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Artikel: THE COMPLEX GEOMETRY OF THE LAGRANGE TOP
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To compute the limit we use (46), (47) and

$$\lim_{P\to\infty^-} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^-)\big) = \theta_{11}'(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta_{11}'(0)$$

$$\lim_{P\to\infty^+} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^+)\big) = \theta_{11}'(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta_{11}'(0)$$

(see Lemma 3.5). □

3.3 Effectivization

Let $\wp, \zeta, \sigma$ be the Weierstrass functions related to the elliptic curve Γ defined by

(51) $$\eta^2 = 4\xi^3 - g_2\xi - g_3$$

(we use the standard notations of [4]).

Consider also the *real* elliptic curve C with affine equation

(52) $$\mu^2 + \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

and natural anti-holomorphic involution $(\lambda,\mu) \to \left(\overline{\lambda},\overline{\mu}\right)$, and put

(53) $$g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^4 - 4\frac{a_1}{4}\frac{a_3}{4}, \qquad g_3 = \det\begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}.$$

It is well known that the curves C and Γ are isomorphic over $\mathbf{C}$ and that under this isomorphism

(54) $$\frac{d\lambda}{\mu} = \frac{d\xi}{\eta}.$$

Following Weil [25] we call Γ the Jacobian $J(C)$ of the elliptic curve C and we write $J(C) = \Gamma$. Note that $J(C)$ and Γ are real isomorphic and that $J(C)$ and C are not real isomorphic.

Further we make the substitution (23) and C becomes the spectral curve $\widetilde{C}_h$ of Adler and van Moerbeke $\left\{\mu^2 + f(\lambda) = 0\right\}$, where

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + \left(2h_3 + m(m+1)h_4^2\right)\lambda^2 - 2h_2\lambda + 1$$

and Γ becomes the Lagrange curve Γ_h. Recall that, as we explained at the end of Section 2, the curve C_h with an equation $\{\mu^2 = f(\lambda)\}$ and antiholomorphic involution $(\lambda,\mu) \to (\overline{\lambda}, -\overline{\mu})$, is isomorphic over $\mathbf{R}$ to $\widetilde{C}_h$, so we write $C_h = \widetilde{C}_h$. The Jacobian curve $J(C_h) = \Gamma_h$ was computed by

Lagrange [17], while C_h appeared first in [1, 21] as a spectral curve of a Lax pair associated to the Lagrange top.

Recall that $\sigma(z)$ is an entire function in z related to $\zeta(z)$, $\wp(z)$ and the already defined function $\theta_{11}(z \mid \tau_1)$ on C_h as follows:

$$\zeta'(z) = -\wp(z)\,, \qquad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z)\,, \qquad {}' = \frac{d}{dz}$$

$$\sigma(z) = \frac{\theta_{11}(zU)}{U\theta'_{11}(0)} \exp\left\{\frac{z^2U^2\theta'''_{11}(0)}{6\theta'_{11}(0)}\right\} = z - \frac{g_2 z^5}{240} + \cdots\,, \tag{55}$$

where U is a constant depending on g_2 and g_3. We shall also use the "addition formula"

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)} = \wp(v) - \wp(u)\,.$$

To state our result let us introduce the notations

$$\begin{aligned} &2x_1 = \epsilon\,\Omega_1 + \bar\epsilon\,\Omega_2\,, && 2x_2 = \bar\epsilon\,\Omega_1 + \epsilon\,\Omega_2\,, && \epsilon^2 = \sqrt{-1} \\ &2y_1 = \epsilon^3\Gamma_1 + \epsilon\,\Gamma_2\,, && 2y_2 = \epsilon\,\Gamma_1 + \epsilon^3\Gamma_2\,, && i^2 = -1 \\ &\rho_1 = -i\,m\,\Omega_3\,, && \rho_2 = -i\,\Omega_3\,. \end{aligned} \tag{56}$$

The system (2) is equivalent to

$$\begin{aligned} &\dot x_1 = +\rho_1 x_1 - y_1\,, && \dot y_1 = -\rho_2 y_1 + x_1\Gamma_3 \\ &\dot x_2 = -\rho_1 x_2 + y_2\,, && \dot y_2 = +\rho_2 y_2 - x_2\Gamma_3 \\ &\rho_1\,,\ \rho_2 = \text{constants}\,, && \dot\Gamma_3 = 2x_1y_2 - 2x_2y_1 \end{aligned} \tag{57}$$

with first integrals $I_0 = 4x_1x_2 - 2\Gamma_3$, $I_1 = 4x_1y_2 + 4x_2y_1 - 2(\rho_1+\rho_2)\Gamma_3$ and $I_2 = \Gamma_3^2 - 4y_1y_2$.

THEOREM 3.6. *The general solution of the Lagrange top* (2) *can be written in the form*

$$x_1(t) = -\frac{\sigma(t-k-l)}{\sigma(t)\,\sigma(k+l)}\,e^{at+b} \qquad x_2(t) = -\frac{\sigma(t+k+l)}{\sigma(t)\,\sigma(k+l)}\,e^{-at-b}$$

$$y_1(t) = \frac{\sigma(t-k)\,\sigma(t-l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}\,e^{at+b} \qquad y_2(t) = \frac{\sigma(t+k)\,\sigma(t+l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}\,e^{-at-b}$$

$$\Gamma_3(t) = \frac{\sigma(t+k)\,\sigma(t-k)}{\sigma^2(k)\,\sigma^2(t)} + \frac{\sigma(t+l)\,\sigma(t-l)}{\sigma^2(l)\,\sigma^2(t)} = -2\wp(t) + \wp(l) + \wp(k)$$

$$\rho_1 = a - \zeta(l) - \zeta(k) \qquad \rho_2 = -a - \zeta(k) - \zeta(l) + 2\zeta(k+l)\,,$$

where g_2, g_3, a, b, k, l are arbitrary constants subject to the relation $g_2^3 - 27g_3^2 \neq 0$.

