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and hence $\Psi = \widetilde{\Psi}$. Finally, the reader may check that the functions (46) and (47) have the analyticity properties from Proposition 3.2 and hence they coincide with the Baker-Akhiezer function defined in Proposition 3.1. □

## 3.2 Solutions of the Lagrange top

Let $z = (z_1, z_2) \in J(C_h; \infty^{\pm})$. It is easy to check that the functions

$$\frac{\theta_{11}(z_1 \pm \tau_2)}{\theta_{11}(z_1)} e^{\mp z_2}$$

live on $J(C_h; \infty^{\pm})$. We shall see that they give solutions of the Lagrange top. By (16) we compute that $\frac{d}{dt} z = \text{constant}$, where

$$\frac{dz}{dt} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 2\pi i \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} & \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda d\lambda}{\mu} & \int_{A_2} \frac{\lambda d\lambda}{\mu} \end{pmatrix}^{-1} \begin{pmatrix} -i \\ -ai \end{pmatrix},$$

$$\int_{A_2} \frac{d\lambda}{\mu} = 0, \qquad \int_{A_2} \frac{\lambda d\lambda}{\mu} = -2\pi i$$

so

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \begin{pmatrix} 2\pi \\ -i \int_{A_1} \frac{\lambda d\lambda}{\mu} + ai \int_{A_1} \frac{d\lambda}{\mu} \end{pmatrix}, \qquad a = -m\Omega_3 .$$

THEOREM 3.4. *The following equations hold*

$$\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t) = \text{const}_3 \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} e^{-z_2}, \tag{48}$$

$$\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t) = \text{const}_4 \frac{\theta_{11}(z_1 + \tau_2)}{\theta_{11}(z_1)} e^{+z_2}, \tag{49}$$

*where*

$$\begin{aligned} z_2 &= tV_2, \quad z_1 = tV_1 + \mathcal{A}(\infty^+ + \infty^- - P_1 - P_2), \\ \tau_2 &= \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2 \end{aligned} \tag{50}$$

*and*

$$\text{const}_3 = \frac{2i\,V_1\,\theta_{11}'(0)}{\theta_{11}\big(\mathcal{A}(\infty^- - \infty^+)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^+ - P_1)\big)}{\theta_{11}\big(\mathcal{A}(\infty^- - P_1)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^+ - P_2)\big)}{\theta_{11}\big(\mathcal{A}(\infty^- - P_2)\big)},$$

$$\text{const}_4 = \frac{2i\,V_1\,\theta_{11}'(0)}{\theta_{11}\big(\mathcal{A}(\infty^+ - \infty^-)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^- - P_1)\big)}{\theta_{11}\big(\mathcal{A}(\infty^+ - P_1)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^- - P_2)\big)}{\theta_{11}\big(\mathcal{A}(\infty^+ - P_2)\big)}.$$

Let us denote

$$\omega_1 = \pm\big(\omega_1^0 + O(\lambda^{-1})\big)\, d\big(\lambda^{-1}\big), \qquad P = (\lambda, \mu) \to \infty^{\pm},$$
$$\omega_2 = \pm\big(\omega_2^1 \lambda + \omega_2^0 + O(\lambda^{-1})\big)\, d\big(\lambda^{-1}\big), \qquad P = (\lambda, \mu) \to \infty^{\pm}.$$

To prove Theorem 3.4 we shall need the following

LEMMA 3.5. *The above defined differentials are such that*

$$\omega_1^0 = -i \int_{B_1} \Omega = -iV_1, \qquad \omega_2^0 = i\,(c^+ - c^-),$$
$$V_2 = -c^+ + c^- + i\Omega_3, \qquad \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2 .$$

