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The map $D \mapsto (\mathcal{L}(D), s_D)$ defines a one-to-one correspondence between the set of relative effective Cartier divisors on X/T and the isomorphism classes of pairs (\mathcal{L}, s) , where \mathcal{L} is an invertible sheaf on X and s is a global section of \mathcal{L} such that the map $s: \mathcal{O}_X \rightarrow \mathcal{L}$ induced by the section s is injective and $\mathcal{L}/s\mathcal{O}_X$ is \mathcal{O}_T -flat.

The proof of the following lemma is straightforward and is left to the reader:

LEMMA 2.2.

(a) *If D_1 and D_2 are relative effective Cartier divisors on X/T , then so is $D_1 + D_2$.*

(b) *Let D_1 and D_2 be two relative effective Cartier divisors on X/T and let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be their ideal sheaves. If $\mathcal{I}(D_1) \subset \mathcal{I}(D_2)$, then $D_1 - D_2$ is also a relative effective Cartier divisor on X/T .*

(c) *Let $T' \rightarrow T$ be a base extension and let $X' = X \times_T T'$. If D is a relative effective Cartier divisor on X/T , then its pull-back to a closed subscheme D' of X' is a relative effective Cartier divisor on X'/T' .*

LEMMA 2.3. *Assume $q: X \rightarrow T$ is flat. Let \mathcal{I} be a coherent sheaf of ideals of \mathcal{O}_X and let D be the closed subscheme of X defined by \mathcal{I} . If for every point $x \in D$, the ideal \mathcal{I}_x of $\mathcal{O}_{X,x}$ is generated by one element g_x whose image in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{T,q(x)}} k(q(x))$ is not a zero divisor, then D is a relative effective Cartier divisor.*

Proof. It suffices to show that g_x is not a zero divisor in $\mathcal{O}_{X,x}$ and that $\mathcal{O}_{X,x}/(g_x)$ is flat over $\mathcal{O}_{T,q(x)}$. This follows from [EGA] §0.10.2.4 by taking $A = \mathcal{O}_{T,q(x)}$, $B = \mathcal{O}_{X,x}$, $M = N = \mathcal{O}_{X,x}$, and $u: M \rightarrow N$ to be the homomorphism $g_x: \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$ defined by the multiplication by g_x .

3. THE CONSTRUCTION OF A BIRATIONAL GROUP

Let X be a nonsingular irreducible projective curve over an algebraically closed field k . A *modulus* \mathfrak{m} supported on a finite subset S of X is a divisor of the form $\mathfrak{m} = \sum_{P \in S} n_P P$ with each $n_P > 0$. For any rational function f on X , we write $f \equiv 0 \pmod{\mathfrak{m}}$ if $v_P(f) \geq n_P$ for every $P \in S$, where v_P is the valuation defined by P . Two divisors D_1 and D_2 on X prime to S are called *\mathfrak{m} -equivalent* if there exists a rational function f satisfying $f - 1 \equiv 0 \pmod{\mathfrak{m}}$ such that $D_1 - D_2 = (f)$. If this holds, we write $D_1 \sim_{\mathfrak{m}} D_2$. Define a ringed

space (X_m, \mathcal{O}_{X_m}) as follows: The underlying set of X_m is $(X - S) \cup \{Q\}$. Define

$$\mathcal{O}_{X_m, Q} = k + \{f \mid f \equiv 0 \pmod{m}\}$$

and for every $x \in X - S$, define $\mathcal{O}_{X_m, x} = \mathcal{O}_{X, x}$. One can show that when $\deg(m) \geq 2$, the ringed space X_m is a singular curve with a unique singular point Q and its normalization is X . (It is easy to see that when $\deg(m) < 2$, the ringed space X_m is identified with X itself.) For a divisor D of X prime to S , we put

$$L_m(D) = H^0(X_m, \mathcal{L}_m), \quad I_m(D) = H^1(X_m, \mathcal{L}_m),$$

where \mathcal{L}_m is the invertible sheaf on X_m corresponding to D . Denote the dimensions of $L_m(D)$ and $I_m(D)$ by $l_m(D)$ and $i_m(D)$, respectively. The Riemann-Roch theorem states that

$$l_m(D) - i_m(D) = \deg(D) + 1 - \pi.$$

In this formula, π is the sum $\pi = g + \delta$, where g is the genus of X and $\delta = \deg(m) - 1$. All these results are proved in [S], Chapter IV.

