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3. Explicit solutions

In this section we find explicit solutions for the Lagrange top (2). We compute first the Baker-Akhiezer function of the $\mathfrak{sl}(2,\mathbf{C})$ (or rather $\mathfrak{su}(2)$) Lax pair (14). This implies explicit formulae for the solutions of the Lagrange top in terms of exponentials and theta functions related to the spectral curve C_h (see for example Dubrovin [8], E. D. Belokolos, A. I. Bobenko, V. Z. Enol′skiĭ, A. R. Its, V. B. Matveev [5]).Then we note that the Jacobian $J(C_h)$ of C_h is just the Lagrange elliptic curve used in the classical theory which provides explicit solutions in terms of exponentials and sigma function related to $J(C_h)$.

By performing a unitary operation on the matrix (15) we may put its leading term in diagonal form. Substituting $a = -m\Omega_3$ in (14) and using the change of variables (25) we obtain the following Lax pair representation for the Lagrange top (2)

$$(29) \qquad \left[A\,,\, B - 2i\frac{d}{dt}\right] = 2i\frac{dA}{dt} + [A,B] = 0\,, \qquad \epsilon^2 = i\,, \quad i^2 = -1$$

where

$$A = A(t,\lambda) = \begin{pmatrix} A_{11}(t,\lambda) & A_{12}(t,\lambda) \\ A_{21}(t,\lambda) & A_{22}(t,\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 +$$

$$+ \begin{pmatrix} (1+m)\Omega_3 & \bar\epsilon\,\Omega_1(t) + \epsilon\,\Omega_2(t) \\ \epsilon\,\Omega_1(t) + \bar\epsilon\,\Omega_2(t) & -(m+1)\Omega_3 \end{pmatrix} \lambda - \begin{pmatrix} \Gamma_3 & \bar\epsilon\,\Gamma_1(t) + \epsilon\,\Gamma_2(t) \\ \epsilon\,\Gamma_1(t) + \bar\epsilon\,\Gamma_2(t) & -\Gamma_3 \end{pmatrix}$$

and

$$B = B(t,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \Omega_3 & \bar\epsilon\,\Omega_1(t) + \epsilon\,\Omega_2(t) \\ \epsilon\,\Omega_1(t) + \bar\epsilon\,\Omega_2(t) & -\Omega_3 \end{pmatrix}.$$

The spectral curve of the above Lax representation is defined by

$$\check{C}_h = \left\{\det\bigl(A(\lambda) - \mu I\bigr) = \mu^2 - f(\lambda) = 0\right\}\,,$$

$$f(\lambda) = \lambda^4 + 2(1+m)\,h_4\lambda^3 + \bigl(2h_3 + m(m+1)\,h_4^2\bigr)\lambda^2 - 2h_2\lambda + 1\,.$$

We shall also denote by C_h the Riemann surface of the compactified affine curve $\check{C}_h$. The reader may note the "similarity" between (29) and the Lax pair of the nonlinear Schrödinger equation (for a rigorous statement see Proposition 5.1).

3.1 The Baker-Akhiezer function

Let us fix a solution $A(t,\lambda)$ of (29) defined in a neighbourhood of $t = 0 \in \mathbf{C}$. We shall also suppose that the point $P = (\lambda, \mu)$ is such that $(1,-1)$ is not an eigenvector of the matrix $A(0,\lambda)$.

PROPOSITION 3.1. *For any $t \in \mathbf{C}$ in a sufficiently small neighbourhood of the origin, there exists a unique eigenfunction*

$$\Psi = \Psi(t, P) = \begin{pmatrix} \Psi^1(t, P) \\ \Psi^2(t, P) \end{pmatrix}, \qquad P = (\lambda, \mu) \in \check{C} \tag{30}$$

of $A(t, \lambda)$ (called the Baker-Akhiezer function) *satisfying the conditions*

$$2i \frac{d}{dt} \Psi(t, P) = B(t, \lambda)\Psi(t, P) \tag{31}$$

$$A(t, \lambda)\Psi(t, P) = \mu\Psi(t, P) \tag{32}$$

and normalized by

$$\Psi^1(0, P) + \Psi^2(0, P) = 1\,. \tag{33}$$

In terms of the coefficients $A_{ij}(t, \lambda)$ of the matrix $A = (A_{ij})$ we have

$$\Psi^1(0, P) = \frac{A_{12}(0, \lambda)}{A_{12}(0, \lambda) + \mu - A_{11}(0, \lambda)} = \frac{\mu - A_{22}(0, \lambda)}{A_{21}(0, \lambda) + \mu - A_{22}(0, \lambda)} \tag{34}$$

$$\Psi^2(0, P) = \frac{\mu - A_{11}(0, \lambda)}{A_{12}(0, \lambda) + \mu - A_{11}(0, \lambda)} = \frac{A_{21}(0, \lambda)}{A_{21}(0, \lambda) + \mu - A_{22}(0, \lambda)}\,. \tag{35}$$

