

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COMPLEX GEOMETRY OF THE LAGRANGE TOP
Autor: Gavrilov, Lubomir / ZHIVKOV, Angel
Kapitel: 3.2 Solutions of the Lagrange top
DOI: <https://doi.org/10.5169/seals-63901>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 08.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

and hence $\Psi = \tilde{\Psi}$. Finally, the reader may check that the functions (46) and (47) have the analyticity properties from Proposition 3.2 and hence they coincide with the Baker-Akhiezer function defined in Proposition 3.1. \square

3.2 SOLUTIONS OF THE LAGRANGE TOP

Let $z = (z_1, z_2) \in J(C_h; \infty^\pm)$. It is easy to check that the functions

$$\frac{\theta_{11}(z_1 \pm \tau_2)}{\theta_{11}(z_1)} e^{\mp z_2}$$

live on $J(C_h; \infty^\pm)$. We shall see that they give solutions of the Lagrange top. By (16) we compute that $\frac{d}{dt}z = \text{constant}$, where

$$\begin{aligned} \frac{dz}{dt} &= \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 2\pi i \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} & \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda d\lambda}{\mu} & \int_{A_2} \frac{\lambda d\lambda}{\mu} \end{pmatrix}^{-1} \begin{pmatrix} -i \\ -ai \end{pmatrix}, \\ &\quad \int_{A_2} \frac{d\lambda}{\mu} = 0, \quad \int_{A_2} \frac{\lambda d\lambda}{\mu} = -2\pi i \end{aligned}$$

so

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left(\int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \begin{pmatrix} 2\pi \\ -i \int_{A_1} \frac{\lambda d\lambda}{\mu} + ai \int_{A_1} \frac{d\lambda}{\mu} \end{pmatrix}, \quad a = -m\Omega_3.$$

THEOREM 3.4. *The following equations hold*

$$(48) \quad \bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t) = \text{const}_3 \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} e^{-z_2},$$

$$(49) \quad \epsilon\Omega_1(t) + \bar{\epsilon}\Omega_2(t) = \text{const}_4 \frac{\theta_{11}(z_1 + \tau_2)}{\theta_{11}(z_1)} e^{+z_2},$$

where

$$(50) \quad \begin{aligned} z_2 &= tV_2, \quad z_1 = tV_1 + \mathcal{A}(\infty^+ + \infty^- - P_1 - P_2), \\ \tau_2 &= \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2 \end{aligned}$$

and

$$\text{const}_3 = \frac{2iV_1\theta'_{11}(0)}{\theta_{11}(\mathcal{A}(\infty^- - \infty^+))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_1))}{\theta_{11}(\mathcal{A}(\infty^- - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}{\theta_{11}(\mathcal{A}(\infty^- - P_2))},$$

$$\text{const}_4 = \frac{2iV_1\theta'_{11}(0)}{\theta_{11}(\mathcal{A}(\infty^+ - \infty^-))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_1))}{\theta_{11}(\mathcal{A}(\infty^+ - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_2))}{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}.$$

Let us denote

$$\begin{aligned}\omega_1 &= \pm(\omega_1^0 + O(\lambda^{-1})) d(\lambda^{-1}), & P = (\lambda, \mu) \rightarrow \infty^\pm, \\ \omega_2 &= \pm(\omega_2^1 \lambda + \omega_2^0 + O(\lambda^{-1})) d(\lambda^{-1}), & P = (\lambda, \mu) \rightarrow \infty^\pm.\end{aligned}$$

To prove Theorem 3.4 we shall need the following

LEMMA 3.5. *The above defined differentials are such that*

$$\begin{aligned}\omega_1^0 &= -i \int_{B_1} \Omega = -iV_1, & \omega_2^0 = i(c^+ - c^-), \\ V_2 &= -c^+ + c^- + i\Omega_3, & \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2.\end{aligned}$$

Proof. The identity $\omega_1^0 = -i \int_{B_1} \Omega$ is a reciprocity law between the differential of the first kind ω_1 and the differential of the second kind Ω [13]. It is obtained by integrating $\pi(P)\omega_1$, where $\pi(P) = \int_{P_0}^P \Omega$, along the border of C_h cut along its homology basis A_1, B_1 . On the other hand

$$\omega_1 = 2\pi i \left(\int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \frac{d\lambda}{\mu}$$

and hence

$$\omega_1^0 = -2\pi i \left(\int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} = -iV_1.$$

