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3. EXPLICIT SOLUTIONS

In this section we find explicit solutions for the Lagrange top (2). We compute first the Baker-Akhiezer function of the $\mathfrak{sl}(2,\mathbf{C})$ (or rather $\mathfrak{su}(2)$) Lax pair (14). This implies explicit formulae for the solutions of the Lagrange top in terms of exponentials and theta functions related to the spectral curve C_h (see for example Dubrovin [8], E. D. Belokolos, A. I. Bobenko, V. Z. Enol′skiĭ, A. R. Its, V. B. Matveev [5]).Then we note that the Jacobian $J(C_h)$ of C_h is just the Lagrange elliptic curve used in the classical theory which provides explicit solutions in terms of exponentials and sigma function related to $J(C_h)$.

By performing a unitary operation on the matrix (15) we may put its leading term in diagonal form. Substituting $a = -m\Omega_3$ in (14) and using the change of variables (25) we obtain the following Lax pair representation for the Lagrange top (2)

$$\text{(29)} \qquad \left[A\,,\,B - 2i\frac{d}{dt}\right] = 2i\frac{dA}{dt} + [A,B] = 0\,, \qquad \epsilon^2 = i\,,\quad i^2 = -1$$

where

$$A = A(t,\lambda) = \begin{pmatrix} A_{11}(t,\lambda) & A_{12}(t,\lambda) \\ A_{21}(t,\lambda) & A_{22}(t,\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 +$$

$$+ \begin{pmatrix} (1+m)\Omega_3 & \bar\epsilon\,\Omega_1(t)+\epsilon\,\Omega_2(t) \\ \epsilon\,\Omega_1(t)+\bar\epsilon\,\Omega_2(t) & -(m+1)\Omega_3 \end{pmatrix}\lambda - \begin{pmatrix} \Gamma_3 & \bar\epsilon\,\Gamma_1(t)+\epsilon\,\Gamma_2(t) \\ \epsilon\,\Gamma_1(t)+\bar\epsilon\,\Gamma_2(t) & -\Gamma_3 \end{pmatrix}$$

and

$$B = B(t,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \Omega_3 & \bar\epsilon\,\Omega_1(t) + \epsilon\,\Omega_2(t) \\ \epsilon\,\Omega_1(t) + \bar\epsilon\,\Omega_2(t) & -\Omega_3 \end{pmatrix}.$$

The spectral curve of the above Lax representation is defined by

$$\check{C}_h = \left\{\det\bigl(A(\lambda) - \mu I\bigr) = \mu^2 - f(\lambda) = 0\right\}\,,$$

$$f(\lambda) = \lambda^4 + 2(1+m)\,h_4\lambda^3 + \bigl(2h_3 + m(m+1)\,h_4^2\bigr)\lambda^2 - 2h_2\lambda + 1\,.$$

We shall also denote by C_h the Riemann surface of the compactified affine curve $\check{C}_h$. The reader may note the "similarity" between (29) and the Lax pair of the nonlinear Schrödinger equation (for a rigorous statement see Proposition 5.1).

3.1 THE BAKER-AKHIEZER FUNCTION

Let us fix a solution $A(t,\lambda)$ of (29) defined in a neighbourhood of $t = 0 \in \mathbf{C}$. We shall also suppose that the point $P = (\lambda,\mu)$ is such that $(1,-1)$ is not an eigenvector of the matrix $A(0,\lambda)$.

PROPOSITION 3.1. *For any $t \in \mathbf{C}$ in a sufficiently small neighbourhood of the origin, there exists a unique eigenfunction*

$$\Psi = \Psi(t,P) = \begin{pmatrix} \Psi^1(t,P) \\ \Psi^2(t,P) \end{pmatrix}, \qquad P = (\lambda,\mu) \in \check{C} \tag{30}$$

of $A(t,\lambda)$ (called the Baker-Akhiezer function) *satisfying the conditions*

$$2i\frac{d}{dt}\Psi(t,P) = B(t,\lambda)\Psi(t,P) \tag{31}$$

$$A(t,\lambda)\Psi(t,P) = \mu\Psi(t,P) \tag{32}$$

and normalized by

$$\Psi^1(0,P) + \Psi^2(0,P) = 1\,. \tag{33}$$

In terms of the coefficients $A_{ij}(t,\lambda)$ of the matrix $A = (A_{ij})$ we have

$$\Psi^1(0,P) = \frac{A_{12}(0,\lambda)}{A_{12}(0,\lambda)+\mu-A_{11}(0,\lambda)} = \frac{\mu - A_{22}(0,\lambda)}{A_{21}(0,\lambda)+\mu-A_{22}(0,\lambda)} \tag{34}$$

$$\Psi^2(0,P) = \frac{\mu - A_{11}(0,\lambda)}{A_{12}(0,\lambda)+\mu-A_{11}(0,\lambda)} = \frac{A_{21}(0,\lambda)}{A_{21}(0,\lambda)+\mu-A_{22}(0,\lambda)}\,. \tag{35}$$

