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The map  $D \mapsto (\mathcal{L}(D), s_D)$  defines a one-to-one correspondence between the set of relative effective Cartier divisors on X/T and the isomorphism classes of pairs  $(\mathcal{L}, s)$ , where  $\mathcal{L}$  is an invertible sheaf on X and s is a global section of  $\mathcal{L}$  such that the map  $s \colon \mathcal{O}_X \to \mathcal{L}$  induced by the section s is injective and  $\mathcal{L}/s\mathcal{O}_X$  is  $\mathcal{O}_T$ -flat.

The proof of the following lemma is straightforward and is left to the reader:

# LEMMA 2.2.

- (a) If  $D_1$  and  $D_2$  are relative effective Cartier divisors on X/T, then so is  $D_1 + D_2$ .
- (b) Let  $D_1$  and  $D_2$  be two relative effective Cartier divisors on X/T and let  $\mathcal{I}(D_1)$  and  $\mathcal{I}(D_2)$  be their ideal sheaves. If  $\mathcal{I}(D_1) \subset \mathcal{I}(D_2)$ , then  $D_1 D_2$  is also a relative effective Cartier divisor on X/T.
- (c) Let  $T' \to T$  be a base extension and let  $X' = X \times_T T'$ . If D is a relative effective Cartier divisor on X/T, then its pull-back to a closed subscheme D' of X' is a relative effective Cartier divisor on X'/T'.
- LEMMA 2.3. Assume  $q: X \to T$  is flat. Let  $\mathcal{I}$  be a coherent sheaf of ideals of  $\mathcal{O}_X$  and let D be the closed subscheme of X defined by  $\mathcal{I}$ . If for every point  $x \in D$ , the ideal  $\mathcal{I}_x$  of  $\mathcal{O}_{X,x}$  is generated by one element  $g_x$  whose image in  $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{T,q(x)}} k(q(x))$  is not a zero divisor, then D is a relative effective Cartier divisor.

*Proof.* It suffices to show that  $g_x$  is not a zero divisor in  $\mathcal{O}_{X,x}$  and that  $\mathcal{O}_{X,x}/(g_x)$  is flat over  $\mathcal{O}_{T,q(x)}$ . This follows from [EGA] §0.10.2.4 by taking  $A = \mathcal{O}_{T,q(x)}$ ,  $B = \mathcal{O}_{X,x}$ ,  $M = N = \mathcal{O}_{X,x}$ , and  $u: M \to N$  to be the homomorphism  $g_x: \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$  defined by the multiplication by  $g_x$ .

# 3. THE CONSTRUCTION OF A BIRATIONAL GROUP

Let X be a nonsingular irreducible projective curve over an algebraically closed field k. A modulus m supported on a finite subset S of X is a divisor of the form  $m = \sum_{P \in S} n_P P$  with each  $n_P > 0$ . For any rational function f on X, we write  $f \equiv 0 \mod m$  if  $v_P(f) \geq n_P$  for every  $P \in S$ , where  $v_P$  is the valuation defined by P. Two divisors  $D_1$  and  $D_2$  on X prime to S are called m-equivalent if there exists a rational function f satisfying  $f - 1 \equiv 0 \mod m$  such that  $D_1 - D_2 = (f)$ . If this holds, we write  $D_1 \sim_m D_2$ . Define a ringed

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space  $(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}})$  as follows: The underlying set of  $X_{\mathfrak{m}}$  is  $(X - S) \cup \{Q\}$ . Define

$$\mathcal{O}_{X_{\mathfrak{m}},Q} = k + \{ f \mid f \equiv 0 \mod \mathfrak{m} \}$$

and for every  $x \in X - S$ , define  $\mathcal{O}_{X_{\mathfrak{m}},x} = \mathcal{O}_{X,x}$ . One can show that when  $\deg(\mathfrak{m}) \geq 2$ , the ringed space  $X_{\mathfrak{m}}$  is a singular curve with a unique singular point Q and its normalization is X. (It is easy to see that when  $\deg(\mathfrak{m}) < 2$ , the ringed space  $X_{\mathfrak{m}}$  is identified with X itself.) For a divisor D of X prime to S, we put

