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## 4. REAL STRUCTURES

Recall that a *real algebraic variety* is a pair  $(X, S)$  where  $X$  is a complex algebraic variety and  $S: X \rightarrow X$  is an anti-holomorphic involution on it. The set of fixed points of  $S$  is the *real part* of  $(X, S)$ .  $S$  acts on the group of divisors  $\text{Div}(X)$ : if  $D \in \text{Div}(X)$  is defined locally by analytic functions  $f_\alpha$ , then  $S(D)$  is defined by the analytic functions  $\overline{f_\alpha \circ S}$ . Thus it is natural to define an involution  $S^*$  on the sheaf of analytic functions  $\mathcal{O}_X$

$$S^*: \Gamma(S(U), \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X) : f \mapsto \overline{f \circ S}.$$

This also induces an involution on the groups of one-forms and one-cycles. If  $\omega \in H^0(X, \Omega^1)$ ,  $c \in H_1(X, \mathbf{Z})$ , then  $\int_c S^* \omega = \overline{\int_{S(c)} \omega}$ . A form  $\omega$  is  $S$ -real if and only if  $S^* \omega = \omega$  and one may always choose a basis of  $S$ -real forms. In the case when  $X = C_h$  is the spectral curve of the Lagrange top, the action of  $S$  on  $\text{Div}(X)$  induces an involution on  $J(C_h; \infty^\pm)$ . This, however, does not suffice to determine the real structure of the invariant manifold  $T_h \sim J(C_h; \infty^\pm) \setminus \phi^{-1}(p)$  (Theorem 2.2), as it will also depend on the point  $p \in J(C_h)$ . Recall that the symmetric product  $S^2 \check{C}_h$  is bi-rational to  $T_h$ . Thus the generalized Jacobian and the invariant manifold  $T_h$  are identified by the Abel map

$$(59) \quad \mathcal{A}: S^2 \check{C}_h \rightarrow J(C_h; \infty^\pm) : P_1 + P_2 \mapsto \int_{W_1 + W_2}^{P_1 + P_2} \omega, \quad \omega = (\omega_1, \omega_2).$$

This induces an involution on  $J(C_h; \infty^\pm)$ ,  $z \mapsto S(z)$ , where

$$z = \int_{W_1 + W_2}^{P_1 + P_2} \omega, \quad S(z) = \int_{W_1 + W_2}^{S(P_1 + P_2)} \omega.$$

Of course this depends on the fixed points  $W_1, W_2 \in J(C_h; \infty^\pm)$ . Let  $\omega_1, \omega_2$  be  $S$ -real. Then

$$S(z) = \int_{W_1 + W_2}^{S(W_1 + W_2)} \omega + \int_{S(W_1 + W_2)}^{S(P_1 + P_2)} \omega = \int_{W_1 + W_2}^{S(W_1 + W_2)} \omega + \overline{\int_{W_1 + W_2}^{P_1 + P_2} \omega} = S(0) + \bar{z}.$$

If  $S$  has a fixed point on  $J(C_h; \infty^\pm)$  (this does not depend on  $W_1, W_2$ ) then one may always choose it for origin, and hence  $S(z) = \bar{z}$  becomes a group homomorphism.

Denote by  $S$  the anti-holomorphic involution on the spectral curve  $C_h$  defined by  $S(\lambda, \mu) = (\bar{\lambda}, -\bar{\mu})$ . This involution comes from the real Lax pair of Adler and van Moerbeke defined in Section 2. We shall also suppose that the real polynomial  $f(\lambda)$  has distinct roots.  $S$  induces an involution on the

usual Jacobian  $J(C_h)$  which we also denote by  $S$ , and an involution on the generalized Jacobian  $J(C_h; \infty^\pm)$  which we denote by  $S^+$ . If we use (59), then in terms of the Jacobi polynomials  $U, V, W$ , it is given by

$$S^+ : (U, V, W) \mapsto (\bar{U}, -\bar{V}, \bar{W}).$$

There is another natural anti-holomorphic involution on  $T_h$  given by the usual complex conjugation

$$(\Omega_i, \Gamma_i) \mapsto (\bar{\Omega}_i, \bar{\Gamma}_i),$$

which we denote by  $S^-$ . In terms of the Jacobi polynomials (12) it is

$$S^- : (U, V, W) \mapsto (\bar{W}, \bar{V}, \bar{U}).$$

**PROPOSITION 4.1.** *The holomorphic involution  $S^+ \circ S^- = S^- \circ S^+$  on  $J(C_h; \infty^\pm)$  is a translation on the half-period  $\frac{1}{2}\Lambda_2$ , where  $\phi(\frac{1}{2}\Lambda_2) = 0 \in J(C_h)$  (see (7), (9)).*

The proof of the above Proposition will be given later in this section. If  $\phi$  is the projection homomorphism defined in (7), then it implies

$$\phi \circ S^+ = \phi \circ S^- = S \circ \phi.$$

In other words the anti-holomorphic involutions  $S^+$  and  $S^-$  “look alike” in the same way on the usual Jacobian  $J(C_h)$  and differ in a half-period in the “vertical” direction with respect to  $\phi$  on the generalized Jacobian  $J(C_h; \infty^\pm)$ .