Proof. Let $\Phi(t, \lambda)$ be a fundamental matrix for the operator $B(t, \lambda) - 2i\frac{d}{dt}$ normalized at $t = 0$. Then the general solution of (31) is written as

$$\Psi(t, P) = \Phi(t, \lambda)\Psi(0, P), \qquad \Phi(0, \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = (\lambda, \mu)\,. \tag{36}$$

As A and $B - 2i\frac{d}{dt}$ commute, we have

$$\left(B(t, \lambda) - 2i\frac{d}{dt}\right) A(t, \lambda)\Phi(t, \lambda) = A(t, \lambda)\left(B(t, \lambda) - 2i\frac{d}{dt}\right) \Phi(t, \lambda) = 0$$

and hence $A(t, \lambda)\Phi(t, \lambda) = \Phi(t, \lambda)M(P)$ for some constant matrix $M(P)$ computed by substituting $t = 0$. Thus $M(P) = A(0, \lambda)$ and

$$A(0, \lambda) = \Phi^{-1}(t, \lambda)A(t, \lambda)\,\Phi(t, \lambda)\,.$$

The constants $\Psi^1(0, P), \Psi^2(0, P)$ are uniquely defined by (32) and (33). Finally,

$$\begin{aligned} A(t, \lambda)\Psi(t, P) &= \Phi(t, \lambda)A(0, \lambda)\,\Phi^{-1}(t, \lambda)\,\Phi(t, \lambda)\,\Psi(0, P) \\ &= \Phi(t, \lambda) \cdot \mu \cdot \Psi(0, P) \\ &= \mu\Psi(t, P)\,. \end{aligned}$$

The formulae (34), (35) follow from (32), (33). □

Denote by ∞^+ (respectively ∞^-) the point on $C_h - \check{C}_h$ such that in its neighbourhood $\mu/\lambda^2 \sim +1$ (resp. (-1)).

PROPOSITION 3.2. *There exists $t_0 > 0$ such that for any fixed $t \in \mathbf{C}$, $|t| < t_0$, the Baker-Akhiezer vector-function $\Psi(t, P)$ is meromorphic in P on the affine curve $\check{C}_h$ and has two poles at $P_1, P_2 \in C_h$ which do not depend on t. In a neighbourhood of the two infinite points $\infty^\pm$ on C_h we have*

$$\Psi^1(t,P) = \begin{cases} \left(1 + O(\lambda^{-1})\right) \exp\left(-\frac{i}{2}(\lambda + \Omega_3)t\right), & P \to \infty^+ \\ O(\lambda^{-1}) \exp\left(+\frac{i}{2}(\lambda + \Omega_3)t\right), & P \to \infty^- \end{cases} \tag{37}$$

$$\Psi^2(t,P) = \begin{cases} O(\lambda^{-1}) \exp\left(-\frac{i}{2}(\lambda + \Omega_3)t\right), & P \to \infty^+ \\ \left(1 + O(\lambda)^{-1}\right) \exp\left(+\frac{i}{2}(\lambda + \Omega_3)t\right), & P \to \infty^-, \end{cases} \tag{38}$$

where $i = \sqrt{-1}$. Moreover, $\Psi^1(t, P)$ $(\Psi^2(t, P))$ has exactly one zero on $\check{C}_h$ and the refined asymptotic estimates of Ψ^1 at ∞^- and of Ψ^2 at ∞^+ read

$$\Psi^1(t,P) = \left[-\frac{\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t)}{2\lambda} + O(\lambda^{-2})\right] \exp\left(+\tfrac{i}{2}(\lambda + \Omega_3)t\right), \quad P \to \infty^- \tag{39}$$

$$\Psi^2(t,P) = \left[+\frac{\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t)}{2\lambda} + O(\lambda^{-2})\right] \exp\left(-\tfrac{i}{2}(\lambda + \Omega_3)t\right), \quad P \to \infty^+. \tag{40}$$

