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To compute the limit we use (46), (47) and

$$\lim_{P \rightarrow \infty^-} \lambda(P) \theta_{11}(\mathcal{A}(P - \infty^-)) = \theta'_{11}(0) \frac{d}{ds} \Big|_{s=0} \int^s \omega_1 = \omega_1^0 \theta'_{11}(0)$$

$$\lim_{P \rightarrow \infty^+} \lambda(P) \theta_{11}(\mathcal{A}(P - \infty^+)) = \theta'_{11}(0) \frac{d}{ds} \Big|_{s=0} \int^s \omega_1 = \omega_1^0 \theta'_{11}(0)$$

(see Lemma 3.5). \square

3.3 EFFECTIVIZATION

Let \wp, ζ, σ be the Weierstrass functions related to the elliptic curve Γ defined by

$$(51) \quad \eta^2 = 4\xi^3 - g_2\xi - g_3$$

(we use the standard notations of [4]).

Consider also the *real* elliptic curve C with affine equation

$$(52) \quad \mu^2 + \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

and natural anti-holomorphic involution $(\lambda, \mu) \rightarrow (\bar{\lambda}, \bar{\mu})$, and put

$$(53) \quad g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^4 - 4\frac{a_1}{4}\frac{a_3}{4}, \quad g_3 = \det \begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}.$$

It is well known that the curves C and Γ are isomorphic over \mathbf{C} and that under this isomorphism

$$(54) \quad \frac{d\lambda}{\mu} = \frac{d\xi}{\eta}.$$

Following Weil [25] we call Γ the Jacobian $J(C)$ of the elliptic curve C and we write $J(C) = \Gamma$. Note that $J(C)$ and Γ are real isomorphic and that $J(C)$ and C are not real isomorphic.

Further we make the substitution (23) and C becomes the spectral curve \tilde{C}_h of Adler and van Moerbeke $\{\mu^2 + f(\lambda) = 0\}$, where

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + (2h_3 + m(m+1)h_4^2)\lambda^2 - 2h_2\lambda + 1$$

and Γ becomes the Lagrange curve Γ_h . Recall that, as we explained at the end of Section 2, the curve C_h with an equation $\{\mu^2 = f(\lambda)\}$ and antiholomorphic involution $(\lambda, \mu) \rightarrow (\bar{\lambda}, -\bar{\mu})$, is isomorphic over \mathbf{R} to \tilde{C}_h , so we write $C_h = \tilde{C}_h$. The Jacobian curve $J(C_h) = \Gamma_h$ was computed by

Lagrange [17], while C_h appeared first in [1, 21] as a spectral curve of a Lax pair associated to the Lagrange top.

Recall that $\sigma(z)$ is an entire function in z related to $\zeta(z)$, $\wp(z)$ and the already defined function $\theta_{11}(z | \tau_1)$ on C_h as follows:

$$\zeta'(z) = -\wp(z) , \quad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z) , \quad ' = \frac{d}{dz}$$

$$(55) \quad \sigma(z) = \frac{\theta_{11}(zU)}{U\theta'_{11}(0)} \exp \left\{ \frac{z^2 U^2 \theta'''_{11}(0)}{6\theta'_{11}(0)} \right\} = z - \frac{g_2 z^5}{240} + \dots ,$$

where U is a constant depending on g_2 and g_3 . We shall also use the ‘‘addition formula’’

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)} = \wp(v) - \wp(u) .$$

To state our result let us introduce the notations

$$(56) \quad \begin{aligned} 2x_1 &= \epsilon \Omega_1 + \bar{\epsilon} \Omega_2 , & 2x_2 &= \bar{\epsilon} \Omega_1 + \epsilon \Omega_2 , & \epsilon^2 &= \sqrt{-1} \\ 2y_1 &= \epsilon^3 \Gamma_1 + \epsilon \Gamma_2 , & 2y_2 &= \epsilon \Gamma_1 + \epsilon^3 \Gamma_2 , & i^2 &= -1 \\ \rho_1 &= -im \Omega_3 , & \rho_2 &= -i \Omega_3 . \end{aligned}$$

The system (2) is equivalent to

$$(57) \quad \begin{aligned} \dot{x}_1 &= +\rho_1 x_1 - y_1 , & \dot{y}_1 &= -\rho_2 y_1 + x_1 \Gamma_3 \\ \dot{x}_2 &= -\rho_1 x_2 + y_2 , & \dot{y}_2 &= +\rho_2 y_2 - x_2 \Gamma_3 \\ \rho_1 , \rho_2 &= \text{constants} , & \dot{\Gamma}_3 &= 2x_1 y_2 - 2x_2 y_1 \end{aligned}$$

with first integrals $I_0 = 4x_1 x_2 - 2\Gamma_3$, $I_1 = 4x_1 y_2 + 4x_2 y_1 - 2(\rho_1 + \rho_2)\Gamma_3$ and $I_2 = \Gamma_3^2 - 4y_1 y_2$.

THEOREM 3.6. *The general solution of the Lagrange top (2) can be written in the form*

$$\begin{aligned} x_1(t) &= -\frac{\sigma(t-k-l)}{\sigma(t)\sigma(k+l)} e^{at+b} & x_2(t) &= -\frac{\sigma(t+k+l)}{\sigma(t)\sigma(k+l)} e^{-at-b} \\ y_1(t) &= \frac{\sigma(t-k)\sigma(t-l)}{\sigma^2(t)\sigma(k)\sigma(l)} e^{at+b} & y_2(t) &= \frac{\sigma(t+k)\sigma(t+l)}{\sigma^2(t)\sigma(k)\sigma(l)} e^{-at-b} \\ \Gamma_3(t) &= \frac{\sigma(t+k)\sigma(t-k)}{\sigma^2(k)\sigma^2(t)} + \frac{\sigma(t+l)\sigma(t-l)}{\sigma^2(l)\sigma^2(t)} = -2\wp(t) + \wp(l) + \wp(k) \\ \rho_1 &= a - \zeta(l) - \zeta(k) & \rho_2 &= -a - \zeta(k) - \zeta(l) + 2\zeta(k+l) , \end{aligned}$$

where g_2, g_3, a, b, k, l are arbitrary constants subject to the relation $g_2^3 - 27g_3^2 \neq 0$.