*Proof.* The identity $\omega_1^0 = -i \int_{B_1} \Omega$ is a reciprocity law between the differential of the first kind $\omega_1$ and the differential of the second kind $\Omega$ [13]. It is obtained by integrating $\pi(P)\omega_1$, where $\pi(P) = \int_{P_0}^{P} \Omega$, along the border of $C_h$ cut along its homology basis $A_1$, $B_1$. On the other hand

$$\omega_1 = 2\pi i \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \frac{d\lambda}{\mu}$$

and hence

$$\omega_1^0 = -2\pi i \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} = -iV_1 .$$

Similarly the identity $\omega_2^0 = i(c^+ - c^-)$ is a reciprocity law between the differential of the third kind $\omega_2$ and the differential of the second kind $\Omega$, and $\mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2$ is a reciprocity law between the differential of the third kind $\omega_2$ and the differential of the first kind $\omega_1$. Finally, as $\omega_2 = \dfrac{\int_{A_1} \frac{\lambda d\lambda}{\mu}}{\int_{A_1} \frac{d\lambda}{\mu}} \dfrac{d\lambda}{\mu} - \dfrac{\lambda\, d\lambda}{\mu}$ we have $\omega_2^0 = -\dfrac{\int_{A_1} \frac{\lambda d\lambda}{\mu}}{\int_{A_1} \frac{d\lambda}{\mu}} - (1+m)\Omega_3 = -iV_1 - \Omega_3$ and hence $V_2 = -c^+ + c^- + i\Omega_3$. $\square$

*Proof of Theorem 3.4.* According to (42), (43)

$$\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t) = -2 \lim_{P \to \infty^-} \frac{\lambda \Psi^1(t, P)}{\Psi^2(t, P)}$$

and

$$\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t) = +2 \lim_{P \to \infty^+} \frac{\lambda \Psi^2(t, P)}{\Psi^1(t, P)} .$$

To compute the limit we use (46), (47) and

$$\lim_{P\to\infty^-} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^-)\big) = \theta_{11}'(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta_{11}'(0)$$

$$\lim_{P\to\infty^+} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^+)\big) = \theta_{11}'(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta_{11}'(0)$$

(see Lemma 3.5). □

### 3.3 Effectivization

Let $\wp, \zeta, \sigma$ be the Weierstrass functions related to the elliptic curve $\Gamma$ defined by

$$\eta^2 = 4\xi^3 - g_2\xi - g_3 \tag{51}$$

(we use the standard notations of [4]).

Consider also the *real* elliptic curve $C$ with affine equation

$$\mu^2 + \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{52}$$

and natural anti-holomorphic involution $(\lambda,\mu) \to (\overline{\lambda},\overline{\mu})$, and put

$$g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^4 - 4\frac{a_1}{4}\frac{a_3}{4}, \quad g_3 = \det\begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}. \tag{53}$$

It is well known that the curves $C$ and $\Gamma$ are isomorphic over $\mathbf{C}$ and that under this isomorphism

$$\frac{d\lambda}{\mu} = \frac{d\xi}{\eta}. \tag{54}$$

Following Weil [25] we call $\Gamma$ the Jacobian $J(C)$ of the elliptic curve $C$ and we write $J(C) = \Gamma$. Note that $J(C)$ and $\Gamma$ are real isomorphic and that $J(C)$ and $C$ are not real isomorphic.

Further we make the substitution (23) and $C$ becomes the spectral curve $\widetilde{C}_h$ of Adler and van Moerbeke $\{\mu^2 + f(\lambda) = 0\}$, where

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + \big(2h_3 + m(m+1)h_4^2\big)\lambda^2 - 2h_2\lambda + 1$$

and $\Gamma$ becomes the Lagrange curve $\Gamma_h$. Recall that, as we explained at the end of Section 2, the curve $C_h$ with an equation $\{\mu^2 = f(\lambda)\}$ and antiholomorphic involution $(\lambda,\mu) \to (\overline{\lambda},-\overline{\mu})$, is isomorphic over $\mathbf{R}$ to $\widetilde{C}_h$, so we write $C_h = \widetilde{C}_h$. The Jacobian curve $J(C_h) = \Gamma_h$ was computed by