Proof. Let $\Phi(t,\lambda)$ be a fundamental matrix for the operator $B(t,\lambda) - 2i\frac{d}{dt}$ normalized at $t=0$. Then the general solution of (31) is written as

$$\Psi(t,P) = \Phi(t,\lambda)\Psi(0,P), \qquad \Phi(0,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = (\lambda,\mu)\,. \tag{36}$$

As A and $B - 2i\frac{d}{dt}$ commute, we have

$$\left(B(t,\lambda) - 2i\frac{d}{dt}\right)A(t,\lambda)\Phi(t,\lambda) = A(t,\lambda)\left(B(t,\lambda) - 2i\frac{d}{dt}\right)\Phi(t,\lambda) = 0$$

and hence $A(t,\lambda)\Phi(t,\lambda) = \Phi(t,\lambda)M(P)$ for some constant matrix $M(P)$ computed by substituting $t=0$. Thus $M(P) = A(0,\lambda)$ and

$$A(0,\lambda) = \Phi^{-1}(t,\lambda)A(t,\lambda)\,\Phi(t,\lambda)\,.$$

The constants $\Psi^1(0,P), \Psi^2(0,P)$ are uniquely defined by (32) and (33). Finally,

$$\begin{aligned} A(t,\lambda)\Psi(t,P) &= \Phi(t,\lambda)A(0,\lambda)\,\Phi^{-1}(t,\lambda)\,\Phi(t,\lambda)\,\Psi(0,P) \\ &= \Phi(t,\lambda)\cdot\mu\cdot\Psi(0,P) \\ &= \mu\Psi(t,P)\,. \end{aligned}$$

The formulae (34), (35) follow from (32), (33). □

Denote by ∞^+ (respectively ∞^-) the point on $C_h - \check{C}_h$ such that in its neighbourhood $\mu/\lambda^2 \sim +1$ (resp. (-1)).

PROPOSITION 3.2. *There exists $t_0 > 0$ such that for any fixed $t \in \mathbf{C}$, $|t| < t_0$, the Baker-Akhiezer vector-function $\Psi(t, P)$ is meromorphic in P on the affine curve $\check{C}_h$ and has two poles at $P_1, P_2 \in C_h$ which do not depend on t. In a neighbourhood of the two infinite points $\infty^{\pm}$ on C_h we have*

$$\Psi^1(t,P) = \begin{cases} \left(1+O(\lambda^{-1})\right)\exp\left(-\frac{i}{2}(\lambda+\Omega_3)t\right), & P\to\infty^+ \\ O(\lambda^{-1})\exp\left(+\frac{i}{2}(\lambda+\Omega_3)t\right), & P\to\infty^- \end{cases} \tag{37}$$

$$\Psi^2(t,P) = \begin{cases} O(\lambda^{-1})\exp\left(-\frac{i}{2}(\lambda+\Omega_3)t\right), & P\to\infty^+ \\ \left(1+O(\lambda)^{-1}\right)\exp\left(+\frac{i}{2}(\lambda+\Omega_3)t\right), & P\to\infty^-, \end{cases} \tag{38}$$

where $i = \sqrt{-1}$. Moreover, $\Psi^1(t, P)$ $(\Psi^2(t, P))$ has exactly one zero on $\check{C}_h$ and the refined asymptotic estimates of Ψ^1 at ∞^- and of Ψ^2 at ∞^+ read

$$\Psi^1(t,P) = \left[-\frac{\bar{\epsilon}\,\Omega_1(t)+\epsilon\,\Omega_2(t)}{2\lambda}+O(\lambda^{-2})\right]\exp\left(+\tfrac{i}{2}(\lambda+\Omega_3)t\right), \quad P\to\infty^- \tag{39}$$

$$\Psi^2(t,P) = \left[+\frac{\epsilon\,\Omega_1(t)+\bar{\epsilon}\,\Omega_2(t)}{2\lambda}+O(\lambda^{-2})\right]\exp\left(-\tfrac{i}{2}(\lambda+\Omega_3)t\right), \quad P\to\infty^+. \tag{40}$$