An important feature of  $S^+$  is that the  $S^+$ -real part of the invariant level set  $T_h$  is preserved by the flow of (2). Indeed, changing the variables as

$$\Omega_1 \rightarrow i\Omega_1, \quad \Omega_2 \rightarrow i\Omega_2, \quad \Omega_3 \rightarrow \Omega_3,$$

$$\Gamma_1 \rightarrow i\Gamma_1, \quad \Gamma_2 \rightarrow i\Gamma_2, \quad \Gamma_3 \rightarrow \Gamma_3,$$

we obtain a new system

$$(60) \quad \begin{aligned} \dot{\Omega}_1 &= -m\Omega_2\Omega_3 - \Gamma_2, & \dot{\Gamma}_1 &= \Gamma_2\Omega_3 - \Gamma_3\Omega_2, \\ \dot{\Omega}_2 &= m\Omega_3\Omega_1 + \Gamma_1, & \dot{\Gamma}_2 &= \Gamma_3\Omega_1 - \Gamma_1\Omega_3, \\ \dot{\Omega}_3 &= 0, & \dot{\Gamma}_3 &= \Gamma_2\Omega_1 - \Gamma_1\Omega_2, \end{aligned}$$

with first integrals

$$\begin{aligned} H_1 &= -\Gamma_1^2 - \Gamma_2^2 + \Gamma_3^2, & H_2 &= -\Omega_1\Gamma_1 - \Omega_2\Gamma_2 + (1+m)\Omega_3\Gamma_3, \\ H_3 &= \frac{1}{2}(-\Omega_1^2 - \Omega_2^2 + (1+m)\Omega_3^2) - \Gamma_3, & H_4 &= \Omega_3. \end{aligned}$$

The anti-holomorphic involution  $S^+$  in these coordinates is given again by the complex conjugation.

**THEOREM 4.2.** *In each of the three connected subdomains of the complement to the discriminant locus of  $f(\lambda)$  the topological type of the real part of the algebraic varieties  $(J(C_h; \infty^\pm), S^\pm)$  and  $(T_h, S^\pm)$  is one and the same and is given in the following table, where  $T^2 = S^1 \times S^1$ .*

roots of $f(\lambda)$	no real roots	two real roots	four real roots
real part of $(J(C_h; \infty^\pm), S^\pm)$	$T^2$	$T^2$	$T^2 \times (\mathbf{Z}/2)$
real part of $(J(C_h; \infty^\pm), S^\pm)$	$T^2$	$\emptyset$	$\emptyset$
real part of $(T_h, S^+)$	$S^1 \times \mathbf{R}$	$S^1 \times \mathbf{R}$	$T^2 \cup (S^1 \times \mathbf{R})$
real part of $(T_h, S^-)$	$T^2$	$\emptyset$	$\emptyset$

**REMARK.** It is easy to check that when the real invariant level set  $T_h^R$  of the Lagrange top is non-empty, then the polynomial  $f(\lambda)$  has no real roots. If we do not use the generalized Jacobian  $J(C_h; \infty^\pm)$ , then it might be difficult to understand the relation between  $T_h^R$  (which has one connected component),  $C_h^R$  (which is empty) and  $J(C_h)^R$  (which has two connected components) (cf. [2], [3, p. 37]).

*Proof of Proposition 4.1.* We have  $S^+ \circ S^- : (U, V, W) \mapsto (W, -V, U)$ . The involution  $(U, V, W) \mapsto (U, -V, W)$  is obviously induced by the elliptic involution  $i : (\lambda, \mu) \mapsto (\lambda, -\mu)$  on  $C_h$  so it is a reflexion. This means that if a fixed point of  $i$  is taken for origin in  $J(C_h; \infty^\pm)$  then  $i = -\text{identity}$ . It remains to prove that  $j : (U, V, W) \mapsto (W, V, U)$  is a reflexion too. The involution  $j$  has the following simple geometrical interpretation. Let  $P_1, P_2$  be two generic points in the  $(\lambda, \mu)$  plane and lying on the affine curve  $\check{C}_h = \{\mu^2 = f(\lambda)\}$ . If  $\{\mu = V(\lambda)\}$  is the straight line through  $P_1$  and  $P_2$  then it intersects  $C_h$  in four points  $P_1, P_2, P_3, P_4$  and then  $j(P_1 + P_2) = P_3 + P_4$ . Indeed, if the zero divisor of the Jacobi polynomial  $U(\lambda)$  on  $C_h$  is  $P_1 + P_2 + i(P_1) + i(P_2)$ , then by (13) the zero divisor of  $W(\lambda)$  is  $P_3 + P_4 + i(P_3) + i(P_4)$  and the involution  $P_1 + P_2 \mapsto P_3 + P_4$  amounts to exchanging the roots of  $U(\lambda)$  and  $V(\lambda)$ .