$$L_{\mathfrak{m}}(D) = H^{0}(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}), \quad I_{\mathfrak{m}}(D) = H^{1}(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}),$$

where  $\mathcal{L}_{\mathfrak{m}}$  is the invertible sheaf on  $X_{\mathfrak{m}}$  corresponding to D. Denote the dimensions of  $L_{\mathfrak{m}}(D)$  and  $I_{\mathfrak{m}}(D)$  by  $l_{\mathfrak{m}}(D)$  and  $i_{\mathfrak{m}}(D)$ , respectively. The Riemann-Roch theorem states that

$$l_{m}(D) - i_{m}(D) = \deg(D) + 1 - \pi$$
.

In this formula,  $\pi$  is the sum  $\pi = g + \delta$ , where g is the genus of X and  $\delta = \deg(\mathfrak{m}) - 1$ . All these results are proved in [S], Chapter IV.

For convenience, a closed point on a scheme is just called a point.

Let T be a connected k-scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \operatorname{spec}(k) \ . \end{array}$$

Since  $X_m$  is proper and flat over  $\operatorname{spec}(k)$ , the morphism q is also proper and flat. Let D be a relative effective Cartier divisor on  $(X_m \times T)/T$  supported on  $(X_m - Q) \times T$  and let  $\mathcal{L}$  be the invertible sheaf corresponding to D. Applying Theorem 1.1 (a) to the morphism q and the invertible sheaf  $\mathcal{L}$ , we conclude that  $t \mapsto \chi(\mathcal{L}_t)$  is a constant function on T. By the Riemann-Roch theorem, we have  $\chi(\mathcal{L}_t) = \deg D_t + 1 - \pi$ . So  $\deg(D_t)$  is also a constant. This constant is called the *degree* of D. Denote by  $\operatorname{Div}^{(n)}(T)$  the set of all relative effective Cartier divisors of degree n on  $(X_m \times T)/T$  supported on  $(X_m - Q) \times T$ .

Let  $(X-S)^{(n)}$  be the *n*-th symmetric power of X-S, i.e., the quotient of  $(X-S)^n$  by the action of the *n*-th symmetric group  $\mathfrak{S}_n$ , where  $\mathfrak{S}_n$  acts on  $(X-S)^n$  by permuting the factors. In the Appendix we show that there exists a relative effective Cartier divisor  $\mathcal{D} \in \operatorname{Div}^{(n)}\big((X-S)^{(n)}\big)$ , called the *universal relative effective Cartier divisor*, whose restriction to the fiber of the projection  $X_m \times (X-S)^{(n)} \to (X-S)^{(n)}$  at  $P_1 + \cdots + P_n \in (X-S)^{(n)}$  is the divisor  $P_1 + \cdots + P_n$  of  $X_m$ . Moreover, we have

PROPOSITION 3.1. The functor  $T \mapsto \operatorname{Div}^{(n)}(T)$  from the category of k-schemes to the category of sets is represented by the symmetric power  $(X-S)^{(n)}$ . More precisely, for any relative effective Cartier divisor D of degree n on  $(X_{\mathfrak{m}} \times T)/T$  supported on  $(X_{\mathfrak{m}} - Q) \times T$ , there exists a unique morphism  $f: T \to (X-S)^{(n)}$  such that the pull-back of  $\mathcal{D}$  by  $\operatorname{id} \times f$  is D.

The proof of this proposition is given in the Appendix. The morphism  $T \to (X - S)^{(n)}$  can be described as follows: For every  $t \in T$ , identifying the fiber of  $q: X_{\mathfrak{m}} \times T \to T$  at t with  $X_{\mathfrak{m}}$ , we may regard the restriction  $D_t$  of D to the fiber at t as an effective divisor of degree n on  $X_{\mathfrak{m}}$  supported on  $X_{\mathfrak{m}} - Q$ . But this kind of divisor can be thought of as a point in  $(X - S)^{(n)}$ . The morphism  $T \to (X - S)^{(n)}$  is just  $t \mapsto D_t$ .

