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## 3. EXPLICIT SOLUTIONS

In this section we find explicit solutions for the Lagrange top (2). We compute first the Baker-Akhiezer function of the  $\mathfrak{sl}(2, \mathbf{C})$  (or rather  $\mathfrak{su}(2)$ ) Lax pair (14). This implies explicit formulae for the solutions of the Lagrange top in terms of exponentials and theta functions related to the spectral curve  $C_h$  (see for example Dubrovin [8], E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, V. B. Matveev [5]). Then we note that the Jacobian  $J(C_h)$  of  $C_h$  is just the Lagrange elliptic curve used in the classical theory which provides explicit solutions in terms of exponentials and sigma function related to  $J(C_h)$ .

By performing a unitary operation on the matrix (15) we may put its leading term in diagonal form. Substituting  $a = -m\Omega_3$  in (14) and using the change of variables (25) we obtain the following Lax pair representation for the Lagrange top (2)

$$(29) \quad [A, B - 2i \frac{d}{dt}] = 2i \frac{dA}{dt} + [A, B] = 0, \quad \epsilon^2 = i, \quad i^2 = -1$$

where

$$A = A(t, \lambda) = \begin{pmatrix} A_{11}(t, \lambda) & A_{12}(t, \lambda) \\ A_{21}(t, \lambda) & A_{22}(t, \lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + \\ + \begin{pmatrix} (1+m)\Omega_3 & \bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t) \\ \epsilon\Omega_1(t) + \bar{\epsilon}\Omega_2(t) & -(m+1)\Omega_3 \end{pmatrix} \lambda - \begin{pmatrix} \Gamma_3 & \bar{\epsilon}\Gamma_1(t) + \epsilon\Gamma_2(t) \\ \epsilon\Gamma_1(t) + \bar{\epsilon}\Gamma_2(t) & -\Gamma_3 \end{pmatrix}$$

and

$$B = B(t, \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \Omega_3 & \bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t) \\ \epsilon\Omega_1(t) + \bar{\epsilon}\Omega_2(t) & -\Omega_3 \end{pmatrix}.$$

The spectral curve of the above Lax representation is defined by

$$\check{C}_h = \{ \det(A(\lambda) - \mu I) = \mu^2 - f(\lambda) = 0 \},$$

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + (2h_3 + m(m+1)h_4^2)\lambda^2 - 2h_2\lambda + 1.$$

We shall also denote by  $C_h$  the Riemann surface of the compactified affine curve  $\check{C}_h$ . The reader may note the “similarity” between (29) and the Lax pair of the nonlinear Schrödinger equation (for a rigorous statement see Proposition 5.1).

## 3.1 THE BAKER-AKHIEZER FUNCTION

Let us fix a solution  $A(t, \lambda)$  of (29) defined in a neighbourhood of  $t = 0 \in \mathbf{C}$ . We shall also suppose that the point  $P = (\lambda, \mu)$  is such that  $(1, -1)$  is not an eigenvector of the matrix  $A(0, \lambda)$ .

PROPOSITION 3.1. *For any  $t \in \mathbf{C}$  in a sufficiently small neighbourhood of the origin, there exists a unique eigenfunction*

$$(30) \quad \Psi = \Psi(t, P) = \begin{pmatrix} \Psi^1(t, P) \\ \Psi^2(t, P) \end{pmatrix}, \quad P = (\lambda, \mu) \in \check{C}$$

of  $A(t, \lambda)$  (called the Baker-Akhiezer function) satisfying the conditions

$$(31) \quad 2i \frac{d}{dt} \Psi(t, P) = B(t, \lambda) \Psi(t, P)$$

$$(32) \quad A(t, \lambda) \Psi(t, P) = \mu \Psi(t, P)$$

and normalized by

$$(33) \quad \Psi^1(0, P) + \Psi^2(0, P) = 1.$$

In terms of the coefficients  $A_{ij}(t, \lambda)$  of the matrix  $A = (A_{ij})$  we have

$$(34) \quad \Psi^1(0, P) = \frac{A_{12}(0, \lambda)}{A_{12}(0, \lambda) + \mu - A_{11}(0, \lambda)} = \frac{\mu - A_{22}(0, \lambda)}{A_{21}(0, \lambda) + \mu - A_{22}(0, \lambda)}$$

$$(35) \quad \Psi^2(0, P) = \frac{\mu - A_{11}(0, \lambda)}{A_{12}(0, \lambda) + \mu - A_{11}(0, \lambda)} = \frac{A_{21}(0, \lambda)}{A_{21}(0, \lambda) + \mu - A_{22}(0, \lambda)}.$$