Proof. According to (32), $(\Psi^1, \Psi^2) \in \mathrm{Ker}(A - \mu I)$ and hence

$$\frac{\Psi^2(t,P)}{\Psi^1(t,P)} = \frac{\mu - \lambda^2 - (1+m)\Omega_3\lambda + \Gamma_3(t)}{\left(\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t)\right)\lambda - \bar{\epsilon}\,\Gamma_1(t) + \epsilon\,\Gamma_2(t)}. \tag{41}$$

If $P \to \infty^+$ then $\mu - \lambda^2 - (1+m)\Omega_3\lambda \sim O(1)$ and using (29), (31), (32) and (41) we compute

$$2i\frac{d}{dt}\ln \Psi^1(t,P) = \lambda + \Omega_3 + \left(\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t)\right)\frac{\Psi^2(t,P)}{\Psi^1(t,P)} = \lambda + \Omega_3 + O(\lambda^{-1})$$

and hence

$$\Psi^1(t,P) = \left(1 + O(\lambda^{-1}\right) \exp\left(-\tfrac{i}{2}(\lambda + \Omega_3)t\right).$$

In a similar way if $P \to \infty^-$ we obtain

$$\Psi^2(t,P) = \left(1 + O(\lambda^{-1})\right) \exp\left(+\tfrac{i}{2}(\lambda + \Omega_3)t\right).$$

To compute the remaining asymptotic estimates we use that if $P \to \infty^-$ then

$$\frac{\Psi^1(t,P)}{\Psi^2(t,P)} = \frac{A_{12}(t,\lambda)}{\mu - A_{11}(t,\lambda)} = -\frac{\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t)}{2\lambda} + O(\lambda^{-2}) \tag{42}$$

and if $P \to \infty^+$ then

$$\frac{\Psi^2(t,P)}{\Psi^1(t,P)} = \frac{A_{21}(t,\lambda)}{\mu - A_{22}(t,\lambda)} = \frac{\epsilon\,\Omega_1(t) + \bar\epsilon\,\Omega_2(t)}{2\lambda} + O(\lambda^{-2})\,. \tag{43}$$

To find the poles of $\Psi(t,P)$ in P we note that according to the proof of Proposition 3.1 (and with the same notations) we have

$$\Psi(t,P) = \Phi(t,\lambda)\Psi(0,P), \qquad \Phi(0,\lambda) = I_2\,. \tag{44}$$

If $|t|$ is sufficiently small, the fundamental matrix $\Phi(t,\lambda)$ has no poles and $\det\Phi(t,\lambda) \neq 0$. It follows that the poles of $\Phi(t,\lambda)$ and $\Phi(0,\lambda)$ coincide, and we can obtain them by solving the following quadratic equation

$$\det A(0,\lambda) = \big(A_{11}(0,\lambda) - A_{12}(0,\lambda)\big)^2 = \mu^2$$

$\big($see (29, (34)$\big)$. One gets two time independent poles $P_1, P_2 \in \check C_h$ of $\Psi(t,P)$.

Finally, the meromorphic one-form $d\ln\Psi^1$ has a simple pole at ∞^- with residue $+1$ and is holomorphic in a neighbourhood of ∞^+. On the other hand $\Psi^1(t,P)$ has exactly two poles on $\check C_h$ and hence it has one zero on $\check C_h$. The same arguments hold for $\Psi^2(t,P)$. □

Let A_1, A_2, B_1 be a basis of $H_1(\check C_h, \mathbf{Z})$ as shown in Figure 2 ($A_1 \circ B_1 = 1$), and let ω_1, ω_2 be a basis of $H^0\big(C, \Omega^1(\infty^+ + \infty^-)\big)$, normalized by the conditions

$$\left(\int_{A_i}\omega_j\right)_{i,j=1,2} = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \end{pmatrix}.$$

We shall also suppose that ω_1 is a holomorphic form on the elliptic curve C_h. Define now the period matrix

$$\Pi = \begin{pmatrix} 2\pi i & 0 & \tau_1 \\ 0 & 2\pi i & \tau_2 \end{pmatrix},$$

where

$$\tau_1 = \int_{B_1}\omega_1, \qquad \tau_2 = \int_{B_1}\omega_2, \qquad \operatorname{Re}(\tau_1) < 0\,.$$