For convenience, a closed point on a scheme is just called a point.

Let T be a connected k -scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_m \times T & \longrightarrow & X_m \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k). \end{array}$$

Since X_m is proper and flat over $\text{spec}(k)$, the morphism q is also proper and flat. Let D be a relative effective Cartier divisor on $(X_m \times T)/T$ supported on $(X_m - Q) \times T$ and let \mathcal{L} be the invertible sheaf corresponding to D . Applying Theorem 1.1 (a) to the morphism q and the invertible sheaf \mathcal{L} , we conclude that $t \mapsto \chi(\mathcal{L}_t)$ is a constant function on T . By the Riemann-Roch theorem, we have $\chi(\mathcal{L}_t) = \deg D_t + 1 - \pi$. So $\deg(D_t)$ is also a constant. This constant is called the *degree* of D . Denote by $\text{Div}^{(n)}(T)$ the set of all relative effective Cartier divisors of degree n on $(X_m \times T)/T$ supported on $(X_m - Q) \times T$.

Let $(X - S)^{(n)}$ be the n -th symmetric power of $X - S$, i.e., the quotient of $(X - S)^n$ by the action of the n -th symmetric group \mathfrak{S}_n , where \mathfrak{S}_n acts on $(X - S)^n$ by permuting the factors. In the Appendix we show that there exists a relative effective Cartier divisor $\mathcal{D} \in \text{Div}^{(n)}((X - S)^{(n)})$, called the *universal relative effective Cartier divisor*, whose restriction to the fiber of the projection $X_m \times (X - S)^{(n)} \rightarrow (X - S)^{(n)}$ at $P_1 + \cdots + P_n \in (X - S)^{(n)}$ is the divisor $P_1 + \cdots + P_n$ of X_m . Moreover, we have

PROPOSITION 3.1. *The functor $T \mapsto \text{Div}^{(n)}(T)$ from the category of k -schemes to the category of sets is represented by the symmetric power $(X-S)^{(n)}$. More precisely, for any relative effective Cartier divisor D of degree n on $(X_m \times T)/T$ supported on $(X_m - Q) \times T$, there exists a unique morphism $f: T \rightarrow (X-S)^{(n)}$ such that the pull-back of \mathcal{D} by $\text{id} \times f$ is D .*

The proof of this proposition is given in the Appendix. The morphism $T \rightarrow (X-S)^{(n)}$ can be described as follows: For every $t \in T$, identifying the fiber of $q: X_m \times T \rightarrow T$ at t with X_m , we may regard the restriction D_t of D to the fiber at t as an effective divisor of degree n on X_m supported on $X_m - Q$. But this kind of divisor can be thought of as a point in $(X-S)^{(n)}$. The morphism $T \rightarrow (X-S)^{(n)}$ is just $t \mapsto D_t$.

LEMMA 3.2. *Let D be a divisor of X prime to S such that $i_m(D) \geq 1$. Then there exists an open subset U of $X-S$ such that for every $P \in U$, we have $i_m(D+P) = i_m(D) - 1$.*

Proof. If $P \notin \text{Supp}(D) \cup S$, then the dual vector space $I_m(D+P)^*$ of $I_m(D+P)$ is identified with the subspace of $I_m(D)^*$ formed by differential forms $\omega \in I_m(D)^*$ vanishing at P . Let $\{\omega_1, \dots, \omega_{i_m(D)}\}$ be a basis of $I_m(D)^*$. We can then take U to be the complement of

$$\text{Supp}(D) \cup S \cup \{P \mid \omega_i(P) = 0 \text{ for } i = 1, \dots, i_m(D)\}.$$

LEMMA 3.3. *Let D_0 be a divisor of X prime to S of degree 0. Then the set*

$$V_{D_0} = \{D \in (X-S)^{(\pi)} \mid l_m(D+D_0) = 1 \text{ and } l(D+D_0 - m) = 0\}$$

is non-empty and open in $(X-S)^{(\pi)}$.