*Proof.* Let  $\Phi(t, \lambda)$  be a fundamental matrix for the operator  $B(t, \lambda) - 2i \frac{d}{dt}$  normalized at  $t = 0$ . Then the general solution of (31) is written as

$$(36) \quad \Psi(t, P) = \Phi(t, \lambda) \Psi(0, P), \quad \Phi(0, \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = (\lambda, \mu).$$

As  $A$  and  $B - 2i \frac{d}{dt}$  commute, we have

$$\left( B(t, \lambda) - 2i \frac{d}{dt} \right) A(t, \lambda) \Phi(t, \lambda) = A(t, \lambda) \left( B(t, \lambda) - 2i \frac{d}{dt} \right) \Phi(t, \lambda) = 0$$

and hence  $A(t, \lambda) \Phi(t, \lambda) = \Phi(t, \lambda) M(P)$  for some constant matrix  $M(P)$  computed by substituting  $t = 0$ . Thus  $M(P) = A(0, \lambda)$  and

$$A(0, \lambda) = \Phi^{-1}(t, \lambda) A(t, \lambda) \Phi(t, \lambda).$$

The constants  $\Psi^1(0, P), \Psi^2(0, P)$  are uniquely defined by (32) and (33). Finally,

$$\begin{aligned} A(t, \lambda) \Psi(t, P) &= \Phi(t, \lambda) A(0, \lambda) \Phi^{-1}(t, \lambda) \Phi(t, \lambda) \Psi(0, P) \\ &= \Phi(t, \lambda) \cdot \mu \cdot \Psi(0, P) \\ &= \mu \Psi(t, P). \end{aligned}$$

The formulae (34), (35) follow from (32), (33).  $\square$

Denote by  $\infty^+$  (respectively  $\infty^-$ ) the point on  $C_h - \check{C}_h$  such that in its neighbourhood  $\mu/\lambda^2 \sim +1$  (resp.  $(-1)$ ).

PROPOSITION 3.2. *There exists  $t_0 > 0$  such that for any fixed  $t \in \mathbf{C}$ ,  $|t| < t_0$ , the Baker-Akhiezer vector-function  $\Psi(t, P)$  is meromorphic in  $P$  on the affine curve  $\check{C}_h$  and has two poles at  $P_1, P_2 \in C_h$  which do not depend on  $t$ . In a neighbourhood of the two infinite points  $\infty^\pm$  on  $C_h$  we have*

$$(37) \quad \Psi^1(t, P) = \begin{cases} (1 + O(\lambda^{-1})) \exp\left(-\frac{i}{2}(\lambda + \Omega_3)t\right), & P \rightarrow \infty^+ \\ O(\lambda^{-1}) \exp\left(+\frac{i}{2}(\lambda + \Omega_3)t\right), & P \rightarrow \infty^- \end{cases}$$

$$(38) \quad \Psi^2(t, P) = \begin{cases} O(\lambda^{-1}) \exp\left(-\frac{i}{2}(\lambda + \Omega_3)t\right), & P \rightarrow \infty^+ \\ (1 + O(\lambda^{-1})) \exp\left(+\frac{i}{2}(\lambda + \Omega_3)t\right), & P \rightarrow \infty^- \end{cases}$$

where  $i = \sqrt{-1}$ . Moreover,  $\Psi^1(t, P)$  ( $\Psi^2(t, P)$ ) has exactly one zero on  $\check{C}_h$  and the refined asymptotic estimates of  $\Psi^1$  at  $\infty^-$  and of  $\Psi^2$  at  $\infty^+$  read

$$(39) \quad \Psi^1(t, P) = \left[ -\frac{\bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t)}{2\lambda} + O(\lambda^{-2}) \right] \exp\left(+\frac{i}{2}(\lambda + \Omega_3)t\right), \quad P \rightarrow \infty^-$$

$$(40) \quad \Psi^2(t, P) = \left[ +\frac{\epsilon\Omega_1(t) + \bar{\epsilon}\Omega_2(t)}{2\lambda} + O(\lambda^{-2}) \right] \exp\left(-\frac{i}{2}(\lambda + \Omega_3)t\right), \quad P \rightarrow \infty^+.$$