REMARK. The non-general solutions of the Lagrange top are obtained from the above formulae by taking the limit $g_2^3 - 27g_3^2 \rightarrow 0$. The formulae for the position of the body in space, and in particular for $\Gamma_3(t)$, $y_1(t)$, $y_2(t)$, are due to Jacobi [15]. The expressions for $x_1(t)$, $x_2(t)$ were first deduced by Klein and Sommerfeld [16, p.436]. Note however that in [16] the constant a , and hence the invariant level set on which the solution lives, is not arbitrary.

Proof. To make the solutions of the Lagrange top effective we use the following 4-dimensional Lie group of transformations preserving the system (57):

$$(58) \quad \begin{aligned} x_1 &\rightarrow Ux_1 e^{at+b}, & x_2 &\rightarrow Ux_2 e^{-at-b}, & t &\rightarrow \frac{t}{U} + T \\ y_1 &\rightarrow U^2 y_1 e^{at+b}, & y_2 &\rightarrow U^2 y_2 e^{-at-b}, & \Gamma_3 &\rightarrow U^2 \Gamma_3 \\ \rho_1 &\rightarrow U\rho_1 + a, & \rho_2 &\rightarrow U\rho_2 - a \end{aligned}$$

where $U \neq 0$, T , a , b are constants.

The group (58) transforms x_1 from (48) (see also (56), (55)), where $z_1 = tU - TU$, $z_1 - \tau_2 = (t - k - l)U$ as follows

$$x_1(t) = \text{const} \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} = - \frac{\sigma(t - k - l)}{\sigma(t) \sigma(k + l)} e^{at+b}.$$

(we used the fact that

$$\frac{\theta_{11}(z_1 - \tau_2) \sigma(t)}{\theta_{11}(z_1) \sigma(t - k - l)}$$

is a constant). The variable x_2 is computed in the same way.

If we define the constant k by the condition $y_1(t - k) = 0$, then the first equation of (57) gives

$$\frac{y_1(t)}{x_1(t)} = \rho_1 - \frac{x_1'(t)}{x_1(t)} = \frac{\sigma(t - k) h(t)}{\sigma(t) \sigma(t - k - l)}$$

where $h(t)$ is a meromorphic function on \mathbf{C} , such that $y_1(t)/x_1(t)$ is single valued with poles at $t = 0$ and $t = k + l$, and residues (-1) and $(+1)$ respectively. These three conditions define $h(t)$ uniquely:

$$h(t) = \frac{\sigma(t - l) \sigma(k + l)}{\sigma(k) \sigma(l)},$$

which implies the formula for $y_1(t)$. The expression for $y_2(t)$ is obtained in the same way.

To deduce an expression for $\Gamma_3(t)$ we use the fact that

$$\Gamma_3(t) = 2x_1x_2 - \frac{1}{2}I_0 = -2\wp(t) + 2\wp(k+l) - \frac{1}{2}I_0.$$

The value of I_0 is easily computed by using the third equation of (57) and the formulae deduced for x_1, y_1 . By substituting $t = k$ we obtain

$$\Gamma_3(k) = \frac{\sigma(k-l)\sigma(k+l)}{\sigma^2(k)\sigma^2(l)} = \wp(l) - \wp(k)$$

and in a similar way $\Gamma_3(l) = \wp(k) - \wp(l)$. We conclude that

$$\Gamma_3(t) = -2\wp(t) + \wp(l) + \wp(k).$$

Finally, to compute ρ_1, ρ_2 we shall use once again (57). As $y_1(k) = 0$ we have

$$\begin{aligned} \rho_1 &= \frac{\dot{x}_1(k)}{x_1(k)} = \frac{d}{dt} \ln x_1(t) \Big|_{t=k} \\ &= \frac{d}{dt} \ln \sigma(t-k-l) \Big|_{t=k} - \frac{d}{dt} \ln \sigma(t) \Big|_{t=k} + a \\ &= a - \zeta(l) - \zeta(k). \end{aligned}$$

In a quite similar way we obtain

$$\rho_2 = -\frac{d}{dt} \ln y_1(t) \Big|_{t=k+l} = -a - \zeta(k) - \zeta(l) + 2\zeta(k+l).$$

Theorem 3.6 is proved. \square

REMARK. If we impose the condition

$$\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = \Gamma_3^2 - 4y_1y_2 = 1,$$

then

$$\begin{aligned} &\left(\frac{\sigma(t+k)\sigma(t-k)}{\sigma^2(k)\sigma^2(t)} + \frac{\sigma(t+l)\sigma(t-l)}{\sigma^2(l)\sigma^2(t)} \right)^2 - \frac{\sigma(t-k)\sigma(t-l)}{\sigma^2(t)\sigma(k)\sigma(l)} \frac{\sigma(t+k)\sigma(t+l)}{\sigma^2(t)\sigma(k)\sigma(l)} \\ &= \left(\frac{\sigma(t+k)\sigma(t-k)}{\sigma^2(k)\sigma^2(t)} - \frac{\sigma(t+l)\sigma(t-l)}{\sigma^2(l)\sigma^2(t)} \right)^2 = (\wp(k) - \wp(l))^2 = 1 \end{aligned}$$

and hence $\wp(k) - \wp(l) = \pm 1$.