Similarly the identity $\omega_2^0 = i(c^+ - c^-)$ is a reciprocity law between the differential of the third kind ω_2 and the differential of the second kind Ω , and $\mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2$ is a reciprocity law between the differential of the third kind ω_2 and the differential of the first kind ω_1 . Finally, as

$$\omega_2 = \frac{\int_{A_1} \frac{\lambda d\lambda}{\mu}}{\int_{A_1} \frac{d\lambda}{\mu}} \frac{d\lambda}{\mu} - \frac{\lambda d\lambda}{\mu} \text{ we have } \omega_2^0 = -\frac{\int_{A_1} \frac{\lambda d\lambda}{\mu}}{\int_{A_1} \frac{d\lambda}{\mu}} - (1+m)\Omega_3 = -iV_1 - \Omega_3$$

and hence $V_2 = -c^+ + c^- + i\Omega_3$. \square

Proof of Theorem 3.4. According to (42), (43)

$$\bar{\epsilon} \Omega_1(t) + \epsilon \Omega_2(t) = -2 \lim_{P \rightarrow \infty^-} \frac{\lambda \Psi^1(t, P)}{\Psi^2(t, P)}$$

and

$$\epsilon \Omega_1(t) + \bar{\epsilon} \Omega_2(t) = +2 \lim_{P \rightarrow \infty^+} \frac{\lambda \Psi^2(t, P)}{\Psi^1(t, P)}.$$

To compute the limit we use (46), (47) and

$$\lim_{P \rightarrow \infty^-} \lambda(P) \theta_{11}(\mathcal{A}(P - \infty^-)) = \theta'_{11}(0) \frac{d}{ds} \Big|_{s=0} \int^s \omega_1 = \omega_1^0 \theta'_{11}(0)$$

$$\lim_{P \rightarrow \infty^+} \lambda(P) \theta_{11}(\mathcal{A}(P - \infty^+)) = \theta'_{11}(0) \frac{d}{ds} \Big|_{s=0} \int^s \omega_1 = \omega_1^0 \theta'_{11}(0)$$

(see Lemma 3.5). \square

3.3 EFFECTIVIZATION

Let \wp, ζ, σ be the Weierstrass functions related to the elliptic curve Γ defined by

$$(51) \quad \eta^2 = 4\xi^3 - g_2\xi - g_3$$

(we use the standard notations of [4]).

Consider also the *real* elliptic curve C with affine equation

$$(52) \quad \mu^2 + \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

and natural anti-holomorphic involution $(\lambda, \mu) \rightarrow (\bar{\lambda}, \bar{\mu})$, and put

$$(53) \quad g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^4 - 4\frac{a_1}{4}\frac{a_3}{4}, \quad g_3 = \det \begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}.$$

It is well known that the curves C and Γ are isomorphic over \mathbf{C} and that under this isomorphism

$$(54) \quad \frac{d\lambda}{\mu} = \frac{d\xi}{\eta}.$$

Following Weil [25] we call Γ the Jacobian $J(C)$ of the elliptic curve C and we write $J(C) = \Gamma$. Note that $J(C)$ and Γ are real isomorphic and that $J(C)$ and C are not real isomorphic.

Further we make the substitution (23) and C becomes the spectral curve \tilde{C}_h of Adler and van Moerbeke $\{\mu^2 + f(\lambda) = 0\}$, where

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + (2h_3 + m(m+1)h_4^2)\lambda^2 - 2h_2\lambda + 1$$

and Γ becomes the Lagrange curve Γ_h . Recall that, as we explained at the end of Section 2, the curve C_h with an equation $\{\mu^2 = f(\lambda)\}$ and antiholomorphic involution $(\lambda, \mu) \rightarrow (\bar{\lambda}, -\bar{\mu})$, is isomorphic over \mathbf{R} to \tilde{C}_h , so we write $C_h = \tilde{C}_h$. The Jacobian curve $J(C_h) = \Gamma_h$ was computed by