Recall that the generalized Jacobian $J(C_h; \infty^\pm)$ of C_h relative to the modulus $m = \infty^+ + \infty^-$ is identified with $\mathbf{C}^2/\Lambda$ where Λ is the lattice in $\mathbf{C}^2$ generated by the columns of Π. Let

$$\theta_{11}(z) = \theta_{11}(z \mid \tau_1) = \sum_{n=-\infty}^{\infty} \exp\{\tfrac12\tau_1(n+\tfrac12)^2 + (z+\pi\sqrt{-1})(n+\tfrac12)\}, \qquad z \in \mathbf{C}$$

be the Jacobi theta function with characteristics $\left[\frac12, \frac12\right]$,

$$\theta_{11}(0) = 0, \quad \theta_{11}(z + 2\pi i) = -\theta_{11}(z), \quad \theta_{11}(z + \tau_1) = -\exp\big(-z - \tfrac12\tau_1\big)\,\theta_{11}(z)\,.$$

Denote by Ω the unique Abelian differential of second kind on C_h with poles at $\infty^{\pm}$, principal parts $\pm\frac{i}{2}\,d\lambda$ where $P=(\lambda,\mu)$, $i=\sqrt{-1}$, and normalized by $\int_{A_1}\Omega=0$. Let $P_0\in\check{C}_h$ be a fixed initial point, $c^{\pm}$, U be the constants defined by

$$(45)\qquad \int_{P_0}^{P}\Omega=\begin{cases}-\frac{i}{2}\lambda+c^{-}+0(\lambda^{-1}), & P\to\infty^{+}\\ +\frac{i}{2}\lambda+c^{+}+0(\lambda^{-1}), & P\to\infty^{-}\end{cases},\qquad \int_{B_1}\Omega=U\,.$$

Define the Abel-Jacobi map

$$\mathcal{A}\colon \operatorname{Div}^0(C_h)\to J(C_h)\,:\,\sum P_i-\sum Q_i\;\mapsto\;\int_{\sum\,Q_i}^{\sum\,P_i}\omega_1\,.$$

Here, and henceforth, we make the convention that the paths of integration between divisors are taken within C_h cut along its homology basis A_1, B_1, which we assume does not contain points of these divisors.

PROPOSITION 3.3. *The Baker-Akhiezer function is explicitly given by*

$$(46)\quad \Psi^1(t,P)=\mathrm{const}_1\cdot\exp\Big[t\Big(\int_{P_0}^{P}\Omega-c^{-}-\tfrac{i}{2}\Omega_3\Big)\Big]\frac{\theta_{11}\big(\mathcal{A}(P+\infty^{-}-P_1-P_2)+tU\big)}{\theta_{11}\big(\mathcal{A}(\infty^{+}+\infty^{-}-P_1-P_2)+tU\big)}$$

$$(47)\quad \Psi^2(t,P)=\mathrm{const}_2\cdot\exp\Big[t\Big(\int_{P_0}^{P}\Omega-c^{+}+\tfrac{i}{2}\Omega_3\Big)\Big]\frac{\theta_{11}\big(\mathcal{A}(P+\infty^{+}-P_1-P_2)+tU\big)}{\theta_{11}\big(\mathcal{A}(\infty^{+}+\infty^{-}-P_1-P_2)+tU\big)}$$

where

$$\mathrm{const}_1=\frac{\theta_{11}\big(\mathcal{A}(P-\infty^{-})\big)}{\theta_{11}\big(\mathcal{A}(\infty^{+}-\infty^{-})\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{+}-P_1)\big)}{\theta_{11}\big(\mathcal{A}(P-P_1)\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{+}-P_2)\big)}{\theta_{11}\big(\mathcal{A}(P-P_2)\big)}$$

$$\mathrm{const}_2=\frac{\theta_{11}\big(\mathcal{A}(P-\infty^{+})\big)}{\theta_{11}\big(\mathcal{A}(\infty^{-}-\infty^{+})\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{-}-P_1)\big)}{\theta_{11}\big(\mathcal{A}(P-P_1)\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{-}-P_2)\big)}{\theta_{11}\big(\mathcal{A}(P-P_2)\big)}$$

and P_1, P_2 *are the poles of* Ψ.