Proof. According to (32), $(\Psi^1, \Psi^2) \in \mathrm{Ker}(A - \mu I)$ and hence

$$\frac{\Psi^2(t,P)}{\Psi^1(t,P)} = \frac{\mu-\lambda^2-(1+m)\Omega_3\lambda+\Gamma_3(t)}{\left(\bar{\epsilon}\,\Omega_1(t)+\epsilon\,\Omega_2(t)\right)\lambda-\bar{\epsilon}\,\Gamma_1(t)+\epsilon\,\Gamma_2(t)}. \tag{41}$$

If $P \to \infty^+$ then $\mu - \lambda^2 - (1+m)\Omega_3\lambda \sim O(1)$ and using (29), (31), (32) and (41) we compute

$$2i\frac{d}{dt}\ln\Psi^1(t,P) = \lambda+\Omega_3+\left(\bar{\epsilon}\,\Omega_1(t)+\epsilon\,\Omega_2(t)\right)\frac{\Psi^2(t,P)}{\Psi^1(t,P)} = \lambda+\Omega_3+O(\lambda^{-1})$$

and hence

$$\Psi^1(t,P) = \left(1+O(\lambda^{-1})\right)\exp\left(-\tfrac{i}{2}(\lambda+\Omega_3)t\right).$$

In a similar way if $P \to \infty^-$ we obtain

$$\Psi^2(t,P) = \left(1+O(\lambda^{-1})\right)\exp\left(+\tfrac{i}{2}(\lambda+\Omega_3)t\right).$$

To compute the remaining asymptotic estimates we use that if $P \to \infty^-$ then

$$\frac{\Psi^1(t,P)}{\Psi^2(t,P)} = \frac{A_{12}(t,\lambda)}{\mu-A_{11}(t,\lambda)} = -\frac{\bar{\epsilon}\,\Omega_1(t)+\epsilon\,\Omega_2(t)}{2\lambda}+O(\lambda^{-2}) \tag{42}$$

and if $P \to \infty^+$ then

$$(43)\qquad \frac{\Psi^2(t,P)}{\Psi^1(t,P)} = \frac{A_{21}(t,\lambda)}{\mu - A_{22}(t,\lambda)} = \frac{\epsilon\,\Omega_1(t) + \bar\epsilon\,\Omega_2(t)}{2\lambda} + O(\lambda^{-2}).$$

To find the poles of $\Psi(t,P)$ in P we note that according to the proof of Proposition 3.1 (and with the same notations) we have

$$(44)\qquad \Psi(t,P) = \Phi(t,\lambda)\Psi(0,P), \qquad \Phi(0,\lambda) = I_2 .$$

If $|t|$ is sufficiently small, the fundamental matrix $\Phi(t,\lambda)$ has no poles and $\det \Phi(t,\lambda) \neq 0$. It follows that the poles of $\Phi(t,\lambda)$ and $\Phi(0,\lambda)$ coincide, and we can obtain them by solving the following quadratic equation

$$\det A(0,\lambda) = \big(A_{11}(0,\lambda) - A_{12}(0,\lambda)\big)^2 = \mu^2$$

$\big($see (29, (34)$\big)$. One gets two time independent poles $P_1, P_2 \in \check{C}_h$ of $\Psi(t,P)$.

Finally, the meromorphic one-form $d\ln\Psi^1$ has a simple pole at ∞^- with residue $+1$ and is holomorphic in a neighbourhood of ∞^+. On the other hand $\Psi^1(t,P)$ has exactly two poles on $\check{C}_h$ and hence it has one zero on $\check{C}_h$. The same arguments hold for $\Psi^2(t,P)$. □

Let A_1, A_2, B_1 be a basis of $H_1(\check{C}_h, \mathbf{Z})$ as shown in Figure 2 ($A_1 \circ B_1 = 1$), and let ω_1, ω_2 be a basis of $H^0\big(C, \Omega^1(\infty^+ + \infty^-)\big)$, normalized by the conditions

$$\left(\int_{A_i} \omega_j\right)_{i,j=1,2} = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \end{pmatrix}.$$