*Proof.* According to (32),  $(\Psi^1, \Psi^2) \in \text{Ker}(A - \mu I)$  and hence

$$(41) \quad \frac{\Psi^2(t, P)}{\Psi^1(t, P)} = \frac{\mu - \lambda^2 - (1 + m)\Omega_3\lambda + \Gamma_3(t)}{(\bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t))\lambda - \bar{\epsilon}\Gamma_1(t) + \epsilon\Gamma_2(t)}.$$

If  $P \rightarrow \infty^+$  then  $\mu - \lambda^2 - (1 + m)\Omega_3\lambda \sim O(1)$  and using (29), (31), (32) and (41) we compute

$$2i \frac{d}{dt} \ln \Psi^1(t, P) = \lambda + \Omega_3 + (\bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t)) \frac{\Psi^2(t, P)}{\Psi^1(t, P)} = \lambda + \Omega_3 + O(\lambda^{-1})$$

and hence

$$\Psi^1(t, P) = (1 + O(\lambda^{-1})) \exp\left(-\frac{i}{2}(\lambda + \Omega_3)t\right).$$

In a similar way if  $P \rightarrow \infty^-$  we obtain

$$\Psi^2(t, P) = (1 + O(\lambda^{-1})) \exp\left(+\frac{i}{2}(\lambda + \Omega_3)t\right).$$

To compute the remaining asymptotic estimates we use that if  $P \rightarrow \infty^-$  then

$$(42) \quad \frac{\Psi^1(t, P)}{\Psi^2(t, P)} = \frac{A_{12}(t, \lambda)}{\mu - A_{11}(t, \lambda)} = -\frac{\bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t)}{2\lambda} + O(\lambda^{-2})$$

and if  $P \rightarrow \infty^+$  then

$$(43) \quad \frac{\Psi^2(t, P)}{\Psi^1(t, P)} = \frac{A_{21}(t, \lambda)}{\mu - A_{22}(t, \lambda)} = \frac{\epsilon \Omega_1(t) + \bar{\epsilon} \Omega_2(t)}{2\lambda} + O(\lambda^{-2}).$$

To find the poles of  $\Psi(t, P)$  in  $P$  we note that according to the proof of Proposition 3.1 (and with the same notations) we have

$$(44) \quad \Psi(t, P) = \Phi(t, \lambda) \Psi(0, P), \quad \Phi(0, \lambda) = I_2.$$

If  $|t|$  is sufficiently small, the fundamental matrix  $\Phi(t, \lambda)$  has no poles and  $\det \Phi(t, \lambda) \neq 0$ . It follows that the poles of  $\Phi(t, \lambda)$  and  $\Phi(0, \lambda)$  coincide, and we can obtain them by solving the following quadratic equation

$$\det A(0, \lambda) = (A_{11}(0, \lambda) - A_{12}(0, \lambda))^2 = \mu^2$$

(see (29), (34)). One gets two time independent poles  $P_1, P_2 \in \check{C}_h$  of  $\Psi(t, P)$ .

Finally, the meromorphic one-form  $d \ln \Psi^1$  has a simple pole at  $\infty^-$  with residue  $+1$  and is holomorphic in a neighbourhood of  $\infty^+$ . On the other hand  $\Psi^1(t, P)$  has exactly two poles on  $\check{C}_h$  and hence it has one zero on  $\check{C}_h$ . The same arguments hold for  $\Psi^2(t, P)$ .  $\square$

Let  $A_1, A_2, B_1$  be a basis of  $H_1(\check{C}_h, \mathbf{Z})$  as shown in Figure 2 ( $A_1 \circ B_1 = 1$ ), and let  $\omega_1, \omega_2$  be a basis of  $H^0(C, \Omega^1(\infty^+ + \infty^-))$ , normalized by the conditions

$$\left( \int_{A_i} \omega_j \right)_{i,j=1,2} = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \end{pmatrix}.$$

We shall also suppose that  $\omega_1$  is a holomorphic form on the elliptic curve  $C_h$ . Define now the period matrix

$$\Pi = \begin{pmatrix} 2\pi i & 0 & \tau_1 \\ 0 & 2\pi i & \tau_2 \end{pmatrix},$$

where

$$\tau_1 = \int_{B_1} \omega_1, \quad \tau_2 = \int_{B_1} \omega_2, \quad \operatorname{Re}(\tau_1) < 0.$$