The proof of the above proposition is based on a general fact: the properties of Ψ enumerated in Proposition 3.2 define it uniquely. Indeed, if Ψ and $\widetilde{\Psi}$ are vector functions both satisfying the assumptions of Proposition 3.2, then the functions Ψ^1 and $\widetilde{\Psi}^1$ (resp. Ψ^2 and $\widetilde{\Psi}^2$) meromorphic on C_h have the same poles. Using this and the asymptotic estimates at infinity we conclude that $\Psi^1/\widetilde{\Psi}^1$ and $\Psi^2/\widetilde{\Psi}^2$ are meromorphic functions on C_h which have one pole (at $\widetilde{\Psi}^i=0$). Moreover

$$\Psi_1(t,\infty^{-})/\widetilde{\Psi}_1(t,\infty^{-})=1,\qquad \Psi_2(t,\infty^{-})/\widetilde{\Psi}_2(t,\infty^{-})=1$$

and hence $\Psi = \widetilde{\Psi}$. Finally, the reader may check that the functions (46) and (47) have the analyticity properties from Proposition 3.2 and hence they coincide with the Baker-Akhiezer function defined in Proposition 3.1. □

3.2 Solutions of the Lagrange top

Let $z = (z_1, z_2) \in J(C_h; \infty^{\pm})$. It is easy to check that the functions

$$\frac{\theta_{11}(z_1 \pm \tau_2)}{\theta_{11}(z_1)}\, e^{\mp z_2}$$

live on $J(C_h; \infty^{\pm})$. We shall see that they give solutions of the Lagrange top. By (16) we compute that $\frac{d}{dt} z = \text{constant}$, where

$$\frac{dz}{dt} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 2\pi i \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} & \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda d\lambda}{\mu} & \int_{A_2} \frac{\lambda d\lambda}{\mu} \end{pmatrix}^{-1} \begin{pmatrix} -i \\ -ai \end{pmatrix},$$

$$\int_{A_2} \frac{d\lambda}{\mu} = 0, \qquad \int_{A_2} \frac{\lambda d\lambda}{\mu} = -2\pi i$$

so

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left(\int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \begin{pmatrix} 2\pi \\ -i \int_{A_1} \frac{\lambda d\lambda}{\mu} + ai \int_{A_1} \frac{d\lambda}{\mu} \end{pmatrix}, \qquad a = -m\Omega_3 .$$

THEOREM 3.4. *The following equations hold*

$$\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t) = \text{const}_3\, \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)}\, e^{-z_2}, \tag{48}$$

$$\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t) = \text{const}_4\, \frac{\theta_{11}(z_1 + \tau_2)}{\theta_{11}(z_1)}\, e^{+z_2}, \tag{49}$$

where

$$\begin{aligned} z_2 &= tV_2, \quad z_1 = tV_1 + \mathcal{A}(\infty^+ + \infty^- - P_1 - P_2), \\ \tau_2 &= \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2 \end{aligned} \tag{50}$$

and

$$\text{const}_3 = \frac{2i\, V_1\, \theta'_{11}(0)}{\theta_{11}\big(\mathcal{A}(\infty^- - \infty^+)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^+ - P_1)\big)}{\theta_{11}\big(\mathcal{A}(\infty^- - P_1)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^+ - P_2)\big)}{\theta_{11}\big(\mathcal{A}(\infty^- - P_2)\big)},$$

$$\text{const}_4 = \frac{2i\, V_1\, \theta'_{11}(0)}{\theta_{11}\big(\mathcal{A}(\infty^+ - \infty^-)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^- - P_1)\big)}{\theta_{11}\big(\mathcal{A}(\infty^+ - P_1)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^- - P_2)\big)}{\theta_{11}\big(\mathcal{A}(\infty^+ - P_2)\big)}.$$

Let us denote

$$\omega_1 = \pm\big(\omega_1^0 + O(\lambda^{-1})\big)\, d\big(\lambda^{-1}\big), \qquad P = (\lambda, \mu) \to \infty^{\pm},$$
$$\omega_2 = \pm\big(\omega_2^1\lambda + \omega_2^0 + O(\lambda^{-1})\big)\, d\big(\lambda^{-1}\big), \qquad P = (\lambda, \mu) \to \infty^{\pm}.$$

To prove Theorem 3.4 we shall need the following

LEMMA 3.5. *The above defined differentials are such that*

$$\omega_1^0 = -i\int_{B_1} \Omega = -iV_1\,, \qquad \omega_2^0 = i\,(c^+ - c^-)\,,$$

$$V_2 = -c^+ + c^- + i\Omega_3\,, \qquad \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2\,.$$

