

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COMPLEX GEOMETRY OF THE LAGRANGE TOP
Autor: Gavrilov, Lubomir / ZHIVKOV, Angel
Kapitel: 4. Real structures
DOI: <https://doi.org/10.5169/seals-63901>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

4. REAL STRUCTURES

Recall that a *real algebraic variety* is a pair (X, S) where X is a complex algebraic variety and $S: X \rightarrow X$ is an anti-holomorphic involution on it. The set of fixed points of S is the *real part* of (X, S) . S acts on the group of divisors $\text{Div}(X)$: if $D \in \text{Div}(X)$ is defined locally by analytic functions f_α , then $S(D)$ is defined by the analytic functions $\overline{f_\alpha} \circ S$. Thus it is natural to define an involution S^* on the sheaf of analytic functions \mathcal{O}_X

$$S^*: \Gamma(S(U), \mathcal{O}_X) \rightarrow \Gamma(U, \mathcal{O}_X) : f \mapsto \overline{f \circ S}.$$

This also induces an involution on the groups of one-forms and one-cycles. If $\omega \in H^0(X, \Omega^1)$, $c \in H_1(X, \mathbf{Z})$, then $\int_c S^* \omega = \overline{\int_{S(c)} \omega}$. A form ω is *S-real* if and only if $S^* \omega = \omega$ and one may always choose a basis of *S-real* forms. In the case when $X = C_h$ is the spectral curve of the Lagrange top, the action of S on $\text{Div}(X)$ induces an involution on $J(C_h; \infty^\pm)$. This, however, does not suffice to determine the real structure of the invariant manifold $T_h \sim J(C_h; \infty^\pm) \setminus \phi^{-1}(p)$ (Theorem 2.2), as it will also depend on the point $p \in J(C_h)$. Recall that the symmetric product $S^2 \check{C}_h$ is bi-rational to T_h . Thus the generalized Jacobian and the invariant manifold T_h are identified by the Abel map

$$(59) \quad \mathcal{A}: S^2 \check{C}_h \rightarrow J(C_h; \infty^\pm) : P_1 + P_2 \mapsto \int_{W_1+W_2}^{P_1+P_2} \omega, \quad \omega = (\omega_1, \omega_2).$$

This induces an involution on $J(C_h; \infty^\pm)$, $z \rightarrow S(z)$, where

$$z = \int_{W_1+W_2}^{P_1+P_2} \omega, \quad S(z) = \int_{W_1+W_2}^{S(P_1+P_2)} \omega.$$

Of course this depends on the fixed points $W_1, W_2 \in J(C_h; \infty^\pm)$. Let ω_1, ω_2 be *S-real*. Then

$$S(z) = \int_{W_1+W_2}^{S(W_1+W_2)} \omega + \int_{S(W_1+W_2)}^{S(P_1+P_2)} \omega = \int_{W_1+W_2}^{S(W_1+W_2)} \omega + \overline{\int_{W_1+W_2}^{P_1+P_2} \omega} = S(0) + \bar{z}.$$

If S has a fixed point on $J(C_h; \infty^\pm)$ (this does not depend on W_1, W_2) then one may always choose it for origin, and hence $S(z) = \bar{z}$ becomes a group homomorphism.

Denote by S the anti-holomorphic involution on the spectral curve C_h defined by $S(\lambda, \mu) = (\bar{\lambda}, -\bar{\mu})$. This involution comes from the real Lax pair of Adler and van Moerbeke defined in Section 2. We shall also suppose that the real polynomial $f(\lambda)$ has distinct roots. S induces an involution on the

usual Jacobian $J(C_h)$ which we also denote by S , and an involution on the generalized Jacobian $J(C_h; \infty^\pm)$ which we denote by S^+ . If we use (59), then in terms of the Jacobi polynomials U, V, W , it is given by

$$S^+ : (U, V, W) \mapsto (\bar{U}, -\bar{V}, \bar{W}).$$

There is another natural anti-holomorphic involution on T_h given by the usual complex conjugation

$$(\Omega_i, \Gamma_i) \mapsto (\bar{\Omega}_i, \bar{\Gamma}_i),$$

which we denote by S^- . In terms of the Jacobi polynomials (12) it is

$$S^- : (U, V, W) \mapsto (\bar{W}, \bar{V}, \bar{U}).$$

PROPOSITION 4.1. *The holomorphic involution $S^+ \circ S^- = S^- \circ S^+$ on $J(C_h; \infty^\pm)$ is a translation on the half-period $\frac{1}{2}\Lambda_2$, where $\phi(\frac{1}{2}\Lambda_2) = 0 \in J(C_h)$ (see (7), (9)).*