Recall that the generalized Jacobian  $J(C_h; \infty^\pm)$  of  $C_h$  relative to the modulus  $m = \infty^+ + \infty^-$  is identified with  $\mathbf{C}^2/\Lambda$  where  $\Lambda$  is the lattice in  $\mathbf{C}^2$  generated by the columns of  $\Pi$ . Let

$$\theta_{11}(z) = \theta_{11}(z \mid \tau_1) = \sum_{n=-\infty}^{\infty} \exp\left\{\frac{1}{2}\tau_1(n + \frac{1}{2})^2 + (z + \pi\sqrt{-1})(n + \frac{1}{2})\right\}, \quad z \in \mathbf{C}$$

be the Jacobi theta function with characteristics  $[\frac{1}{2}, \frac{1}{2}]$ ,

$$\theta_{11}(0) = 0, \quad \theta_{11}(z + 2\pi i) = -\theta_{11}(z), \quad \theta_{11}(z + \tau_1) = -\exp(-z - \frac{1}{2}\tau_1) \theta_{11}(z).$$

Denote by  $\Omega$  the unique Abelian differential of second kind on  $C_h$  with poles at  $\infty^\pm$ , principal parts  $\pm\frac{i}{2}d\lambda$  where  $P = (\lambda, \mu)$ ,  $i = \sqrt{-1}$ , and normalized by  $\int_{A_1} \Omega = 0$ . Let  $P_0 \in \check{C}_h$  be a fixed initial point,  $c^\pm$ ,  $U$  be the constants defined by

$$(45) \quad \int_{P_0}^P \Omega = \begin{cases} -\frac{i}{2}\lambda + c^- + 0(\lambda^{-1}), & P \rightarrow \infty^+ \\ +\frac{i}{2}\lambda + c^+ + 0(\lambda^{-1}), & P \rightarrow \infty^- \end{cases}, \quad \int_{B_1} \Omega = U.$$

Define the Abel-Jacobi map

$$\mathcal{A}: \text{Div}^0(C_h) \rightarrow J(C_h) : \sum P_i - \sum Q_i \mapsto \int_{\Sigma} \sum_{Q_i} P_i \omega_1.$$

Here, and henceforth, we make the convention that the paths of integration between divisors are taken within  $C_h$  cut along its homology basis  $A_1, B_1$ , which we assume does not contain points of these divisors.

PROPOSITION 3.3. *The Baker-Akhiezer function is explicitly given by*

$$(46) \quad \Psi^1(t, P) = \text{const}_1 \cdot \exp \left[ t \left( \int_{P_0}^P \Omega - c^- - \frac{i}{2} \Omega_3 \right) \right] \frac{\theta_{11}(\mathcal{A}(P + \infty^- - P_1 - P_2) + tU)}{\theta_{11}(\mathcal{A}(\infty^+ + \infty^- - P_1 - P_2) + tU)}$$

$$(47) \quad \Psi^2(t, P) = \text{const}_2 \cdot \exp \left[ t \left( \int_{P_0}^P \Omega - c^+ + \frac{i}{2} \Omega_3 \right) \right] \frac{\theta_{11}(\mathcal{A}(P + \infty^+ - P_1 - P_2) + tU)}{\theta_{11}(\mathcal{A}(\infty^+ + \infty^- - P_1 - P_2) + tU)}$$

where

$$\text{const}_1 = \frac{\theta_{11}(\mathcal{A}(P - \infty^-))}{\theta_{11}(\mathcal{A}(\infty^+ - \infty^-))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_1))}{\theta_{11}(\mathcal{A}(P - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}{\theta_{11}(\mathcal{A}(P - P_2))}$$

$$\text{const}_2 = \frac{\theta_{11}(\mathcal{A}(P - \infty^+))}{\theta_{11}(\mathcal{A}(\infty^- - \infty^+))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_1))}{\theta_{11}(\mathcal{A}(P - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_2))}{\theta_{11}(\mathcal{A}(P - P_2))}$$

and  $P_1, P_2$  are the poles of  $\Psi$ .

The proof of the above proposition is based on a general fact: the properties of  $\Psi$  enumerated in Proposition 3.2 define it uniquely. Indeed, if  $\Psi$  and  $\tilde{\Psi}$  are vector functions both satisfying the assumptions of Proposition 3.2, then the functions  $\Psi^1$  and  $\tilde{\Psi}^1$  (resp.  $\Psi^2$  and  $\tilde{\Psi}^2$ ) meromorphic on  $C_h$  have the same poles. Using this and the asymptotic estimates at infinity we conclude that  $\Psi^1/\tilde{\Psi}^1$  and  $\Psi^2/\tilde{\Psi}^2$  are meromorphic functions on  $C_h$  which have one pole (at  $\tilde{\Psi}^i = 0$ ). Moreover

$$\Psi_1(t, \infty^-)/\tilde{\Psi}_1(t, \infty^-) = 1, \quad \Psi_2(t, \infty^-)/\tilde{\Psi}_2(t, \infty^-) = 1$$

and hence  $\Psi = \tilde{\Psi}$ . Finally, the reader may check that the functions (46) and (47) have the analyticity properties from Proposition 3.2 and hence they coincide with the Baker-Akhiezer function defined in Proposition 3.1.  $\square$