Proof. The identity $\omega_1^0 = -i\int_{B_1}\Omega$ is a reciprocity law between the differential of the first kind ω_1 and the differential of the second kind Ω [13]. It is obtained by integrating $\pi(P)\omega_1$, where $\pi(P) = \int_{P_0}^{P}\Omega$, along the border of C_h cut along its homology basis A_1, B_1. On the other hand

$$\omega_1 = 2\pi i\left(\int_{A_1}\frac{d\lambda}{\mu}\right)^{-1}\frac{d\lambda}{\mu}$$

and hence

$$\omega_1^0 = -2\pi i\left(\int_{A_1}\frac{d\lambda}{\mu}\right)^{-1} = -iV_1\,.$$

Similarly the identity $\omega_2^0 = i(c^+ - c^-)$ is a reciprocity law between the differential of the third kind ω_2 and the differential of the second kind Ω, and $\mathcal{A}(\infty^+ - \infty^-) = \int_{B_1}\omega_2$ is a reciprocity law between the differential of the third kind ω_2 and the differential of the first kind ω_1. Finally, as $\omega_2 = \frac{\int_{A_1}\frac{\lambda d\lambda}{\mu}}{\int_{A_1}\frac{d\lambda}{\mu}}\frac{d\lambda}{\mu} - \frac{\lambda\, d\lambda}{\mu}$ we have $\omega_2^0 = -\frac{\int_{A_1}\frac{\lambda d\lambda}{\mu}}{\int_{A_1}\frac{d\lambda}{\mu}} - (1+m)\Omega_3 = -iV_1 - \Omega_3$ and hence $V_2 = -c^+ + c^- + i\Omega_3$. □

Proof of Theorem 3.4. According to (42), (43)

$$\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t) = -2\lim_{P\to\infty^-}\frac{\lambda\Psi^1(t,P)}{\Psi^2(t,P)}$$

and

$$\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t) = +2\lim_{P\to\infty^+}\frac{\lambda\Psi^2(t,P)}{\Psi^1(t,P)}\,.$$

To compute the limit we use (46), (47) and

$$\lim_{P\to\infty^-} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^-)\big) = \theta_{11}'(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta_{11}'(0)$$

$$\lim_{P\to\infty^+} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^+)\big) = \theta_{11}'(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta_{11}'(0)$$

(see Lemma 3.5). □

3.3 Effectivization

Let $\wp, \zeta, \sigma$ be the Weierstrass functions related to the elliptic curve Γ defined by

(51) $$\eta^2 = 4\xi^3 - g_2\xi - g_3$$

(we use the standard notations of [4]).

Consider also the *real* elliptic curve C with affine equation

(52) $$\mu^2 + \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

and natural anti-holomorphic involution $(\lambda, \mu) \to (\overline{\lambda}, \overline{\mu})$, and put

(53) $$g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^4 - 4\frac{a_1}{4}\frac{a_3}{4}, \quad g_3 = \det\begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}.$$

It is well known that the curves C and Γ are isomorphic over $\mathbf{C}$ and that under this isomorphism

(54) $$\frac{d\lambda}{\mu} = \frac{d\xi}{\eta}.$$

Following Weil [25] we call Γ the Jacobian $J(C)$ of the elliptic curve C and we write $J(C) = \Gamma$. Note that $J(C)$ and Γ are real isomorphic and that $J(C)$ and C are not real isomorphic.

Further we make the substitution (23) and C becomes the spectral curve $\widetilde{C}_h$ of Adler and van Moerbeke $\{\mu^2 + f(\lambda) = 0\}$, where

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + \left(2h_3 + m(m+1)h_4^2\right)\lambda^2 - 2h_2\lambda + 1$$

and Γ becomes the Lagrange curve Γ_h. Recall that, as we explained at the end of Section 2, the curve C_h with an equation $\{\mu^2 = f(\lambda)\}$ and antiholomorphic involution $(\lambda, \mu) \to (\overline{\lambda}, -\overline{\mu})$, is isomorphic over $\mathbf{R}$ to $\widetilde{C}_h$, so we write $C_h = \widetilde{C}_h$. The Jacobian curve $J(C_h) = \Gamma_h$ was computed by

Lagrange [17], while C_h appeared first in [1, 21] as a spectral curve of a Lax pair associated to the Lagrange top.