Proof. Consider the Cartesian square

$$\begin{array}{ccc} X_m \times (X-S)^{(\pi)} & \xrightarrow{p} & X_m \\ q \downarrow & & \downarrow \\ (X-S)^{(\pi)} & \longrightarrow & \text{spec}(k) . \end{array}$$

Applying Theorem 1.1 (b) to q and the invertible sheaf \mathcal{L} on $X_m \times (X-S)^{(\pi)}$ corresponding to the divisor $\mathcal{D} + p^*(D_0)$, where \mathcal{D} is the universal relative effective Cartier divisor, we conclude that the set

$$V_1 = \{t \in (X - S)^{(\pi)} \mid \dim H^0(X_m, \mathcal{L}_t) \leq 1\}$$

is open, that is,

$$V_1 = \{D \in (X - S)^{(\pi)} \mid l_m(D + D_0) \leq 1\}$$

is open. By the Riemann-Roch theorem we have, for any $D \in (X - S)^{(\pi)}$,

$$l_m(D + D_0) \geq \deg(D + D_0) + 1 - \pi = 1.$$

So we must have

$$V_1 = \{D \in (X - S)^{(\pi)} \mid l_m(D + D_0) = 1\}.$$

If $l_m(D_0) \neq 0$, then there exists a rational function f on X such that $(f) + D_0$ is an effective divisor on X prime to S . This effective divisor must be 0 since it is of degree 0. Hence $l_m(D_0) = l_m((f) + D_0) = l_m(0) = 1$. So in any case we have $l_m(D_0) \leq 1$. By the Riemann-Roch theorem, we have $i_m(D_0) \leq \pi$. Applying Lemma 3.2 repeatedly, we can find $P_1, \dots, P_{i_m(D_0)}$ in $X - S$ so that $i_m(D_0 + P_1 + \dots + P_{i_m(D_0)}) = 0$. Choose $P_{i_m(D_0)+1}, \dots, P_\pi$ in $X - S$ arbitrarily. We have

$$i_m(D_0 + P_1 + \dots + P_{i_m(D_0)}) \geq i_m(D_0 + P_1 + \dots + P_{i_m(D_0)} + P_{i_m(D_0)+1} + \dots + P_\pi).$$

(This can be seen by interpreting $i_m(D)$ as the dimension of the vector space of differential forms ω regular at Q satisfying $(\omega) \geq D$.) So we have $i_m(D_0 + P_1 + \dots + P_\pi) = 0$. By the Riemann-Roch theorem, we have $l_m(D_0 + P_1 + \dots + P_\pi) = 1$. Hence $P_1 + \dots + P_\pi$ is in the set V_1 and V_1 is not empty.

Similarly by Theorem 1.1 (b) applied to the projection $q: X \times (X - S)^{(\pi)} \rightarrow (X - S)^{(\pi)}$ and the invertible sheaf on $X \times (X - S)^{(\pi)}$ corresponding to the divisor $\mathcal{D} + p^*(D_0 - m)$, where $p: X \times (X - S)^{(\pi)} \rightarrow X$ is another projection, we see that the set

$$V_2 = \{D \in (X - S)^{(\pi)} \mid l(D + D_0 - m) = 0\}$$

is open. Since $\deg(D_0 - m) < 0$, we have $l(D_0 - m) = 0$. By the Riemann-Roch theorem, we have $i(D_0 - m) = \pi$. Applying Lemma 3.2 repeatedly (but taking $m = 0$), we can find $P_1, \dots, P_\pi \in X - S$ such that $i(D_0 - m + P_1 + \dots + P_\pi) = 0$. Then by the Riemann-Roch theorem we have $l(D_0 - m + P_1 + \dots + P_\pi) = 0$. So $P_1 + \dots + P_\pi$ is in V_2 and V_2 is not empty.

Since $(X - S)^{(\pi)}$ is irreducible, the set $V_{D_0} = V_1 \cap V_2$ is open and non-empty.

LEMMA 3.4. *Fix a point P_0 in S .*

(a) *The set*

$$U = \{(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_m(D_1 + D_2 - \pi P_0) = 1, \quad l(D_1 + D_2 - \pi P_0 - m) = 0\}$$

is a non-empty open subset of $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$.