### 3.2 SOLUTIONS OF THE LAGRANGE TOP

Let  $z = (z_1, z_2) \in J(C_h; \infty^\pm)$ . It is easy to check that the functions

$$\frac{\theta_{11}(z_1 \pm \tau_2)}{\theta_{11}(z_1)} e^{\mp z_2}$$

live on  $J(C_h; \infty^\pm)$ . We shall see that they give solutions of the Lagrange top. By (16) we compute that  $\frac{d}{dt}z = \text{constant}$ , where

$$\begin{aligned} \frac{dz}{dt} &= \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 2\pi i \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} & \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda d\lambda}{\mu} & \int_{A_2} \frac{\lambda d\lambda}{\mu} \end{pmatrix}^{-1} \begin{pmatrix} -i \\ -ai \end{pmatrix}, \\ &\int_{A_2} \frac{d\lambda}{\mu} = 0, \quad \int_{A_2} \frac{\lambda d\lambda}{\mu} = -2\pi i \end{aligned}$$

so

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \begin{pmatrix} 2\pi \\ -i \int_{A_1} \frac{\lambda d\lambda}{\mu} + ai \int_{A_1} \frac{d\lambda}{\mu} \end{pmatrix}, \quad a = -m\Omega_3.$$

**THEOREM 3.4.** *The following equations hold*

$$(48) \quad \bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t) = \text{const}_3 \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} e^{-z_2},$$

$$(49) \quad \epsilon\Omega_1(t) + \bar{\epsilon}\Omega_2(t) = \text{const}_4 \frac{\theta_{11}(z_1 + \tau_2)}{\theta_{11}(z_1)} e^{+z_2},$$

where

$$(50) \quad \begin{aligned} z_2 &= tV_2, \quad z_1 = tV_1 + \mathcal{A}(\infty^+ + \infty^- - P_1 - P_2), \\ \tau_2 &= \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2 \end{aligned}$$

and

$$\text{const}_3 = \frac{2iV_1\theta'_{11}(0)}{\theta_{11}(\mathcal{A}(\infty^- - \infty^+))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_1))}{\theta_{11}(\mathcal{A}(\infty^- - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}{\theta_{11}(\mathcal{A}(\infty^- - P_2))},$$

$$\text{const}_4 = \frac{2iV_1\theta'_{11}(0)}{\theta_{11}(\mathcal{A}(\infty^+ - \infty^-))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_1))}{\theta_{11}(\mathcal{A}(\infty^+ - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_2))}{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}.$$

Let us denote

$$\begin{aligned}\omega_1 &= \pm(\omega_1^0 + O(\lambda^{-1})) d(\lambda^{-1}), & P = (\lambda, \mu) \rightarrow \infty^\pm, \\ \omega_2 &= \pm(\omega_2^1 \lambda + \omega_2^0 + O(\lambda^{-1})) d(\lambda^{-1}), & P = (\lambda, \mu) \rightarrow \infty^\pm.\end{aligned}$$

To prove Theorem 3.4 we shall need the following

LEMMA 3.5. *The above defined differentials are such that*

$$\begin{aligned}\omega_1^0 &= -i \int_{B_1} \Omega = -iV_1, & \omega_2^0 = i(c^+ - c^-), \\ V_2 &= -c^+ + c^- + i\Omega_3, & \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2.\end{aligned}$$

*Proof.* The identity  $\omega_1^0 = -i \int_{B_1} \Omega$  is a reciprocity law between the differential of the first kind  $\omega_1$  and the differential of the second kind  $\Omega$  [13]. It is obtained by integrating  $\pi(P)\omega_1$ , where  $\pi(P) = \int_{P_0}^P \Omega$ , along the border of  $C_h$  cut along its homology basis  $A_1, B_1$ . On the other hand

$$\omega_1 = 2\pi i \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \frac{d\lambda}{\mu}$$

and hence

$$\omega_1^0 = -2\pi i \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} = -iV_1.$$