The proof of the above Proposition will be given later in this section. If ϕ is the projection homomorphism defined in (7), then it implies

$$\phi \circ S^+ = \phi \circ S^- = S \circ \phi.$$

In other words the anti-holomorphic involutions S^+ and S^- “look alike” in the same way on the usual Jacobian $J(C_h)$ and differ in a half-period in the “vertical” direction with respect to ϕ on the generalized Jacobian $J(C_h; \infty^\pm)$.

An important feature of S^+ is that the S^+ -real part of the invariant level set T_h is preserved by the flow of (2). Indeed, changing the variables as

$$\begin{aligned} \Omega_1 &\rightarrow i\Omega_1, & \Omega_2 &\rightarrow i\Omega_2, & \Omega_3 &\rightarrow \Omega_3, \\ \Gamma_1 &\rightarrow i\Gamma_1, & \Gamma_2 &\rightarrow i\Gamma_2, & \Gamma_3 &\rightarrow \Gamma_3, \end{aligned}$$

we obtain a new system

$$(60) \quad \begin{aligned} \dot{\Omega}_1 &= -m\Omega_2\Omega_3 - \Gamma_2, & \dot{\Gamma}_1 &= \Gamma_2\Omega_3 - \Gamma_3\Omega_2, \\ \dot{\Omega}_2 &= m\Omega_3\Omega_1 + \Gamma_1, & \dot{\Gamma}_2 &= \Gamma_3\Omega_1 - \Gamma_1\Omega_3, \\ \dot{\Omega}_3 &= 0, & \dot{\Gamma}_3 &= \Gamma_2\Omega_1 - \Gamma_1\Omega_2, \end{aligned}$$

with first integrals

$$\begin{aligned} H_1 &= -\Gamma_1^2 - \Gamma_2^2 + \Gamma_3^2, & H_2 &= -\Omega_1\Gamma_1 - \Omega_2\Gamma_2 + (1+m)\Omega_3\Gamma_3, \\ H_3 &= \frac{1}{2}(-\Omega_1^2 - \Omega_2^2 + (1+m)\Omega_3^2) - \Gamma_3, & H_4 &= \Omega_3. \end{aligned}$$

The anti-holomorphic involution S^+ in these coordinates is given again by the complex conjugation.

THEOREM 4.2. *In each of the three connected subdomains of the complement to the discriminant locus of $f(\lambda)$ the topological type of the real part of the algebraic varieties $(J(C_h; \infty^\pm), S^\pm)$ and (T_h, S^\pm) is one and the same and is given in the following table, where $T^2 = S^1 \times S^1$.*

roots of $f(\lambda)$	no real roots	two real roots	four real roots
real part of $(J(C_h; \infty^\pm), S^+)$	T^2	T^2	$T^2 \times (\mathbf{Z}/2)$
real part of $(J(C_h; \infty^\pm), S^-)$	T^2	\emptyset	\emptyset
real part of (T_h, S^+)	$S^1 \times \mathbf{R}$	$S^1 \times \mathbf{R}$	$T^2 \cup (S^1 \times \mathbf{R})$
real part of (T_h, S^-)	T^2	\emptyset	\emptyset

REMARK. It is easy to check that when the real invariant level set $T_h^{\mathbf{R}}$ of the Lagrange top is non-empty, then the polynomial $f(\lambda)$ has no real roots. If we do not use the generalized Jacobian $J(C_h; \infty^\pm)$, then it might be difficult to understand the relation between $T_h^{\mathbf{R}}$ (which has one connected component), $C_h^{\mathbf{R}}$ (which is empty) and $J(C_h)^{\mathbf{R}}$ (which has two connected components) (cf. [2], [3, p. 37]).