(b) *The set*

$$V = \{(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_m(D_2 - D_1 + \pi P_0) = 1, \quad l(D_2 - D_1 + \pi P_0 - m) = 0\}$$

is a non-empty open subset of $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$.

Proof. (a) Let $p_1, p_2: (X - S)^{(\pi)} \times (X - S)^{(\pi)} \rightarrow (X - S)^{(\pi)}$ be the projections and let E_i ($i = 1, 2$) be the pull-backs by $\text{id} \times p_i$ of the universal relative effective Cartier divisor \mathcal{D} on $X_m \times (X - S)^{(\pi)}$. Put $E = E_1 + E_2$. This is a divisor on $X_m \times (X - S)^{(\pi)} \times (X - S)^{(\pi)}$.

Consider the Cartesian square

$$\begin{array}{ccc} X_m \times (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \xrightarrow{p} & X_m \\ q \downarrow & & \downarrow \\ (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \longrightarrow & \text{spec}(k) . \end{array}$$

By the Riemann-Roch theorem, for any $(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)}$, we have

$$l_m(D_1 + D_2 - \pi P_0) \geq \deg(D_1 + D_2 - \pi P_0) + 1 - \pi = 1 ,$$

that is, for any $t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)}$, we have $l_m(E_t - \pi P_0) \geq 1$. Applying Theorem 1.1 (b) to the projection q and the invertible sheaf corresponding to the divisor $E - p^*(P_0)$, we see that the set

$$U_1 = \{t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_m(E_t - \pi P_0) = 1\}$$

is open. Similarly the set

$$U_2 = \{t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l(E_t - \pi P_0 - m) = 0\}$$

is also open. Hence the set $U = U_1 \cap U_2$ is open.

Applying Lemma 3.3 to $D_0 = 0$, we see that there exists a $D \in (X - S)^{(\pi)}$ such that $l_m(D) = 1$ and $l(D - m) = 0$. Then $(D, \pi P_0)$ is in U . So U is non-empty. This proves (a).

The proof of (b) is similar and is omitted.

DEFINITION 3.5. A *birational group* over k is a nonsingular variety V together with a rational map $m: V \times V \rightarrow V$, $(a, b) \mapsto ab$ such that

- (a) $(ab)c = a(bc)$ when both sides are defined;
- (b) the rational maps $\Phi: (a, b) \mapsto (a, ab)$ and $\Psi: (a, b) \mapsto (b, ab)$ on $V \times V$ are birational.

PROPOSITION 3.6. *There exists a unique rational map*

$$m: (X - S)^{(\pi)} \times (X - S)^{(\pi)} \rightarrow (X - S)^{(\pi)}$$

whose domain of definition contains the set U in 3.4(a) such that $m(D_1, D_2)$ is the unique effective divisor that is \mathfrak{m} -equivalent to $D_1 + D_2 - \pi P_0$ for any $(D_1, D_2) \in U$. Moreover m makes $(X - S)^{(\pi)}$ a birational group.

Proof. Keep the notations in the proof of Lemma 3.4. Consider the Cartesian squares

$$\begin{array}{ccccccc} X_{\mathfrak{m}} = q^{-1}(t) & \longrightarrow & X_{\mathfrak{m}} \times U & \subset & X_{\mathfrak{m}} \times (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \xrightarrow{p} & X_{\mathfrak{m}} \\ \downarrow & & q \downarrow & & \downarrow & & \downarrow \\ \text{spec}(k(t)) & \longrightarrow & U & \subset & (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \longrightarrow & \text{spec}(k). \end{array}$$