LEMMA 3.2. Let D be a divisor of X prime to S such that  $i_{\mathfrak{m}}(D) \geq 1$ . Then there exists an open subset U of X-S such that for every  $P \in U$ , we have  $i_{\mathfrak{m}}(D+P)=i_{\mathfrak{m}}(D)-1$ .

*Proof.* If  $P \notin \operatorname{Supp}(D) \cup S$ , then the dual vector space  $I_{\mathfrak{m}}(D+P)^*$  of  $I_{\mathfrak{m}}(D+P)$  is identified with the subspace of  $I_{\mathfrak{m}}(D)^*$  formed by differential forms  $\omega \in I_{\mathfrak{m}}(D)^*$  vanishing at P. Let  $\{\omega_1, \ldots, \omega_{i_{\mathfrak{m}}(D)}\}$  be a basis of  $I_{\mathfrak{m}}(D)^*$ . We can then take U to be the complement of

$$\text{Supp}(D) \cup S \cup \{P \mid \omega_i(P) = 0 \text{ for } i = 1, ..., i_{\mathfrak{m}}(D)\}.$$

LEMMA 3.3. Let  $D_0$  be a divisor of X prime to S of degree 0. Then the set

$$V_{D_0} = \{ D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D + D_0) = 1 \text{ and } l(D + D_0 - \mathfrak{m}) = 0 \}$$
 is non-empty and open in  $(X - S)^{(\pi)}$ .

Proof. Consider the Cartesian square

$$X_{\mathfrak{m}} \times (X - S)^{(\pi)} \xrightarrow{p} X_{\mathfrak{m}}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad$$

Applying Theorem 1.1 (b) to q and the invertible sheaf  $\mathcal{L}$  on  $X_{\mathfrak{m}} \times (X - S)^{(\pi)}$  corresponding to the divisor  $\mathcal{D} + p^*(D_0)$ , where  $\mathcal{D}$  is the universal relative effective Cartier divisor, we conclude that the set

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$$V_1 = \{ t \in (X - S)^{(\pi)} \mid \dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t) \le 1 \}$$

is open, that is,

$$V_1 = \{ D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D + D_0) \le 1 \}$$

is open. By the Riemann-Roch theorem we have, for any  $D \in (X - S)^{(\pi)}$ ,

$$l_{\mathfrak{m}}(D+D_0) \ge \deg(D+D_0) + 1 - \pi = 1.$$

So we must have

$$V_1 = \{ D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D + D_0) = 1 \}.$$

If  $l_{\mathfrak{m}}(D_0) \neq 0$ , then there exists a rational function f on X such that  $(f) + D_0$  is an effective divisor on X prime to S. This effective divisor must be 0 since it is of degree 0. Hence  $l_{\mathfrak{m}}(D_0) = l_{\mathfrak{m}}((f) + D_0) = l_{\mathfrak{m}}(0) = 1$ . So in any case we have  $l_{\mathfrak{m}}(D_0) \leq 1$ . By the Riemann-Roch theorem, we have  $i_{\mathfrak{m}}(D_0) \leq \pi$ . Applying Lemma 3.2 repeatedly, we can find  $P_1, \ldots, P_{i_{\mathfrak{m}}(D_0)}$  in X - S so that  $i_{\mathfrak{m}}(D_0 + P_1 + \cdots + P_{i_{\mathfrak{m}}(D_0)}) = 0$ . Choose  $P_{i_{\mathfrak{m}}(D_0)+1}, \ldots, P_{\pi}$  in X - S arbitrarily. We have

$$i_{\mathfrak{m}}(D_0 + P_1 + \cdots + P_{i_{\mathfrak{m}}(D_0)}) \ge i_{\mathfrak{m}}(D_0 + P_1 + \cdots + P_{i_{\mathfrak{m}}(D_0)} + P_{i_{\mathfrak{m}}(D_0)+1} + \cdots + P_{\pi}).$$