Proof of Proposition 4.1. We have $S^+ \circ S^- : (U, V, W) \mapsto (W, -V, U)$. The involution $(U, V, W) \mapsto (U, -V, W)$ is obviously induced by the elliptic involution $i : (\lambda, \mu) \mapsto (\lambda, -\mu)$ on C_h so it is a reflexion. This means that if a fixed point of i is taken for origin in $J(C_h; \infty^\pm)$ then $i = -\text{identity}$. It remains to prove that $j : (U, V, W) \mapsto (W, V, U)$ is a reflexion too. The involution j has the following simple geometrical interpretation. Let P_1, P_2 be two generic points in the (λ, μ) plane and lying on the affine curve $\check{C}_h = \{\mu^2 = f(\lambda)\}$. If $\{\mu = V(\lambda)\}$ is the straight line through P_1 and P_2 then it intersects C_h in four points P_1, P_2, P_3, P_4 and then $j(P_1 + P_2) = P_3 + P_4$. Indeed, if the zero divisor of the Jacobi polynomial $U(\lambda)$ on C_h is $P_1 + P_2 + i(P_1) + i(P_2)$, then by (13) the zero divisor of $W(\lambda)$ is $P_3 + P_4 + i(P_3) + i(P_4)$ and the involution $P_1 + P_2 \mapsto P_3 + P_4$ amounts to exchanging the roots of $U(\lambda)$ and $V(\lambda)$.

Let W_i , $i = 1, \dots, 4$ be the Weierstrass points on C_h . Then

$$\left(\frac{\mu - V(\lambda)}{\mu} \right) = \sum_{i=1}^4 P_i - \sum_{i=1}^4 W_i, \quad \frac{\mu - V(\lambda)}{\mu} \approx 1$$

and hence on $J(C_h; \infty^\pm) \sim \text{Div}^0(\check{C}_h) / \sim^m$ we have $P_1 + P_2 = -P_3 - P_4 + \text{constant}$. This implies that j is a reflexion. Thus we have proved that $S^+ \circ S^-$

is a translation $(S^+ \circ S^-)(z) = z + a$. Finally, a is easily computed. We have $i(W_k) = W_k$, $j(W_1 + W_2) = W_3 + W_4$ and hence $a \stackrel{m}{\sim} W_1 + W_2 - W_3 - W_4$. Further if λ_1, λ_2 are zeros of $f(\lambda)$, then $(g) = W_1 + W_2 - W_3 - W_4$, where $g(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)/\mu$. Moreover $g(\infty^\pm) = \pm 1$, $g^2(\infty^\pm) = 1$ and hence

$$W_1 + W_2 - W_3 - W_4 \sim 0, \quad W_1 + W_2 - W_3 - W_4 \not\stackrel{m}{\sim} 0, \quad 2(W_1 + W_2 - W_3 - W_4) \stackrel{m}{\sim} 0.$$

This shows that a is a half-period and $\phi(a) = 0 \in J(C_h)$. \square

Proof of Theorem 4.2. The proof will consist of two steps. First we determine the action of S^\pm on $H_1(\check{C}_h, \mathbf{Z})$ and hence on the period lattice Λ . From that we deduce the first two lines of the table. Second, we determine the action of $S^\pm: D_\infty \mapsto D_\infty$ on the infinity divisor $D_\infty = \phi^{-1}(p) = \mathbf{C}^2/\Lambda_2 \sim C^*$ and then we use that

$$\text{real part of } (T_h, S^\pm) = \text{real part of } (J(C_h; \infty^\pm), S^\pm) - \text{real part of } D_\infty.$$

It is easier to determine the action of S^+ on Λ . Indeed, S^+ is induced by an anti-holomorphic involution on C_h , $S^+: (\lambda, \mu) \mapsto (\bar{\lambda}, -\bar{\mu})$. Note that S^+ always has fixed points on $J(C_h; \infty^\pm)$: if W_1, W_2 are two Weierstrass points on C_h such that either $W_1 = \bar{W}_2$, or W_1 and W_2 are S^+ -real, then $S^+(W_1 + W_2) = W_1 + W_2$. On the other hand S^- has fixed points only if $f(\lambda)$ has no real roots. Indeed, in this last case let W_i , $i = 1, \dots, 4$, be the Weierstrass points of C_h where $W_1 = \bar{W}_2$, $W_3 = \bar{W}_4$. Then $j(W_1 + W_3) = W_2 + W_4$ (see the proof of Proposition 4.1) and hence $S^-(W_1 + W_3) = W_1 + W_3$. On the other hand if $U = \bar{W}$ and $V = \bar{V}$, then

$$V^2(\lambda) + U(\lambda)W(\lambda) = |V(\lambda)|^2 + |U(\lambda)|^2 = f(\lambda) > 0 \quad \forall \lambda \in \mathbf{R},$$

and hence $f(\lambda)$ has no real roots.