Recall that $\sigma(z)$ is an entire function in z related to $\zeta(z)$, $\wp(z)$ and the already defined function $\theta_{11}(z \mid \tau_1)$ on C_h as follows:

$$\zeta'(z) = -\wp(z)\,, \qquad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z)\,, \qquad {}' = \frac{d}{dz}$$

$$\sigma(z) = \frac{\theta_{11}(zU)}{U\theta'_{11}(0)} \exp\left\{\frac{z^2U^2\theta'''_{11}(0)}{6\theta'_{11}(0)}\right\} = z - \frac{g_2 z^5}{240} + \cdots\,, \tag{55}$$

where U is a constant depending on g_2 and g_3. We shall also use the "addition formula"

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)} = \wp(v) - \wp(u)\,.$$

To state our result let us introduce the notations

$$\begin{aligned} &2x_1 = \epsilon\,\Omega_1 + \bar\epsilon\,\Omega_2\,, && 2x_2 = \bar\epsilon\,\Omega_1 + \epsilon\,\Omega_2\,, && \epsilon^2 = \sqrt{-1} \\ &2y_1 = \epsilon^3\Gamma_1 + \epsilon\,\Gamma_2\,, && 2y_2 = \epsilon\,\Gamma_1 + \epsilon^3\Gamma_2\,, && i^2 = -1 \\ &\rho_1 = -i\,m\,\Omega_3\,, && \rho_2 = -i\,\Omega_3\,. \end{aligned} \tag{56}$$

The system (2) is equivalent to

$$\begin{aligned} &\dot x_1 = +\rho_1 x_1 - y_1\,, && \dot y_1 = -\rho_2 y_1 + x_1\Gamma_3 \\ &\dot x_2 = -\rho_1 x_2 + y_2\,, && \dot y_2 = +\rho_2 y_2 - x_2\Gamma_3 \\ &\rho_1\,,\ \rho_2 = \text{constants}\,, && \dot\Gamma_3 = 2x_1y_2 - 2x_2y_1 \end{aligned} \tag{57}$$

with first integrals $I_0 = 4x_1x_2 - 2\Gamma_3$, $I_1 = 4x_1y_2 + 4x_2y_1 - 2(\rho_1+\rho_2)\Gamma_3$ and $I_2 = \Gamma_3^2 - 4y_1y_2$.

THEOREM 3.6. *The general solution of the Lagrange top* (2) *can be written in the form*

$$x_1(t) = -\frac{\sigma(t-k-l)}{\sigma(t)\,\sigma(k+l)}\,e^{at+b} \qquad x_2(t) = -\frac{\sigma(t+k+l)}{\sigma(t)\,\sigma(k+l)}\,e^{-at-b}$$

$$y_1(t) = \frac{\sigma(t-k)\,\sigma(t-l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}\,e^{at+b} \qquad y_2(t) = \frac{\sigma(t+k)\,\sigma(t+l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}\,e^{-at-b}$$

$$\Gamma_3(t) = \frac{\sigma(t+k)\,\sigma(t-k)}{\sigma^2(k)\,\sigma^2(t)} + \frac{\sigma(t+l)\,\sigma(t-l)}{\sigma^2(l)\,\sigma^2(t)} = -2\wp(t) + \wp(l) + \wp(k)$$

$$\rho_1 = a - \zeta(l) - \zeta(k) \qquad \rho_2 = -a - \zeta(k) - \zeta(l) + 2\zeta(k+l)\,,$$

where g_2, g_3, a, b, k, l are arbitrary constants subject to the relation $g_2^3 - 27g_3^2 \neq 0$.

REMARK. The non-general solutions of the Lagrange top are obtained from the above formulae by taking the limit $g_2^3 - 27g_3^2 \to 0$. The formulae for the position of the body in space, and in particular for $\Gamma_3(t)$, $y_1(t)$, $y_2(t)$, are due to Jacobi [15]. The expressions for $x_1(t)$, $x_2(t)$ were first deduced by Klein and Sommerfeld [16, p. 436]. Note however that in [16] the constant a, and hence the invariant level set on which the solution lives, is not arbitrary.

Proof. To make the solutions of the Lagrange top effective we use the following 4-dimensional Lie group of transformations preserving the system (57):

$$
\begin{array}{llll}
 & x_1 \to U x_1 e^{at+b}, & x_2 \to U x_2 e^{-at-b}, & t \to \dfrac{t}{U} + T \\
(58) & y_1 \to U^2 y_1\, e^{at+b}, & y_2 \to U^2 y_2\, e^{-at-b}, & \Gamma_3 \to U^2\, \Gamma_3 \\
 & \rho_1 \to U\rho_1 + a, & \rho_2 \to U\rho_2 - a &
\end{array}
$$

where $U \neq 0$, T, a, b are constants.