We shall also suppose that ω_1 is a holomorphic form on the elliptic curve C_h. Define now the period matrix

$$\Pi = \begin{pmatrix} 2\pi i & 0 & \tau_1 \\ 0 & 2\pi i & \tau_2 \end{pmatrix},$$

where

$$\tau_1 = \int_{B_1} \omega_1, \qquad \tau_2 = \int_{B_1} \omega_2, \qquad \mathrm{Re}(\tau_1) < 0 .$$

Recall that the generalized Jacobian $J(C_h; \infty^{\pm})$ of C_h relative to the modulus $m = \infty^+ + \infty^-$ is identified with $\mathbf{C}^2/\Lambda$ where Λ is the lattice in $\mathbf{C}^2$ generated by the columns of Π. Let

$$\theta_{11}(z) = \theta_{11}(z \mid \tau_1) = \sum_{n=-\infty}^{\infty} \exp\{\tfrac{1}{2}\tau_1(n+\tfrac{1}{2})^2 + (z+\pi\sqrt{-1})(n+\tfrac{1}{2})\}, \qquad z \in \mathbf{C}$$

be the Jacobi theta function with characteristics $\left[\tfrac{1}{2}, \tfrac{1}{2}\right]$,

$$\theta_{11}(0) = 0, \quad \theta_{11}(z + 2\pi i) = -\theta_{11}(z), \quad \theta_{11}(z+\tau_1) = -\exp\big(-z - \tfrac{1}{2}\tau_1\big)\,\theta_{11}(z).$$

Denote by Ω the unique Abelian differential of second kind on C_h with poles at $\infty^{\pm}$, principal parts $\pm\frac{i}{2}\,d\lambda$ where $P=(\lambda,\mu)$, $i=\sqrt{-1}$, and normalized by $\int_{A_1}\Omega=0$. Let $P_0\in\check{C}_h$ be a fixed initial point, $c^{\pm}$, U be the constants defined by

$$(45)\qquad \int_{P_0}^{P}\Omega=\begin{cases}-\frac{i}{2}\lambda+c^{-}+0(\lambda^{-1}), & P\to\infty^{+}\\ +\frac{i}{2}\lambda+c^{+}+0(\lambda^{-1}), & P\to\infty^{-}\end{cases},\qquad \int_{B_1}\Omega=U.$$

Define the Abel-Jacobi map

$$\mathcal{A}\colon \operatorname{Div}^0(C_h)\to J(C_h)\ :\ \sum P_i-\sum Q_i\ \mapsto\ \int_{\Sigma\ Q_i}^{\Sigma\ P_i}\omega_1\,.$$

Here, and henceforth, we make the convention that the paths of integration between divisors are taken within C_h cut along its homology basis A_1, B_1, which we assume does not contain points of these divisors.

PROPOSITION 3.3. *The Baker-Akhiezer function is explicitly given by*

$$(46)\quad \Psi^1(t,P)=\mathrm{const}_1\cdot\exp\Big[t\Big(\int_{P_0}^{P}\Omega-c^{-}-\tfrac{i}{2}\Omega_3\Big)\Big]\frac{\theta_{11}\big(\mathcal{A}(P+\infty^{-}-P_1-P_2)+tU\big)}{\theta_{11}\big(\mathcal{A}(\infty^{+}+\infty^{-}-P_1-P_2)+tU\big)}$$

$$(47)\quad \Psi^2(t,P)=\mathrm{const}_2\cdot\exp\Big[t\Big(\int_{P_0}^{P}\Omega-c^{+}+\tfrac{i}{2}\Omega_3\Big)\Big]\frac{\theta_{11}\big(\mathcal{A}(P+\infty^{+}-P_1-P_2)+tU\big)}{\theta_{11}\big(\mathcal{A}(\infty^{+}+\infty^{-}-P_1-P_2)+tU\big)}$$

where

$$\mathrm{const}_1=\frac{\theta_{11}\big(\mathcal{A}(P-\infty^{-})\big)}{\theta_{11}\big(\mathcal{A}(\infty^{+}-\infty^{-})\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{+}-P_1)\big)}{\theta_{11}\big(\mathcal{A}(P-P_1)\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{+}-P_2)\big)}{\theta_{11}\big(\mathcal{A}(P-P_2)\big)}$$

$$\mathrm{const}_2=\frac{\theta_{11}\big(\mathcal{A}(P-\infty^{+})\big)}{\theta_{11}\big(\mathcal{A}(\infty^{-}-\infty^{+})\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{-}-P_1)\big)}{\theta_{11}\big(\mathcal{A}(P-P_1)\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{-}-P_2)\big)}{\theta_{11}\big(\mathcal{A}(P-P_2)\big)}$$

and P_1, P_2 are the poles of Ψ.

