2$ . This is a divisor on  $X_m \times (X - S)^{(\pi)} \times (X - S)^{(\pi)}$ .

Consider the Cartesian square

$$\begin{array}{ccc} X_m \times (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \xrightarrow{p} & X_m \\ q \downarrow & & \downarrow \\ (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \longrightarrow & \text{spec}(k) . \end{array}$$

By the Riemann-Roch theorem, for any  $(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)}$ , we have

$$l_m(D_1 + D_2 - \pi P_0) \geq \deg(D_1 + D_2 - \pi P_0) + 1 - \pi = 1 ,$$

that is, for any  $t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)}$ , we have  $l_m(E_t - \pi P_0) \geq 1$ . Applying Theorem 1.1 (b) to the projection  $q$  and the invertible sheaf corresponding to the divisor  $E - p^*(P_0)$ , we see that the set

$$U_1 = \{t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_m(E_t - \pi P_0) = 1\}$$

is open. Similarly the set

$$U_2 = \{t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l(E_t - \pi P_0 - m) = 0\}$$

is also open. Hence the set  $U = U_1 \cap U_2$  is open.

Applying Lemma 3.3 to  $D_0 = 0$ , we see that there exists a  $D \in (X - S)^{(\pi)}$  such that  $l_m(D) = 1$  and  $l(D - m) = 0$ . Then  $(D, \pi P_0)$  is in  $U$ . So  $U$  is non-empty. This proves (a).

The proof of (b) is similar and is omitted.

DEFINITION 3.5. A *birational group* over  $k$  is a nonsingular variety  $V$  together with a rational map  $m: V \times V \rightarrow V$ ,  $(a, b) \mapsto ab$  such that

- (a)  $(ab)c = a(bc)$  when both sides are defined;
- (b) the rational maps  $\Phi: (a, b) \mapsto (a, ab)$  and  $\Psi: (a, b) \mapsto (b, ab)$  on  $V \times V$  are birational.

PROPOSITION 3.6. *There exists a unique rational map*

$$m: (X - S)^{(\pi)} \times (X - S)^{(\pi)} \rightarrow (X - S)^{(\pi)}$$

whose domain of definition contains the set  $U$  in 3.4(a) such that  $m(D_1, D_2)$  is the unique effective divisor that is  $\mathfrak{m}$ -equivalent to  $D_1 + D_2 - \pi P_0$  for any  $(D_1, D_2) \in U$ . Moreover  $m$  makes  $(X - S)^{(\pi)}$  a birational group.

*Proof.* Keep the notations in the proof of Lemma 3.4. Consider the Cartesian squares

$$\begin{array}{ccccccc} X_{\mathfrak{m}} = q^{-1}(t) & \longrightarrow & X_{\mathfrak{m}} \times U & \subset & X_{\mathfrak{m}} \times (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \xrightarrow{p} & X_{\mathfrak{m}} \\ \downarrow & & q \downarrow & & \downarrow & & \downarrow \\ \text{spec}(k(t)) & \longrightarrow & U & \subset & (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \longrightarrow & \text{spec}(k). \end{array}$$