Suppose first that $f(\lambda)$ has no real roots and let us choose a basis A_1, B_1, A_2 of $H_1(\check{C}_h, \mathbf{Z})$ as shown in Figure 2 and in Figure 3 overleaf.

Then $S^+(A_1) = A_1$, $S^+(A_2) = A_2$ and it is easily seen that $S^+(B_1) + B_1$ is homologous to A_2 on $H_1(\check{C}_h, \mathbf{Z})$. Thus in the basis A_1, A_2, B_1 the matrix of the involution $S^+: H_1(\check{C}_h, \mathbf{Z}) \rightarrow H_1(\check{C}_h, \mathbf{Z})$ takes the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

From this and the fact that $(J(C_h; \infty^\pm), S^+)$ is not empty we conclude that the real part of $(J(C_h; \infty^\pm), S^+)$ is a torus with generators the periods $\int_{B_1} \omega$ and

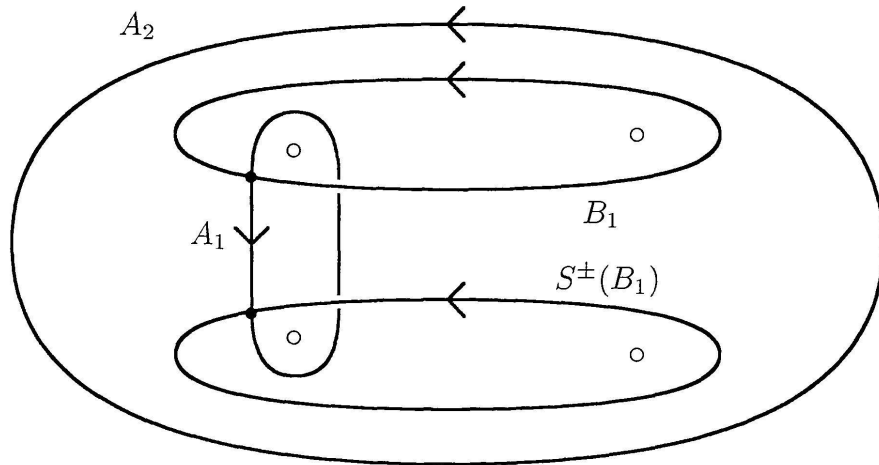


FIGURE 3

Projection of the cycles $A_1, B_1, A_2, S^\pm(B_1)$ on the λ -plane

$\int_{A_2} \omega$. On the other hand the real part of $(J(C_h; \infty^\pm), S^-)$ is also non-empty and $S^+ \circ S^-$ is a translation. We conclude that the real part of $(J(C_h; \infty^\pm), S^-)$ is just a translation of the real part of $(J(C_h; \infty^\pm), S^+)$ and in particular it is generated by the same periods.

In a similar way we find the real part of $(J(C_h; \infty^\pm), S^+)$ in the remaining cases. Note that in an appropriate \mathbf{Z} basis of $H_1(\check{C}_h, \mathbf{Z})$ the matrix of the involution $S^\pm: H_1(\check{C}_h, \mathbf{Z}) \rightarrow H_1(\check{C}_h, \mathbf{Z})$ takes the same form if $f(\lambda)$ has two real roots, and it is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

if $f(\lambda)$ has four real roots. This implies the first two lines of the table.

Let us determine now the real part of (D_∞, S^\pm) . As $D_\infty = \mathbf{C}^*/\Lambda_2$ then we have to compute $S^\pm(\Lambda_2)$. Note that, as the real invariant manifold T_h is compact, then (D_∞, S^-) is always empty. On the other hand (D_∞, S^+) is never empty. Indeed, if $S^+(\lambda, \mu) = (\bar{\lambda}, -\bar{\mu})$ then for $Q \in C_h$ the point $Q + S^+(Q)$ is S^+ -real on $J(C_h; \infty^\pm)$. As $S^+(\infty^+) = \infty^-$ we see that an S^+ -real point of $\phi^{-1}(p)$ is obtained by taking the limit $Q \mapsto \infty^+$ in $S^+(Q) + Q$ along an appropriate real analytic curve on \check{C}_h . Finally, from the computation of the action of S^+ on Λ we get $S^+(\Lambda_2) = \Lambda_2$ which shows that the S^+ -real part of $(\phi^{-1}(p), S^+)$ is always a circle \mathbf{R}/Λ_2 . This gives the last two lines in the table. \square