Let  $W_i$ ,  $i = 1, \dots, 4$  be the Weierstrass points on  $C_h$ . Then

$$\left( \frac{\mu - V(\lambda)}{\mu} \right) = \sum_{i=1}^4 P_i - \sum_{i=1}^4 W_i, \quad \frac{\mu - V(\lambda)}{\mu} \approx 1$$

and hence on  $J(C_h; \infty^\pm) \sim \text{Div}^0(\check{C}_h) / \sim$  we have  $P_1 + P_2 = -P_3 - P_4 + \text{constant}$ . This implies that  $j$  is a reflexion. Thus we have proved that  $S^+ \circ S^-$

is a translation  $(S^+ \circ S^-)(z) = z + a$ . Finally,  $a$  is easily computed. We have  $i(W_k) = W_k$ ,  $j(W_1 + W_2) = W_3 + W_4$  and hence  $a \stackrel{m}{\sim} W_1 + W_2 - W_3 - W_4$ . Further if  $\lambda_1, \lambda_2$  are zeros of  $f(\lambda)$ , then  $(g) = W_1 + W_2 - W_3 - W_4$ , where  $g(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)/\mu$ . Moreover  $g(\infty^\pm) = \pm 1$ ,  $g^2(\infty^\pm) = 1$  and hence

$$W_1 + W_2 - W_3 - W_4 \sim 0, \quad W_1 + W_2 - W_3 - W_4 \stackrel{m}{\not\sim} 0, \quad 2(W_1 + W_2 - W_3 - W_4) \stackrel{m}{\sim} 0.$$

This shows that  $a$  is a half-period and  $\phi(a) = 0 \in J(C_h)$ .  $\square$

*Proof of Theorem 4.2.* The proof will consist of two steps. First we determine the action of  $S^\pm$  on  $H_1(\check{C}_h, \mathbf{Z})$  and hence on the period lattice  $\Lambda$ . From that we deduce the first two lines of the table. Second, we determine the action of  $S^\pm: D_\infty \mapsto D_\infty$  on the infinity divisor  $D_\infty = \phi^{-1}(p) = \mathbf{C}^2/\Lambda_2 \sim C^*$  and then we use that

$$\text{real part of } (T_h, S^\pm) = \text{real part of } (J(C_h; \infty^\pm), S^\pm) - \text{real part of } D_\infty.$$

It is easier to determine the action of  $S^+$  on  $\Lambda$ . Indeed,  $S^+$  is induced by an anti-holomorphic involution on  $C_h$ ,  $S^+: (\lambda, \mu) \mapsto (\bar{\lambda}, -\bar{\mu})$ . Note that  $S^+$  always has fixed points on  $J(C_h; \infty^\pm)$ : if  $W_1, W_2$  are two Weierstrass points on  $C_h$  such that either  $W_1 = \bar{W}_2$ , or  $W_1$  and  $W_2$  are  $S^+$ -real, then  $S^+(W_1 + W_2) = W_1 + W_2$ . On the other hand  $S^-$  has fixed points only if  $f(\lambda)$  has no real roots. Indeed, in this last case let  $W_i$ ,  $i = 1, \dots, 4$ , be the Weierstrass points of  $C_h$  where  $W_1 = \bar{W}_2$ ,  $W_3 = \bar{W}_4$ . Then  $j(W_1 + W_3) = W_2 + W_4$  (see the proof of Proposition 4.1) and hence  $S^-(W_1 + W_3) = W_1 + W_3$ . On the other hand if  $U = \bar{W}$  and  $V = \bar{V}$ , then

$$V^2(\lambda) + U(\lambda)W(\lambda) = |V(\lambda)|^2 + |U(\lambda)|^2 = f(\lambda) > 0 \quad \forall \lambda \in \mathbf{R},$$

and hence  $f(\lambda)$  has no real roots.