Let  $\mathcal{L}$  be the restriction to  $X_{\mathfrak{m}} \times U$  of the invertible sheaf corresponding to the divisor  $E_1 + E_2 - p^*(\pi P_0)$ . By Theorem 1.1(c) and the choice of  $U$ , the sheaf  $q_*\mathcal{L}$  is invertible. The canonical homomorphism  $q^*q_*\mathcal{L} \rightarrow \mathcal{L}$  gives rise to  $s: \mathcal{O}_{X_{\mathfrak{m}} \times U} \rightarrow \mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$ . We claim that the pair  $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$  defines a relative effective Cartier divisor on  $(X_{\mathfrak{m}} \times U)/U$ . According to Remark 2.1, it is enough to check that  $s$  is injective and  $\text{coker}(s)$  is  $\mathcal{O}_U$ -flat. Since  $\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$  is invertible, it is enough to verify  $s_t$  is injective for all  $t \in U$  by [EGA] §0.10.2.4, where  $s_t$  is the homomorphism obtained by restricting  $s$  to the fiber of  $q$  at  $t$ . It suffices to show that the restriction of the canonical homomorphism  $q^*q_*\mathcal{L} \rightarrow \mathcal{L}$  to the fiber of  $q$  at  $t$  is injective. By Theorem 1.1(c) we have  $q_*\mathcal{L} \otimes_{\mathcal{O}_U} k(t) = H^0(X_{\mathfrak{m}}, \mathcal{L}_t)$ . So the restriction of the canonical homomorphism to the fiber is  $H^0(X_{\mathfrak{m}}, \mathcal{L}_t) \otimes_k \mathcal{O}_{X_{\mathfrak{m}}} \rightarrow \mathcal{L}_t$ . Denote this map by  $s'_t$ ; we need to show it is injective. But we have  $\dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t) = 1$  since  $t \in U$ . If we fix a nonzero element  $g \in H^0(X_{\mathfrak{m}}, \mathcal{L}_t)$ , then  $s'_t$  is identified with  $\mathcal{O}_{X_{\mathfrak{m}}} \rightarrow \mathcal{L}_t$ ,  $a \mapsto ag$ . This last map is injective since  $X_{\mathfrak{m}}$  is an integral scheme and  $g$  can be thought of as a rational function. So  $s_t$  is injective. Hence  $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$  defines a relative effective Cartier divisor. The restriction of this divisor to the fiber of  $q$  at  $t$  is the divisor on  $X_{\mathfrak{m}}$  defined by the pair  $(\mathcal{L}_t, g)$ , which is supported on  $X_{\mathfrak{m}} - Q$ . So the divisor defined by

$(\mathcal{L} \otimes (q^* q_* \mathcal{L})^{-1}, s)$  is supported on  $(X_{\mathfrak{m}} - Q) \times U$ . By Proposition 3.1 there exists a unique morphism of varieties  $m: U \rightarrow (X - S)^{(\pi)}$  such that the divisor defined by  $(\mathcal{L} \otimes (q^* q_* \mathcal{L})^{-1}, s)$  is the pull-back by  $\text{id} \times m$  of the universal relative effective Cartier divisor  $\mathcal{D}$  on  $X_{\mathfrak{m}} \times (X - S)^{(\pi)}$ . For any  $(D_1, D_2) \in U$ , we have  $l_{\mathfrak{m}}(D_1 + D_2 - \pi P_0) = 1$  and  $l(D_1 + D_2 - \pi P_0 - \mathfrak{m}) = 0$ . So there is one and only one effective divisor  $\mathfrak{m}$ -equivalent to  $D_1 + D_2 - \pi P_0$  and it is simply  $m(D_1, D_2)$ .

Similarly, using Lemma 3.4 (b) and Proposition 3.1, one can show that there exists a morphism  $r: V \rightarrow (X - S)^{(\pi)}$  such that  $r(D_1, D_2)$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $D_2 - D_1 + \pi P_0$  for any  $(D_1, D_2) \in V$ .

Let us verify that  $m$  defines a birational group on  $(X - S)^{(\pi)}$ . First we show

$$m(m(D_1, D_2), D_3) = m(D_1, m(D_2, D_3))$$

when  $(D_1, D_2)$ ,  $(D_2, D_3)$ ,  $(m(D_1, D_2), D_3)$  and  $(D_1, m(D_2, D_3))$  all belong to  $U$ . Indeed  $m(m(D_1, D_2), D_3)$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $m(D_1, D_2) + D_3 - \pi P_0$ , and  $m(D_1, m(D_2, D_3))$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $D_1 + m(D_2, D_3) - \pi P_0$ . But  $m(D_1, D_2) + D_3 - \pi P_0$  and  $D_1 + m(D_2, D_3) - \pi P_0$  are  $\mathfrak{m}$ -equivalent since both are  $\mathfrak{m}$ -equivalent to  $D_1 + D_2 + D_3 - 2\pi P_0$ . So we have  $m(m(D_1, D_2), D_3) = m(D_1, m(D_2, D_3))$ .

One can also verify  $m(D_1, D_2) = m(D_2, D_1)$  when both  $(D_1, D_2)$  and  $(D_2, D_1)$  are in  $U$ , that is, the operation  $m$  is commutative.

Next we show that  $\Theta: (D_1, D_2) \mapsto (D_1, r(D_1, D_2))$  is the birational inverse of  $\Phi: (D_1, D_2) \mapsto (D_1, m(D_1, D_2))$  so that  $\Phi$  is birational. Since the operation  $m$  is commutative, the rational map  $\Psi: (D_1, D_2) \mapsto (D_2, m(D_1, D_2))$  is also birational. Therefore  $m$  makes  $(X - S)^{(\pi)}$  a birational group.