(This can be seen by interpreting  $i_{\mathfrak{m}}(D)$  as the dimension of the vector space of differential forms  $\omega$  regular at Q satisfying  $(\omega) \geq D$ .) So we have  $i_{\mathfrak{m}}(D_0 + P_1 + \cdots + P_{\pi}) = 0$ . By the Riemann-Roch theorem, we have  $l_{\mathfrak{m}}(D_0 + P_1 + \cdots + P_{\pi}) = 1$ . Hence  $P_1 + \cdots + P_{\pi}$  is in the set  $V_1$  and  $V_1$  is not empty.

Similarly by Theorem 1.1 (b) applied to the projection  $q: X \times (X-S)^{(\pi)} \to (X-S)^{(\pi)}$  and the invertible sheaf on  $X \times (X-S)^{(\pi)}$  corresponding to the divisor  $\mathcal{D} + p^*(D_0 - \mathfrak{m})$ , where  $p: X \times (X-S)^{(\pi)} \to X$  is another projection, we see that the set

$$V_2 = \{ D \in (X - S)^{(\pi)} \mid l(D + D_0 - \mathfrak{m}) = 0 \}$$

is open. Since  $\deg(D_0-\mathfrak{m})<0$ , we have  $l(D_0-\mathfrak{m})=0$ . By the Riemann-Roch theorem, we have  $i(D_0-\mathfrak{m})=\pi$ . Applying Lemma 3.2 repeatedly (but taking  $\mathfrak{m}=0$ ), we can find  $P_1,\ldots,P_\pi\in X-S$  such that  $i(D_0-\mathfrak{m}+P_1+\cdots+P_\pi)=0$ . Then by the Riemann-Roch theorem we have  $l(D_0-\mathfrak{m}+P_1+\cdots+P_\pi)=0$ . So  $P_1+\cdots+P_\pi$  is in  $V_2$  and  $V_2$  is not empty.

Since  $(X-S)^{(\pi)}$  is irreducible, the set  $V_{D_0} = V_1 \cap V_2$  is open and non-empty.

LEMMA 3.4. Fix a point  $P_0$  in S.

(a) The set

$$U = \{ (D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)}$$
  
 
$$| l_{\mathfrak{m}}(D_1 + D_2 - \pi P_0) = 1, \quad l(D_1 + D_2 - \pi P_0 - \mathfrak{m}) = 0 \}$$

is a non-empty open subset of  $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$ .

(b) The set

$$V = \{ (D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)}$$
  
 
$$| l_{\mathfrak{m}}(D_2 - D_1 + \pi P_0) = 1, \quad l(D_2 - D_1 + \pi P_0 - \mathfrak{m}) = 0 \}$$

is a non-empty open subset of  $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$ .

*Proof.* (a) Let  $p_1, p_2: (X-S)^{(\pi)} \times (X-S)^{(\pi)} \to (X-S)^{(\pi)}$  be the projections and let  $E_i$  (i=1,2) be the pull-backs by id  $\times p_i$  of the universal relative effective Cartier divisor  $\mathcal{D}$  on  $X_{\mathfrak{m}} \times (X-S)^{(\pi)}$ . Put  $E=E_1+E_2$ . This is a divisor on  $X_{\mathfrak{m}} \times (X-S)^{(\pi)} \times (X-S)^{(\pi)}$ .

Consider the Cartesian square

By the Riemann-Roch theorem, for any  $(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)}$ , we have

$$l_{\mathfrak{m}}(D_1 + D_2 - \pi P_0) \ge \deg(D_1 + D_2 - \pi P_0) + 1 - \pi = 1$$

that is, for any  $t \in (X-S)^{(\pi)} \times (X-S)^{(\pi)}$ , we have  $l_{\mathfrak{m}}(E_t - \pi P_0) \geq 1$ . Applying Theorem 1.1 (b) to the projection q and the invertible sheaf corresponding to the divisor  $E - p^*(P_0)$ , we see that the set

$$U_1 = \{ t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(E_t - \pi P_0) = 1 \}$$

is open. Similarly the set

$$U_2 = \{t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l(E_t - \pi P_0 - \mathfrak{m}) = 0\}$$

is also open. Hence the set  $U = U_1 \cap U_2$  is open.