Similarly the identity  $\omega_2^0 = i(c^+ - c^-)$  is a reciprocity law between the differential of the third kind  $\omega_2$  and the differential of the second kind  $\Omega$ , and  $\mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2$  is a reciprocity law between the differential of the third kind  $\omega_2$  and the differential of the first kind  $\omega_1$ . Finally, as

$$\omega_2 = \frac{\int_{A_1} \frac{\lambda d\lambda}{\mu}}{\int_{A_1} \frac{d\lambda}{\mu}} \frac{d\lambda}{\mu} - \frac{\lambda d\lambda}{\mu} \text{ we have } \omega_2^0 = -\frac{\int_{A_1} \frac{\lambda d\lambda}{\mu}}{\int_{A_1} \frac{d\lambda}{\mu}} - (1+m)\Omega_3 = -iV_1 - \Omega_3$$

and hence  $V_2 = -c^+ + c^- + i\Omega_3$ .  $\square$

*Proof of Theorem 3.4.* According to (42), (43)

$$\bar{\epsilon} \Omega_1(t) + \epsilon \Omega_2(t) = -2 \lim_{P \rightarrow \infty^-} \frac{\lambda \Psi^1(t, P)}{\Psi^2(t, P)}$$

and

$$\epsilon \Omega_1(t) + \bar{\epsilon} \Omega_2(t) = +2 \lim_{P \rightarrow \infty^+} \frac{\lambda \Psi^2(t, P)}{\Psi^1(t, P)}.$$

To compute the limit we use (46), (47) and

$$\lim_{P \rightarrow \infty^-} \lambda(P) \theta_{11}(\mathcal{A}(P - \infty^-)) = \theta'_{11}(0) \frac{d}{ds} \Big|_{s=0} \int^s \omega_1 = \omega_1^0 \theta'_{11}(0)$$

$$\lim_{P \rightarrow \infty^+} \lambda(P) \theta_{11}(\mathcal{A}(P - \infty^+)) = \theta'_{11}(0) \frac{d}{ds} \Big|_{s=0} \int^s \omega_1 = \omega_1^0 \theta'_{11}(0)$$

(see Lemma 3.5).  $\square$

### 3.3 EFFECTIVIZATION

Let  $\wp, \zeta, \sigma$  be the Weierstrass functions related to the elliptic curve  $\Gamma$  defined by

$$(51) \quad \eta^2 = 4\xi^3 - g_2\xi - g_3$$

(we use the standard notations of [4]).

Consider also the *real* elliptic curve  $C$  with affine equation

$$(52) \quad \mu^2 + \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

and natural anti-holomorphic involution  $(\lambda, \mu) \rightarrow (\bar{\lambda}, \bar{\mu})$ , and put

$$(53) \quad g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^4 - 4\frac{a_1}{4}\frac{a_3}{4}, \quad g_3 = \det \begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}.$$

It is well known that the curves  $C$  and  $\Gamma$  are isomorphic over  $\mathbf{C}$  and that under this isomorphism

$$(54) \quad \frac{d\lambda}{\mu} = \frac{d\xi}{\eta}.$$

Following Weil [25] we call  $\Gamma$  the Jacobian  $J(C)$  of the elliptic curve  $C$  and we write  $J(C) = \Gamma$ . Note that  $J(C)$  and  $\Gamma$  are real isomorphic and that  $J(C)$  and  $C$  are not real isomorphic.

Further we make the substitution (23) and  $C$  becomes the spectral curve  $\tilde{C}_h$  of Adler and van Moerbeke  $\{\mu^2 + f(\lambda) = 0\}$ , where

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + (2h_3 + m(m+1)h_4^2)\lambda^2 - 2h_2\lambda + 1$$

and  $\Gamma$  becomes the Lagrange curve  $\Gamma_h$ . Recall that, as we explained at the end of Section 2, the curve  $C_h$  with an equation  $\{\mu^2 = f(\lambda)\}$  and anti-holomorphic involution  $(\lambda, \mu) \rightarrow (\bar{\lambda}, -\bar{\mu})$ , is isomorphic over  $\mathbf{R}$  to  $\tilde{C}_h$ , so we write  $C_h = \tilde{C}_h$ . The Jacobian curve  $J(C_h) = \Gamma_h$  was computed by

Lagrange [17], while  $C_h$  appeared first in [1, 21] as a spectral curve of a Lax pair associated to the Lagrange top.