Let \mathcal{L} be the restriction to $X_{\mathfrak{m}} \times U$ of the invertible sheaf corresponding to the divisor $E_1 + E_2 - p^*(\pi P_0)$. By Theorem 1.1(c) and the choice of U , the sheaf $q_*\mathcal{L}$ is invertible. The canonical homomorphism $q^*q_*\mathcal{L} \rightarrow \mathcal{L}$ gives rise to $s: \mathcal{O}_{X_{\mathfrak{m}} \times U} \rightarrow \mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$. We claim that the pair $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$ defines a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times U)/U$. According to Remark 2.1, it is enough to check that s is injective and $\text{coker}(s)$ is \mathcal{O}_U -flat. Since $\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$ is invertible, it is enough to verify s_t is injective for all $t \in U$ by [EGA] §0.10.2.4, where s_t is the homomorphism obtained by restricting s to the fiber of q at t . It suffices to show that the restriction of the canonical homomorphism $q^*q_*\mathcal{L} \rightarrow \mathcal{L}$ to the fiber of q at t is injective. By Theorem 1.1(c) we have $q_*\mathcal{L} \otimes_{\mathcal{O}_U} k(t) = H^0(X_{\mathfrak{m}}, \mathcal{L}_t)$. So the restriction of the canonical homomorphism to the fiber is $H^0(X_{\mathfrak{m}}, \mathcal{L}_t) \otimes_k \mathcal{O}_{X_{\mathfrak{m}}} \rightarrow \mathcal{L}_t$. Denote this map by s'_t ; we need to show it is injective. But we have $\dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t) = 1$ since $t \in U$. If we fix a nonzero element $g \in H^0(X_{\mathfrak{m}}, \mathcal{L}_t)$, then s'_t is identified with $\mathcal{O}_{X_{\mathfrak{m}}} \rightarrow \mathcal{L}_t$, $a \mapsto ag$. This last map is injective since $X_{\mathfrak{m}}$ is an integral scheme and g can be thought of as a rational function. So s_t is injective. Hence $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$ defines a relative effective Cartier divisor. The restriction of this divisor to the fiber of q at t is the divisor on $X_{\mathfrak{m}}$ defined by the pair (\mathcal{L}_t, g) , which is supported on $X_{\mathfrak{m}} - Q$. So the divisor defined by

$(\mathcal{L} \otimes (q^* q_* \mathcal{L})^{-1}, s)$ is supported on $(X_m - Q) \times U$. By Proposition 3.1 there exists a unique morphism of varieties $m: U \rightarrow (X - S)^{(\pi)}$ such that the divisor defined by $(\mathcal{L} \otimes (q^* q_* \mathcal{L})^{-1}, s)$ is the pull-back by $\text{id} \times m$ of the universal relative effective Cartier divisor \mathcal{D} on $X_m \times (X - S)^{(\pi)}$. For any $(D_1, D_2) \in U$, we have $l_m(D_1 + D_2 - \pi P_0) = 1$ and $l(D_1 + D_2 - \pi P_0 - m) = 0$. So there is one and only one effective divisor m -equivalent to $D_1 + D_2 - \pi P_0$ and it is simply $m(D_1, D_2)$.

Similarly, using Lemma 3.4 (b) and Proposition 3.1, one can show that there exists a morphism $r: V \rightarrow (X - S)^{(\pi)}$ such that $r(D_1, D_2)$ is the unique effective divisor m -equivalent to $D_2 - D_1 + \pi P_0$ for any $(D_1, D_2) \in V$.

Let us verify that m defines a birational group on $(X - S)^{(\pi)}$. First we show

$$m(m(D_1, D_2), D_3) = m(D_1, m(D_2, D_3))$$

when (D_1, D_2) , (D_2, D_3) , $(m(D_1, D_2), D_3)$ and $(D_1, m(D_2, D_3))$ all belong to U . Indeed $m(m(D_1, D_2), D_3)$ is the unique effective divisor m -equivalent to $m(D_1, D_2) + D_3 - \pi P_0$, and $m(D_1, m(D_2, D_3))$ is the unique effective divisor m -equivalent to $D_1 + m(D_2, D_3) - \pi P_0$. But $m(D_1, D_2) + D_3 - \pi P_0$ and $D_1 + m(D_2, D_3) - \pi P_0$ are m -equivalent since both are m -equivalent to $D_1 + D_2 + D_3 - 2\pi P_0$. So we have $m(m(D_1, D_2), D_3) = m(D_1, m(D_2, D_3))$.

One can also verify $m(D_1, D_2) = m(D_2, D_1)$ when both (D_1, D_2) and (D_2, D_1) are in U , that is, the operation m is commutative.