First we verify  $\Phi \Theta(D_1, D_2) = (D_1, D_2)$  whenever the left-hand side is defined. We have

$$\Phi \Theta(D_1, D_2) = \Phi(D_1, r(D_1, D_2)) = (D_1, m(D_1, r(D_1, D_2))).$$

Moreover  $m(D_1, r(D_1, D_2))$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $D_1 + r(D_1, D_2) - \pi P_0$ . But  $D_2$  is also an effective divisor  $\mathfrak{m}$ -equivalent to  $D_1 + r(D_1, D_2) - \pi P_0$  since we have

$$D_1 + r(D_1, D_2) - \pi P_0 \sim_{\mathfrak{m}} D_1 + (D_2 - D_1 + \pi P_0) - \pi P_0 = D_2.$$

Hence  $m(D_1, r(D_1, D_2)) = D_2$  and  $\Phi \Theta(D_1, D_2) = (D_1, D_2)$ .

Similarly one can show that  $\Theta \Phi(D_1, D_2) = (D_1, D_2)$  when the left-hand side is defined.



Note that  $\Phi$  is a regular morphism defined on  $U$  and  $\Theta$  is a regular morphism defined on  $V$ . Since

$$\Phi \Theta(D_1, D_2) = (D_1, D_2) \quad \text{and} \quad \Theta \Phi(D_1, D_2) = (D_1, D_2)$$

whenever the left-hand sides are defined, the maps  $\Phi$  and  $\Theta$  induce regular morphisms  $\Phi: U \cap \Phi^{-1}(V) \rightarrow V \cap \Theta^{-1}(U)$  and  $\Theta: V \cap \Theta^{-1}(U) \rightarrow U \cap \Phi^{-1}(V)$ . To show that  $\Phi$  and  $\Theta$  are birational inverses to each other, it is enough to check that  $U \cap \Phi^{-1}(V)$  and  $V \cap \Theta^{-1}(U)$  are non-empty.

Note that  $(D_1, D_2) \in U \cap \Phi^{-1}(V)$  if and only if  $(D_1, D_2) \in U$  and

$$l_m(m(D_1, D_2) - D_1 + \pi P_0) = 1, \quad l(m(D_1, D_2) - D_1 + \pi P_0 - m) = 0.$$

Since  $m(D_1, D_2) \sim_m D_1 + D_2 - \pi P_0$ , the above equations are equivalent to

$$l_m(D_2) = 1, \quad l(D_2 - m) = 0.$$

Applying Lemma 3.3 to the divisor  $D_0 = 0$ , we conclude that the set

$$V_0 = \{D \in (X - S)^{(\pi)} \mid l_m(D) = 0, \quad l(D - m) = 0\}$$

is open and non-empty. Since  $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$  is irreducible, the set  $U \cap ((X - S)^{(\pi)} \times V_0)$  is also open and non-empty. This set is exactly  $U \cap \Phi^{-1}(V)$ . So  $U \cap \Phi^{-1}(V)$  is non-empty.

Similarly  $V \cap \Theta^{-1}(U)$  is also non-empty. This completes the proof of the proposition.

#### 4. FROM BIRATIONAL GROUPS TO ALGEBRAIC GROUPS

Let  $k$  be an algebraically closed field, let  $V$  be a connected nonsingular variety over  $k$ , and let  $m: V \times V \rightarrow V$ ,  $(a, b) \mapsto ab$  be a rational map satisfying  $(ab)c = a(bc)$ . Assume the rational maps  $\Phi(a, b) = (a, ab)$  and  $\Psi(a, b) = (b, ab)$  are birational. Then there exist open subsets  $X_\Phi$ ,  $Y_\Phi$ ,  $X_\Psi$  and  $Y_\Psi$  in  $V \times V$  such that  $\Phi$  induces an isomorphism  $X_\Phi \cong Y_\Phi$  and  $\Psi$  induces an isomorphism  $X_\Psi \cong Y_\Psi$ . Put  $Z = X_\Phi \cap Y_\Phi \cap X_\Psi \cap Y_\Psi$ .

It is convenient to write the formulae for  $\Phi^{-1}$  and  $\Psi^{-1}$  as  $\Phi^{-1}(a, b) = (a, a^{-1}b)$  and  $\Psi^{-1}(a, b) = (ba^{-1}, a)$ .