Recall that  $\sigma(z)$  is an entire function in  $z$  related to  $\zeta(z)$ ,  $\wp(z)$  and the already defined function  $\theta_{11}(z \mid \tau_1)$  on  $C_h$  as follows:

$$(55) \quad \begin{aligned} \zeta'(z) &= -\wp(z), \quad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z), \quad ' = \frac{d}{dz} \\ \sigma(z) &= \frac{\theta_{11}(zU)}{U\theta'_{11}(0)} \exp \left\{ \frac{z^2 U^2 \theta'''_{11}(0)}{6\theta'_{11}(0)} \right\} = z - \frac{g_2 z^5}{240} + \dots, \end{aligned}$$

where  $U$  is a constant depending on  $g_2$  and  $g_3$ . We shall also use the “addition formula”

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)} = \wp(v) - \wp(u).$$

To state our result let us introduce the notations

$$(56) \quad \begin{aligned} 2x_1 &= \epsilon \Omega_1 + \bar{\epsilon} \Omega_2, & 2x_2 &= \bar{\epsilon} \Omega_1 + \epsilon \Omega_2, & \epsilon^2 &= \sqrt{-1} \\ 2y_1 &= \epsilon^3 \Gamma_1 + \epsilon \Gamma_2, & 2y_2 &= \epsilon \Gamma_1 + \epsilon^3 \Gamma_2, & i^2 &= -1 \\ \rho_1 &= -im\Omega_3, & \rho_2 &= -i\Omega_3. \end{aligned}$$

The system (2) is equivalent to

$$(57) \quad \begin{aligned} \dot{x}_1 &= +\rho_1 x_1 - y_1, & \dot{y}_1 &= -\rho_2 y_1 + x_1 \Gamma_3 \\ \dot{x}_2 &= -\rho_1 x_2 + y_2, & \dot{y}_2 &= +\rho_2 y_2 - x_2 \Gamma_3 \\ \rho_1, \rho_2 &= \text{constants}, & \dot{\Gamma}_3 &= 2x_1 y_2 - 2x_2 y_1 \end{aligned}$$

with first integrals  $I_0 = 4x_1 x_2 - 2\Gamma_3$ ,  $I_1 = 4x_1 y_2 + 4x_2 y_1 - 2(\rho_1 + \rho_2)\Gamma_3$  and  $I_2 = \Gamma_3^2 - 4y_1 y_2$ .

**THEOREM 3.6.** *The general solution of the Lagrange top (2) can be written in the form*

$$\begin{aligned} x_1(t) &= -\frac{\sigma(t-k-l)}{\sigma(t)\sigma(k+l)} e^{at+b} & x_2(t) &= -\frac{\sigma(t+k+l)}{\sigma(t)\sigma(k+l)} e^{-at-b} \\ y_1(t) &= \frac{\sigma(t-k)\sigma(t-l)}{\sigma^2(t)\sigma(k)\sigma(l)} e^{at+b} & y_2(t) &= \frac{\sigma(t+k)\sigma(t+l)}{\sigma^2(t)\sigma(k)\sigma(l)} e^{-at-b} \\ \Gamma_3(t) &= \frac{\sigma(t+k)\sigma(t-k)}{\sigma^2(k)\sigma^2(t)} + \frac{\sigma(t+l)\sigma(t-l)}{\sigma^2(l)\sigma^2(t)} = -2\wp(t) + \wp(l) + \wp(k) \end{aligned}$$

$$\rho_1 = a - \zeta(l) - \zeta(k) \quad \rho_2 = -a - \zeta(k) - \zeta(l) + 2\zeta(k+l),$$

where  $g_2$ ,  $g_3$ ,  $a$ ,  $b$ ,  $k$ ,  $l$  are arbitrary constants subject to the relation  $g_2^3 - 27g_3^2 \neq 0$ .

REMARK. The non-general solutions of the Lagrange top are obtained from the above formulae by taking the limit  $g_2^3 - 27g_3^2 \rightarrow 0$ . The formulae for the position of the body in space, and in particular for  $\Gamma_3(t)$ ,  $y_1(t)$ ,  $y_2(t)$ , are due to Jacobi [15]. The expressions for  $x_1(t)$ ,  $x_2(t)$  were first deduced by Klein and Sommerfeld [16, p. 436]. Note however that in [16] the constant  $a$ , and hence the invariant level set on which the solution lives, is not arbitrary.

*Proof.* To make the solutions of the Lagrange top effective we use the following 4-dimensional Lie group of transformations preserving the system (57):

$$(58) \quad \begin{aligned} x_1 &\rightarrow Ux_1 e^{at+b}, & x_2 &\rightarrow Ux_2 e^{-at-b}, & t &\rightarrow \frac{t}{U} + T \\ y_1 &\rightarrow U^2 y_1 e^{at+b}, & y_2 &\rightarrow U^2 y_2 e^{-at-b}, & \Gamma_3 &\rightarrow U^2 \Gamma_3 \\ \rho_1 &\rightarrow U\rho_1 + a, & \rho_2 &\rightarrow U\rho_2 - a \end{aligned}$$

where  $U \neq 0$ ,  $T$ ,  $a$ ,  $b$  are constants.