The proof of the above proposition is based on a general fact: the properties of Ψ enumerated in Proposition 3.2 define it uniquely. Indeed, if Ψ and $\widetilde{\Psi}$ are vector functions both satisfying the assumptions of Proposition 3.2, then the functions Ψ^1 and $\widetilde{\Psi}^1$ (resp. Ψ^2 and $\widetilde{\Psi}^2$) meromorphic on C_h have the same poles. Using this and the asymptotic estimates at infinity we conclude that $\Psi^1/\widetilde{\Psi}^1$ and $\Psi^2/\widetilde{\Psi}^2$ are meromorphic functions on C_h which have one pole (at $\widetilde{\Psi}^i=0$). Moreover

$$\Psi_1(t,\infty^{-})/\widetilde{\Psi}_1(t,\infty^{-})=1,\qquad \Psi_2(t,\infty^{-})/\widetilde{\Psi}_2(t,\infty^{-})=1$$

and hence $\Psi = \widetilde{\Psi}$. Finally, the reader may check that the functions (46) and (47) have the analyticity properties from Proposition 3.2 and hence they coincide with the Baker-Akhiezer function defined in Proposition 3.1. □

3.2 Solutions of the Lagrange top

Let $z = (z_1, z_2) \in J(C_h; \infty^{\pm})$. It is easy to check that the functions

$$\frac{\theta_{11}(z_1 \pm \tau_2)}{\theta_{11}(z_1)} e^{\mp z_2}$$

live on $J(C_h; \infty^{\pm})$. We shall see that they give solutions of the Lagrange top. By (16) we compute that $\frac{d}{dt} z = \text{constant}$, where

$$\frac{dz}{dt} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 2\pi i \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} & \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda d\lambda}{\mu} & \int_{A_2} \frac{\lambda d\lambda}{\mu} \end{pmatrix}^{-1} \begin{pmatrix} -i \\ -ai \end{pmatrix},$$

$$\int_{A_2} \frac{d\lambda}{\mu} = 0, \qquad \int_{A_2} \frac{\lambda d\lambda}{\mu} = -2\pi i$$

so

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left(\int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \begin{pmatrix} 2\pi \\ -i \int_{A_1} \frac{\lambda d\lambda}{\mu} + ai \int_{A_1} \frac{d\lambda}{\mu} \end{pmatrix}, \qquad a = -m\Omega_3 .$$

THEOREM 3.4. *The following equations hold*

$$\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t) = \text{const}_3 \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} e^{-z_2}, \tag{48}$$

$$\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t) = \text{const}_4 \frac{\theta_{11}(z_1 + \tau_2)}{\theta_{11}(z_1)} e^{+z_2}, \tag{49}$$

where

$$\begin{aligned} z_2 &= tV_2, \quad z_1 = tV_1 + \mathcal{A}(\infty^+ + \infty^- - P_1 - P_2), \\ \tau_2 &= \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2 \end{aligned} \tag{50}$$

and

$$\text{const}_3 = \frac{2i\, V_1\, \theta'_{11}(0)}{\theta_{11}\left(\mathcal{A}(\infty^- - \infty^+)\right)} \cdot \frac{\theta_{11}\left(\mathcal{A}(\infty^+ - P_1)\right)}{\theta_{11}\left(\mathcal{A}(\infty^- - P_1)\right)} \cdot \frac{\theta_{11}\left(\mathcal{A}(\infty^+ - P_2)\right)}{\theta_{11}\left(\mathcal{A}(\infty^- - P_2)\right)},$$

$$\text{const}_4 = \frac{2i\, V_1\, \theta'_{11}(0)}{\theta_{11}\left(\mathcal{A}(\infty^+ - \infty^-)\right)} \cdot \frac{\theta_{11}\left(\mathcal{A}(\infty^- - P_1)\right)}{\theta_{11}\left(\mathcal{A}(\infty^+ - P_1)\right)} \cdot \frac{\theta_{11}\left(\mathcal{A}(\infty^- - P_2)\right)}{\theta_{11}\left(\mathcal{A}(\infty^+ - P_2)\right)}.$$