Suppose first that  $f(\lambda)$  has no real roots and let us choose a basis  $A_1, B_1, A_2$  of  $H_1(\check{C}_h, \mathbf{Z})$  as shown in Figure 2 and in Figure 3 overleaf.

Then  $S^+(A_1) = A_1$ ,  $S^+(A_2) = A_2$  and it is easily seen that  $S^+(B_1) + B_1$  is homologous to  $A_2$  on  $H_1(\check{C}_h, \mathbf{Z})$ . Thus in the basis  $A_1, A_2, B_1$  the matrix of the involution  $S^+: H_1(\check{C}_h, \mathbf{Z}) \rightarrow H_1(\check{C}_h, \mathbf{Z})$  takes the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

From this and the fact that  $(J(C_h; \infty^\pm), S^+)$  is not empty we conclude that the real part of  $(J(C_h; \infty^\pm), S^+)$  is a torus with generators the periods  $\int_{B_1} \omega$  and

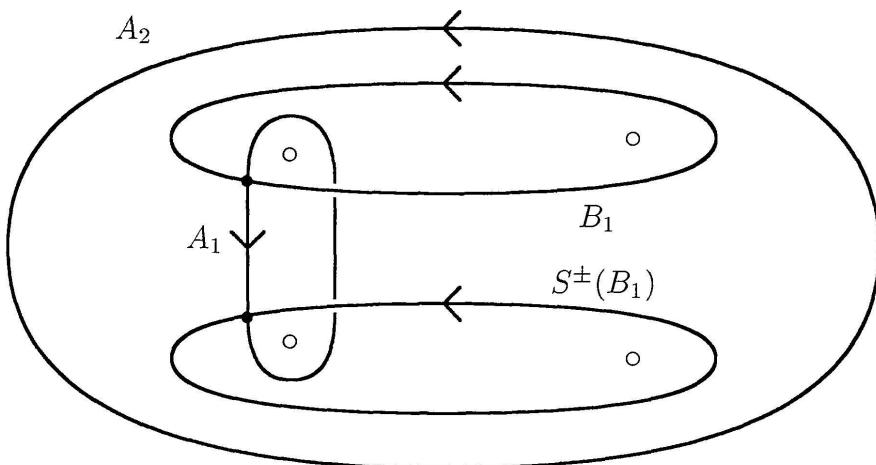


FIGURE 3

Projection of the cycles  $A_1, B_1, A_2, S^\pm(B_1)$  on the  $\lambda$ -plane

$\int_{A_2} \omega$ . On the other hand the real part of  $(J(C_h; \infty^\pm), S^-)$  is also non-empty and  $S^+ \circ S^-$  is a translation. We conclude that the real part of  $(J(C_h; \infty^\pm), S^-)$  is just a translation of the real part of  $(J(C_h; \infty^\pm), S^+)$  and in particular it is generated by the same periods.

In a similar way we find the real part of  $(J(C_h; \infty^\pm), S^+)$  in the remaining cases. Note that in an appropriate  $\mathbf{Z}$  basis of  $H_1(\check{C}_h, \mathbf{Z})$  the matrix of the involution  $S^\pm: H_1(\check{C}_h, \mathbf{Z}) \rightarrow H_1(\check{C}_h, \mathbf{Z})$  takes the same form if  $f(\lambda)$  has two real roots, and it is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

if  $f(\lambda)$  has four real roots. This implies the first two lines of the table.

Let us determine now the real part of  $(D_\infty, S^\pm)$ . As  $D_\infty = \mathbf{C}^*/\Lambda_2$  then we have to compute  $S^\pm(\Lambda_2)$ . Note that, as the real invariant manifold  $T_h$  is compact, then  $(D_\infty, S^-)$  is always empty. On the other hand  $(D_\infty, S^+)$  is never empty. Indeed, if  $S^+(\lambda, \mu) = (\bar{\lambda}, -\bar{\mu})$  then for  $Q \in C_h$  the point  $Q + S^+(Q)$  is  $S^+$ -real on  $J(C_h; \infty^\pm)$ . As  $S^+(\infty^+) = \infty^-$  we see that an  $S^+$ -real point of  $\phi^{-1}(p)$  is obtained by taking the limit  $Q \mapsto \infty^+$  in  $S^+(Q) + Q$  along an appropriate real analytic curve on  $\check{C}_h$ . Finally, from the computation of the action of  $S^+$  on  $\Lambda$  we get  $S^+(\Lambda_2) = \Lambda_2$  which shows that the  $S^+$ -real part of  $(\phi^{-1}(p), S^+)$  is always a circle  $\mathbf{R}/\Lambda_2$ . This gives the last two lines in the table.  $\square$