The group (58) transforms  $x_1$  from (48) (see also (56), (55)), where  $z_1 = tU - TU$ ,  $z_1 - \tau_2 = (t - k - l)U$  as follows

$$x_1(t) = \text{const} \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} = - \frac{\sigma(t - k - l)}{\sigma(t) \sigma(k + l)} e^{at+b}.$$

(we used the fact that

$$\frac{\theta_{11}(z_1 - \tau_2) \sigma(t)}{\theta_{11}(z_1) \sigma(t - k - l)}$$

is a constant). The variable  $x_2$  is computed in the same way.

If we define the constant  $k$  by the condition  $y_1(t - k) = 0$ , then the first equation of (57) gives

$$\frac{y_1(t)}{x_1(t)} = \rho_1 - \frac{x'_1(t)}{x_1(t)} = \frac{\sigma(t - k) h(t)}{\sigma(t) \sigma(t - k - l)}$$

where  $h(t)$  is a meromorphic function on  $\mathbf{C}$ , such that  $y_1(t)/x_1(t)$  is single valued with poles at  $t = 0$  and  $t = k + l$ , and residues  $(-1)$  and  $(+1)$  respectively. These three conditions define  $h(t)$  uniquely:

$$h(t) = \frac{\sigma(t - l) \sigma(k + l)}{\sigma(k) \sigma(l)},$$

which implies the formula for  $y_1(t)$ . The expression for  $y_2(t)$  is obtained in the same way.

To deduce an expression for  $\Gamma_3(t)$  we use the fact that

$$\Gamma_3(t) = 2x_1x_2 - \frac{1}{2}I_0 = -2\wp(t) + 2\wp(k+l) - \frac{1}{2}I_0.$$

The value of  $I_0$  is easily computed by using the third equation of (57) and the formulae deduced for  $x_1, y_1$ . By substituting  $t = k$  we obtain

$$\Gamma_3(k) = \frac{\sigma(k-l)\sigma(k+l)}{\sigma^2(k)\sigma^2(l)} = \wp(l) - \wp(k)$$

and in a similar way  $\Gamma_3(l) = \wp(k) - \wp(l)$ . We conclude that

$$\Gamma_3(t) = -2\wp(t) + \wp(l) + \wp(k).$$

Finally, to compute  $\rho_1, \rho_2$  we shall use once again (57). As  $y_1(k) = 0$  we have

$$\begin{aligned} \rho_1 &= \frac{\dot{x}_1(k)}{x_1(k)} = \frac{d}{dt} \ln x_1(t) \Big|_{t=k} \\ &= \frac{d}{dt} \ln \sigma(t-k-l) \Big|_{t=k} - \frac{d}{dt} \ln \sigma(t) \Big|_{t=k} + a \\ &= a - \zeta(l) - \zeta(k). \end{aligned}$$

In a quite similar way we obtain

$$\rho_2 = -\frac{d}{dt} \ln y_1(t) \Big|_{t=k+l} = -a - \zeta(k) - \zeta(l) + 2\zeta(k+l).$$

Theorem 3.6 is proved.  $\square$

REMARK. If we impose the condition

$$\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = \Gamma_3^2 - 4y_1y_2 = 1,$$

then

$$\begin{aligned} &\left( \frac{\sigma(t+k)\sigma(t-k)}{\sigma^2(k)\sigma^2(t)} + \frac{\sigma(t+l)\sigma(t-l)}{\sigma^2(l)\sigma^2(t)} \right)^2 - \frac{\sigma(t-k)\sigma(t-l)}{\sigma^2(t)\sigma(k)\sigma(l)} \frac{\sigma(t+k)\sigma(t+l)}{\sigma^2(t)\sigma(k)\sigma(l)} \\ &= \left( \frac{\sigma(t+k)\sigma(t-k)}{\sigma^2(k)\sigma^2(t)} - \frac{\sigma(t+l)\sigma(t-l)}{\sigma^2(l)\sigma^2(t)} \right)^2 = (\wp(k) - \wp(l))^2 = 1 \end{aligned}$$

and hence  $\wp(k) - \wp(l) = \pm 1$ .