REMARK. The non-general solutions of the Lagrange top are obtained from the above formulae by taking the limit $g_2^3 - 27g_3^2 \to 0$. The formulae for the position of the body in space, and in particular for $\Gamma_3(t)$, $y_1(t)$, $y_2(t)$, are due to Jacobi [15]. The expressions for $x_1(t)$, $x_2(t)$ were first deduced by Klein and Sommerfeld [16, p. 436]. Note however that in [16] the constant a, and hence the invariant level set on which the solution lives, is not arbitrary.

Proof. To make the solutions of the Lagrange top effective we use the following 4-dimensional Lie group of transformations preserving the system (57):

$$\begin{aligned}
&x_1 \to U x_1 e^{at+b}, \qquad && x_2 \to U x_2 e^{-at-b}, \qquad && t \to \frac{t}{U} + T \\
&y_1 \to U^2 y_1\, e^{at+b}, && y_2 \to U^2 y_2\, e^{-at-b}, && \Gamma_3 \to U^2\,\Gamma_3 \\
&\rho_1 \to U\rho_1 + a, && \rho_2 \to U\rho_2 - a &&
\end{aligned} \tag{58}$$

where $U \neq 0$, T, a, b are constants.

The group (58) transforms x_1 from (48) (see also (56), (55)), where $z_1 = tU - TU$, $z_1 - \tau_2 = (t-k-l)U$ as follows

$$x_1(t) = \text{const}\,\frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} = -\frac{\sigma(t-k-l)}{\sigma(t)\,\sigma(k+l)}\,e^{at+b}\,.$$

(we used the fact that

$$\frac{\theta_{11}(z_1 - \tau_2)\,\sigma(t)}{\theta_{11}(z_1)\,\sigma(t-k-l)}$$

is a constant). The variable x_2 is computed in the same way.

If we define the constant k by the condition $y_1(t-k) = 0$, then the first equation of (57) gives

$$\frac{y_1(t)}{x_1(t)} = \rho_1 - \frac{x_1'(t)}{x_1(t)} = \frac{\sigma(t-k)\,h(t)}{\sigma(t)\,\sigma(t-k-l)}$$

where $h(t)$ is a meromorphic function on $\mathbf{C}$, such that $y_1(t)/x_1(t)$ is single valued with poles at $t = 0$ and $t = k + l$, and residues (-1) and $(+1)$ respectively. These three conditions define $h(t)$ uniquely:

$$h(t) = \frac{\sigma(t-l)\,\sigma(k+l)}{\sigma(k)\,\sigma(l)}\,,$$

which implies the formula for $y_1(t)$. The expression for $y_2(t)$ is obtained in the same way.

To deduce an expression for $\Gamma_3(t)$ we use the fact that

$$\Gamma_3(t) = 2x_1x_2 - \tfrac{1}{2}I_0 = -2\wp(t) + 2\wp(k+l) - \tfrac{1}{2}I_0 .$$

The value of I_0 is easily computed by using the third equation of (57) and the formulae deduced for x_1, y_1. By substituting $t = k$ we obtain

$$\Gamma_3(k) = \frac{\sigma(k-l)\,\sigma(k+l)}{\sigma^2(k)\,\sigma^2(l)} = \wp(l) - \wp(k)$$

and in a similar way $\Gamma_3(l) = \wp(k) - \wp(l)$. We conclude that

$$\Gamma_3(t) = -2\wp(t) + \wp(l) + \wp(k) .$$

Finally, to compute ρ_1, ρ_2 we shall use once again (57). As $y_1(k) = 0$ we have

$$\begin{aligned}
\rho_1 &= \frac{\dot{x}_1(k)}{x_1(k)} = \frac{d}{dt}\ln x_1(t)\Big|_{t=k} \\
&= \frac{d}{dt}\ln\sigma(t-k-l)\Big|_{t=k} - \frac{d}{dt}\ln\sigma(t)\Big|_{t=k} + a \\
&= a - \zeta(l) - \zeta(k) .
\end{aligned}$$

In a quite similar way we obtain

$$\rho_2 = -\frac{d}{dt}\ln y_1(t)\Big|_{t=k+l} = -a - \zeta(k) - \zeta(l) + 2\zeta(k+l) .$$

Theorem 3.6 is proved. □

REMARK. If we impose the condition

$$\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = \Gamma_3^2 - 4y_1y_2 = 1 ,$$

then

$$\left(\frac{\sigma(t+k)\,\sigma(t-k)}{\sigma^2(k)\,\sigma^2(t)} + \frac{\sigma(t+l)\,\sigma(t-l)}{\sigma^2(l)\,\sigma^2(t)}\right)^2 - \frac{\sigma(t-k)\,\sigma(t-l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}\,\frac{\sigma(t+k)\,\sigma(t+l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}$$

$$= \left(\frac{\sigma(t+k)\,\sigma(t-k)}{\sigma^2(k)\,\sigma^2(t)} - \frac{\sigma(t+l)\,\sigma(t-l)}{\sigma^2(l)\,\sigma^2(t)}\right)^2 = \big(\wp(k) - \wp(l)\big)^2 = 1$$

and hence $\wp(k) - \wp(l) = \pm 1$.