Next we show that $\Theta: (D_1, D_2) \mapsto (D_1, r(D_1, D_2))$ is the birational inverse of $\Phi: (D_1, D_2) \mapsto (D_1, m(D_1, D_2))$ so that Φ is birational. Since the operation m is commutative, the rational map $\Psi: (D_1, D_2) \mapsto (D_2, m(D_1, D_2))$ is also birational. Therefore m makes $(X - S)^{(\pi)}$ a birational group.

First we verify $\Phi \Theta(D_1, D_2) = (D_1, D_2)$ whenever the left-hand side is defined. We have

$$\Phi \Theta(D_1, D_2) = \Phi(D_1, r(D_1, D_2)) = (D_1, m(D_1, r(D_1, D_2))).$$

Moreover $m(D_1, r(D_1, D_2))$ is the unique effective divisor m -equivalent to $D_1 + r(D_1, D_2) - \pi P_0$. But D_2 is also an effective divisor m -equivalent to $D_1 + r(D_1, D_2) - \pi P_0$ since we have

$$D_1 + r(D_1, D_2) - \pi P_0 \sim_m D_1 + (D_2 - D_1 + \pi P_0) - \pi P_0 = D_2.$$

Hence $m(D_1, r(D_1, D_2)) = D_2$ and $\Phi \Theta(D_1, D_2) = (D_1, D_2)$.

Similarly one can show that $\Theta \Phi(D_1, D_2) = (D_1, D_2)$ when the left-hand side is defined.

Note that Φ is a regular morphism defined on U and Θ is a regular morphism defined on V . Since

$$\Phi \Theta(D_1, D_2) = (D_1, D_2) \quad \text{and} \quad \Theta \Phi(D_1, D_2) = (D_1, D_2)$$

whenever the left-hand sides are defined, the maps Φ and Θ induce regular morphisms $\Phi: U \cap \Phi^{-1}(V) \rightarrow V \cap \Theta^{-1}(U)$ and $\Theta: V \cap \Theta^{-1}(U) \rightarrow U \cap \Phi^{-1}(V)$. To show that Φ and Θ are birational inverses to each other, it is enough to check that $U \cap \Phi^{-1}(V)$ and $V \cap \Theta^{-1}(U)$ are non-empty.

Note that $(D_1, D_2) \in U \cap \Phi^{-1}(V)$ if and only if $(D_1, D_2) \in U$ and

$$l_m(m(D_1, D_2) - D_1 + \pi P_0) = 1, \quad l(m(D_1, D_2) - D_1 + \pi P_0 - m) = 0.$$

Since $m(D_1, D_2) \sim_m D_1 + D_2 - \pi P_0$, the above equations are equivalent to

$$l_m(D_2) = 1, \quad l(D_2 - m) = 0.$$

Applying Lemma 3.3 to the divisor $D_0 = 0$, we conclude that the set

$$V_0 = \{D \in (X - S)^{(\pi)} \mid l_m(D) = 0, \quad l(D - m) = 0\}$$

is open and non-empty. Since $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$ is irreducible, the set $U \cap ((X - S)^{(\pi)} \times V_0)$ is also open and non-empty. This set is exactly $U \cap \Phi^{-1}(V)$. So $U \cap \Phi^{-1}(V)$ is non-empty.

Similarly $V \cap \Theta^{-1}(U)$ is also non-empty. This completes the proof of the proposition.

4. FROM BIRATIONAL GROUPS TO ALGEBRAIC GROUPS

Let k be an algebraically closed field, let V be a connected nonsingular variety over k , and let $m: V \times V \rightarrow V$, $(a, b) \mapsto ab$ be a rational map satisfying $(ab)c = a(bc)$. Assume the rational maps $\Phi(a, b) = (a, ab)$ and $\Psi(a, b) = (b, ab)$ are birational. Then there exist open subsets X_Φ , Y_Φ , X_Ψ and Y_Ψ in $V \times V$ such that Φ induces an isomorphism $X_\Phi \cong Y_\Phi$ and Ψ induces an isomorphism $X_\Psi \cong Y_\Psi$. Put $Z = X_\Phi \cap Y_\Phi \cap X_\Psi \cap Y_\Psi$.

It is convenient to write the formulae for Φ^{-1} and Ψ^{-1} as $\Phi^{-1}(a, b) = (a, a^{-1}b)$ and $\Psi^{-1}(a, b) = (ba^{-1}, a)$.