Applying Lemma 3.3 to  $D_0 = 0$ , we see that there exists a  $D \in (X - S)^{(\pi)}$  such that  $l_{\mathfrak{m}}(D) = 1$  and  $l(D - \mathfrak{m}) = 0$ . Then  $(D, \pi P_0)$  is in U. So U is non-empty. This proves (a).

The proof of (b) is similar and is omitted.

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DEFINITION 3.5. A birational group over k is a nonsingular variety V together with a rational map  $m: V \times V \to V$ ,  $(a, b) \mapsto ab$  such that

- (a) (ab)c = a(bc) when both sides are defined;
- (b) the rational maps  $\Phi: (a,b) \mapsto (a,ab)$  and  $\Psi: (a,b) \mapsto (b,ab)$  on  $V \times V$  are birational.

PROPOSITION 3.6. There exists a unique rational map

$$m: (X-S)^{(\pi)} \times (X-S)^{(\pi)} \to (X-S)^{(\pi)}$$

whose domain of definition contains the set U in 3.4 (a) such that  $m(D_1, D_2)$  is the unique effective divisor that is  $\mathfrak{m}$ -equivalent to  $D_1 + D_2 - \pi P_0$  for any  $(D_1, D_2) \in U$ . Moreover m makes  $(X - S)^{(\pi)}$  a birational group.

*Proof.* Keep the notations in the proof of Lemma 3.4. Consider the Cartesian squares

$$X_{\mathfrak{m}} = q^{-1}(t) \longrightarrow X_{\mathfrak{m}} \times U \subset X_{\mathfrak{m}} \times (X - S)^{(\pi)} \times (X - S)^{(\pi)} \stackrel{p}{\longrightarrow} X_{\mathfrak{m}}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{spec}(k(t)) \longrightarrow U \subset (X - S)^{(\pi)} \times (X - S)^{(\pi)} \longrightarrow \operatorname{spec}(k).$$

Let  $\mathcal{L}$  be the restriction to  $X_{\mathfrak{m}} \times U$  of the invertible sheaf corresponding to the divisor  $E_1 + E_2 - p^*(\pi P_0)$ . By Theorem 1.1(c) and the choice of U, the sheaf  $q_*\mathcal{L}$  is invertible. The canonical homomorphism  $q^*q_*\mathcal{L} \to \mathcal{L}$  gives rise to  $s: \mathcal{O}_{X_m \times U} \to \mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$ . We claim that the pair  $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$ defines a relative effective Cartier divisor on  $(X_m \times U)/U$ . According to Remark 2.1, it is enough to check that s is injective and coker(s) is  $\mathcal{O}_U$ -flat. Since  $\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$  is invertible, it is enough to verify  $s_t$  is injective for all  $t \in U$  by [EGA] §0.10.2.4, where  $s_t$  is the homomorphism obtained by restricting s to the fiber of q at t. It suffices to show that the restriction of the canonical homomorphism  $q^*q_*\mathcal{L} \to \mathcal{L}$  to the fiber of q at t is injective. By Theorem 1.1 (c) we have  $q_*\mathcal{L} \otimes_{\mathcal{O}_U} k(t) = H^0(X_{\mathfrak{m}}, \mathcal{L}_t)$ . So the restriction of the canonical homomorphism to the fiber is  $H^0(X_{\mathfrak{m}}, \mathcal{L}_t) \otimes_k \mathcal{O}_{X_{\mathfrak{m}}} \to \mathcal{L}_t$ . Denote this map by  $s'_t$ ; we need to show it is injective. But we have  $\dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t) = 1$ since  $t \in U$ . If we fix a nonzero element  $g \in H^0(X_{\mathfrak{m}}, \mathcal{L}_t)$ , then  $s'_t$  is identified with  $\mathcal{O}_{X_{\mathfrak{m}}} \to \mathcal{L}_t$ ,  $a \mapsto ag$ . This last map is injective since  $X_{\mathfrak{m}}$  is an integral scheme and g can be thought of as a rational function. So  $s_t$  is injective. Hence  $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$  defines a relative effective Cartier divisor. The restriction of this divisor to the fiber of q at t is the divisor on  $X_{\mathfrak{m}}$  defined by the pair  $(\mathcal{L}_t, g)$ , which is supported on  $X_{\mathfrak{m}} - Q$ . So the divisor defined by