The group (58) transforms x_1 from (48) (see also (56), (55)), where $z_1 = tU - TU$, $z_1 - \tau_2 = (t - k - l)U$ as follows

$$
x_1(t) = \text{const}\, \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} = -\frac{\sigma(t-k-l)}{\sigma(t)\,\sigma(k+l)}\, e^{at+b}.
$$

(we used the fact that

$$
\frac{\theta_{11}(z_1 - \tau_2)\,\sigma(t)}{\theta_{11}(z_1)\,\sigma(t-k-l)}
$$

is a constant). The variable x_2 is computed in the same way.

If we define the constant k by the condition $y_1(t-k) = 0$, then the first equation of (57) gives

$$
\frac{y_1(t)}{x_1(t)} = \rho_1 - \frac{x_1'(t)}{x_1(t)} = \frac{\sigma(t-k)\,h(t)}{\sigma(t)\,\sigma(t-k-l)}
$$

where $h(t)$ is a meromorphic function on $\mathbf{C}$, such that $y_1(t)/x_1(t)$ is single valued with poles at $t = 0$ and $t = k + l$, and residues (-1) and $(+1)$ respectively. These three conditions define $h(t)$ uniquely:

$$
h(t) = \frac{\sigma(t-l)\,\sigma(k+l)}{\sigma(k)\,\sigma(l)},
$$

which implies the formula for $y_1(t)$. The expression for $y_2(t)$ is obtained in the same way.

To deduce an expression for $\Gamma_3(t)$ we use the fact that

$$\Gamma_3(t) = 2x_1x_2 - \tfrac{1}{2}I_0 = -2\wp(t) + 2\wp(k+l) - \tfrac{1}{2}I_0 \,.$$

The value of I_0 is easily computed by using the third equation of (57) and the formulae deduced for x_1, y_1. By substituting $t = k$ we obtain

$$\Gamma_3(k) = \frac{\sigma(k-l)\,\sigma(k+l)}{\sigma^2(k)\,\sigma^2(l)} = \wp(l) - \wp(k)$$

and in a similar way $\Gamma_3(l) = \wp(k) - \wp(l)$. We conclude that

$$\Gamma_3(t) = -2\wp(t) + \wp(l) + \wp(k)\,.$$

Finally, to compute ρ_1, ρ_2 we shall use once again (57). As $y_1(k) = 0$ we have

$$\begin{aligned}\rho_1 &= \frac{\dot{x}_1(k)}{x_1(k)} = \frac{d}{dt}\ln x_1(t)\Big|_{t=k} \\ &= \frac{d}{dt}\ln\sigma(t-k-l)\Big|_{t=k} - \frac{d}{dt}\ln\sigma(t)\Big|_{t=k} + a \\ &= a - \zeta(l) - \zeta(k)\,.\end{aligned}$$

In a quite similar way we obtain

$$\rho_2 = -\frac{d}{dt}\ln y_1(t)\Big|_{t=k+l} = -a - \zeta(k) - \zeta(l) + 2\zeta(k+l)\,.$$

Theorem 3.6 is proved. $\square$

REMARK. If we impose the condition

$$\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = \Gamma_3^2 - 4y_1y_2 = 1\,,$$

then

$$\begin{aligned}&\left(\frac{\sigma(t+k)\,\sigma(t-k)}{\sigma^2(k)\,\sigma^2(t)} + \frac{\sigma(t+l)\,\sigma(t-l)}{\sigma^2(l)\,\sigma^2(t)}\right)^2 - \frac{\sigma(t-k)\,\sigma(t-l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}\,\frac{\sigma(t+k)\,\sigma(t+l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)} \\ &\qquad = \left(\frac{\sigma(t+k)\,\sigma(t-k)}{\sigma^2(k)\,\sigma^2(t)} - \frac{\sigma(t+l)\,\sigma(t-l)}{\sigma^2(l)\,\sigma^2(t)}\right)^2 = \big(\wp(k) - \wp(l)\big)^2 = 1\end{aligned}$$

and hence $\wp(k) - \wp(l) = \pm 1$.