 $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$  is supported on  $(X_{\mathfrak{m}} - Q) \times U$ . By Proposition 3.1 there exists a unique morphism of varieties  $m \colon U \to (X - S)^{(\pi)}$  such that the divisor defined by  $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$  is the pull-back by  $\mathrm{id} \times m$  of the universal relative effective Cartier divisor  $\mathcal{D}$  on  $X_{\mathfrak{m}} \times (X - S)^{(\pi)}$ . For any  $(D_1, D_2) \in U$ , we have  $l_{\mathfrak{m}}(D_1 + D_2 - \pi P_0) = 1$  and  $l(D_1 + D_2 - \pi P_0 - \mathfrak{m}) = 0$ . So there is one and only one effective divisor  $\mathfrak{m}$ -equivalent to  $D_1 + D_2 - \pi P_0$  and it is simply  $m(D_1, D_2)$ .

Similarly, using Lemma 3.4(b) and Proposition 3.1, one can show that there exists a morphism  $r: V \to (X - S)^{(\pi)}$  such that  $r(D_1, D_2)$  is the unique effective divisor m-equivalent to  $D_2 - D_1 + \pi P_0$  for any  $(D_1, D_2) \in V$ .

Let us verify that m defines a birational group on  $(X - S)^{(\pi)}$ . First we show

$$m(m(D_1, D_2), D_3) = m(D_1, m(D_2, D_3))$$

when  $(D_1,D_2)$ ,  $(D_2,D_3)$ ,  $(m(D_1,D_2),D_3)$  and  $(D_1,m(D_2,D_3))$  all belong to U. Indeed  $m(m(D_1,D_2),D_3)$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $m(D_1,D_2)+D_3-\pi P_0$ , and  $m(D_1,m(D_2,D_3))$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $D_1+m(D_2,D_3)-\pi P_0$ . But  $m(D_1,D_2)+D_3-\pi P_0$  and  $D_1+m(D_2,D_3)-\pi P_0$  are  $\mathfrak{m}$ -equivalent since both are  $\mathfrak{m}$ -equivalent to  $D_1+D_2+D_3-2\pi P_0$ . So we have  $m(m(D_1,D_2),D_3)=m(D_1,m(D_2,D_3))$ .

One can also verify  $m(D_1, D_2) = m(D_2, D_1)$  when both  $(D_1, D_2)$  and  $(D_2, D_1)$  are in U, that is, the operation m is commutative.

Next we show that  $\Theta: (D_1, D_2) \mapsto (D_1, r(D_1, D_2))$  is the birational inverse of  $\Phi: (D_1, D_2) \mapsto (D_1, m(D_1, D_2))$  so that  $\Phi$  is birational. Since the operation m is commutative, the rational map  $\Psi: (D_1, D_2) \mapsto (D_2, m(D_1, D_2))$  is also birational. Therefore m makes  $(X - S)^{(\pi)}$  a birational group.

First we verify  $\Phi\Theta(D_1,D_2)=(D_1,D_2)$  whenever the left-hand side is defined. We have

$$\Phi\Theta(D_1, D_2) = \Phi(D_1, r(D_1, D_2)) = (D_1, m(D_1, r(D_1, D_2))).$$

Moreover  $m(D_1, r(D_1, D_2))$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $D_1 + r(D_1, D_2) - \pi P_0$ . But  $D_2$  is also an effective divisor  $\mathfrak{m}$ -equivalent to  $D_1 + r(D_1, D_2) - \pi P_0$  since we have

$$D_1 + r(D_1, D_2) - \pi P_0 \sim_{\mathfrak{m}} D_1 + (D_2 - D_1 + \pi P_0) - \pi P_0 = D_2$$

Hence  $m(D_1, r(D_1, D_2)) = D_2$  and  $\Phi \Theta(D_1, D_2) = (D_1, D_2)$ .

Similarly one can show that  $\Theta \Phi(D_1, D_2) = (D_1, D_2)$  when the left-hand side is defined.

Note that  $\Phi$  is a regular morphism defined on U and  $\Theta$  is a regular morphism defined on V. Since

$$\Phi \Theta(D_1, D_2) = (D_1, D_2)$$
 and  $\Theta \Phi(D_1, D_2) = (D_1, D_2)$ 

whenever the left-hand sides are defined, the maps  $\Phi$  and  $\Theta$  induce regular morphisms  $\Phi \colon U \cap \Phi^{-1}(V) \to V \cap \Theta^{-1}(U)$  and  $\Theta \colon V \cap \Theta^{-1}(U) \to U \cap \Phi^{-1}(V)$ . To show that  $\Phi$  and  $\Theta$  are birational inverses to each other, it is enough to check that  $U \cap \Phi^{-1}(V)$  and  $V \cap \Theta^{-1}(U)$  are non-empty.

Note that  $(D_1, D_2) \in U \cap \Phi^{-1}(V)$  if and only if  $(D_1, D_2) \in U$  and

$$l_{\mathfrak{m}}(m(D_1, D_2) - D_1 + \pi P_0) = 1, \quad l(m(D_1, D_2) - D_1 + \pi P_0 - \mathfrak{m}) = 0.$$

Since  $m(D_1, D_2) \sim_{\mathfrak{m}} D_1 + D_2 - \pi P_0$ , the above equations are equivalent to

$$l_{\mathfrak{m}}(D_2) = 1, \quad l(D_2 - \mathfrak{m}) = 0.$$

Applying Lemma 3.3 to the divisor  $D_0 = 0$ , we conclude that the set

$$V_0 = \{ D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D) = 0, \quad l(D - \mathfrak{m}) = 0 \}$$

is open and non-empty. Since  $(X-S)^{(\pi)} \times (X-S)^{(\pi)}$  is irreducible, the set  $U \cap ((X-S)^{(\pi)} \times V_0)$  is also open and non-empty. This set is exactly  $U \cap \Phi^{-1}(V)$ . So  $U \cap \Phi^{-1}(V)$  is non-empty.

Similarly  $V \cap \Theta^{-1}(U)$  is also non-empty. This completes the proof of the proposition.

## 4. From Birational Groups to algebraic groups

Let k be an algebraically closed field, let V be a connected nonsingular variety over k, and let  $m\colon V\times V\to V$ ,  $(a,b)\mapsto ab$  be a rational map satisfying (ab)c=a(bc). Assume the rational maps  $\Phi(a,b)=(a,ab)$  and  $\Psi(a,b)=(b,ab)$  are birational. Then there exist open subsets  $X_\Phi$ ,  $Y_\Phi$ ,  $X_\Psi$  and  $Y_\Psi$  in  $V\times V$  such that  $\Phi$  induces an isomorphism  $X_\Phi\cong Y_\Phi$  and  $\Psi$  induces an isomorphism  $X_\Psi\cong Y_\Psi$ . Put  $Z=X_\Phi\cap Y_\Phi\cap X_\Psi\cap Y_\Psi$ .

It is convenient to write the formulae for  $\Phi^{-1}$  and  $\Psi^{-1}$  as  $\Phi^{-1}(a,b) = (a,a^{-1}b)$  and  $\Psi^{-1}(a,b) = (ba^{-1},a)$ .