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THE COMPLEX GEOMETRY OF THE LAGRANGE TOP

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ABSTRACT. We prove that the heavy symmetric top (Lagrange, 1788) linearizes on a two-dimensional non-compact algebraic group – the generalized Jacobian of an elliptic curve with two points identified. This leads to a transparent description of its complex and real invariant level sets. We deduce, by making use of a Baker-Akhiezer function, simple explicit formulae for the general solution of the Lagrange top. Finally, we describe the two real structures of the Lagrange top and their relation with the focusing and the non-focusing non-linear Schrödinger equation.

CONTENTS

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1. Introduction

The motion under gravity of a rigid body one of whose points is fixed is described by a Hamiltonian system on the cotangent bundle $T^*\,\mathrm{SO}(3)$ of its configuration space SO(3), coordinatized by Euler angles and their conjugate momenta. This system was first obtained by Lagrange around 1788 [17], the particular case of free rigid body motion being already known to Euler. After a first reduction, with respect to rotations about the vertical in space, this leads to the following two degrees of freedom Hamiltonian system on T^*S^2, also obtained by Lagrange [17, p. 232 and p. 243]:

$$\frac{dM}{dt} = M \times \Omega + \chi \times \Gamma, \qquad \frac{d\Gamma}{dt} = \Gamma \times \Omega \tag{1}$$

$$M = (M_1, M_2, M_3),\ \ \Omega = (\Omega_1, \Omega_2, \Omega_3),\ \ \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3),\ \ \chi = (\chi_1, \chi_2, \chi_3).$$

Here M, Ω and Γ denote respectively the angular momentum, the angular velocity and the coordinates of the unit vector in the direction of gravity, all expressed in body-coordinates. The constant vector χ is the center of mass in body-coordinates multiplied by the mass of the body and the acceleration. We recall that $M = I\Omega$ where I is the matrix of the inertia operator and we may suppose that $I = \mathrm{diag}\,(I_1, I_2, I_3)$. The system (1) may be viewed as a two degrees of freedom Hamiltonian system on the manifold $\mathfrak{se}^*(3) \sim \mathfrak{se}(3)$ – the Lie algebra of the Euclidean group of three space $\mathrm{SE}(3) = \mathrm{SO}(3) \times \mathbf{R}^3$. Indeed, $\mathfrak{se}^*(3)$ with its usual Kostant-Kirillov-Poisson structure may be identified, via (a multiple of) the Killing form, with $\mathfrak{se}(3)$. This induces the following Lie-Poisson bracket on $\mathfrak{se}(3) \sim \mathbf{R}^3 \times \mathbf{R}^3$

$$\{M_1, M_2\} = -M_3\,, \ldots, \qquad \{M_1, \Gamma_2\} = -\Gamma_3\,, \ldots, \qquad \{\Gamma_i, \Gamma_j\} = 0$$

with coadjoint orbits

$$\mathcal{M}_a = \left\{ (M, \Gamma) \in \mathbf{R}^6 \,:\, \langle \Gamma, \Gamma \rangle = 1,\ \langle \Gamma, M \rangle = a \right\},$$

and on each symplectic leaf (1) is Hamiltonian with Hamiltonian function the energy of the body (see [21])

$$E = \tfrac{1}{2}\langle \Omega, M \rangle - \langle \chi, \Gamma \rangle\,.$$

Further we shall be interested in the case when the body is symmetric about an axis through the center of gravity and the fixed point – the so-called Lagrange top [17, p. 253]. This is equivalent to the conditions $I_1 = I_2$ and $\chi = (0, 0, \chi_3)$. Without loss of generality we may also suppose that $\chi_3 / I_1 = 1$, and if we put $m = (I_3 - I_2)/I_1$ then (1) takes the form

$$(2) \qquad \begin{aligned} \dot\Omega_1 &= -m\,\Omega_2\Omega_3 - \Gamma_2 & \dot\Gamma_1 &= \Gamma_2\Omega_3 - \Gamma_3\Omega_2 \\ \dot\Omega_2 &= m\,\Omega_3\Omega_1 + \Gamma_1 & \dot\Gamma_2 &= \Gamma_3\Omega_1 - \Gamma_1\Omega_3 \\ \dot\Omega_3 &= 0 & \dot\Gamma_3 &= \Gamma_1\Omega_2 - \Gamma_2\Omega_1 \end{aligned}$$

with first integrals

$$\begin{aligned} H_1 &= \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 \\ H_2 &= \Omega_1\Gamma_1 + \Omega_2\Gamma_2 + (1+m)\Omega_3\Gamma_3 \\ E = H_3 &= \tfrac{1}{2}\left(\Omega_1^2 + \Omega_2^2 + (1+m)\Omega_3^2\right) - \Gamma_3 \,. \end{aligned}$$

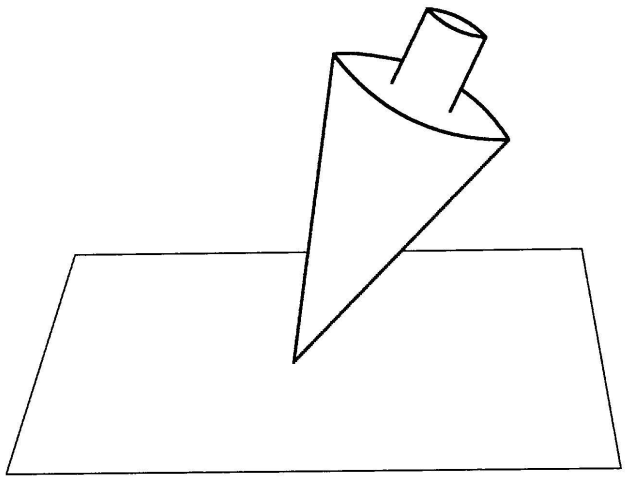

FIGURE 1
The Lagrange top

Due to the symmetry of the body there is an additional integral of motion,

$$H_4 = \Omega_3 \,,$$

which makes (2) Liouville integrable on the symplectic leaf

$$\mathcal{M}_a = \left\{(\Omega,\Gamma) \in \mathbf{R}^6 \,:\, \Gamma_1^2+\Gamma_2^2+\Gamma_3^2 = 1\,,\; \Omega_1\Gamma_1+\Omega_2\Gamma_2+(1+m)\Omega_3\Gamma_3 = a\right\}.$$

The Hamiltonian vector field generated by H_4 on $\mathcal{M}_a$ is given by

$$(3) \qquad \begin{aligned} \dot\Omega_1 &= \Omega_2 & \dot\Gamma_1 &= \Gamma_2 \\ \dot\Omega_2 &= -\Omega_1 & \dot\Gamma_2 &= -\Gamma_1 \\ \dot\Omega_3 &= 0 & \dot\Gamma_3 &= 0 \end{aligned}$$

and it represents uniform rotations about the symmetry axis through the center of gravity and the fixed point in space.

The Lagrange top is one of the most classical examples of integrable systems and it appears in almost all papers on this subject. The explicit formulae for the position of the body in space ($\Gamma_1, \Gamma_2, \Gamma_3$ in our case) were found by Jacobi [15, p. 503–505]. In the last twenty years most of the integrable problems of classical mechanics were revisited by making use of algebro-geometric techniques. From this point of view the Lagrange top takes a somewhat singular place – the results available are either incomplete, or inexact, or even wrong. Consider the complexified group of rotations $\mathbf{C}^* \sim \mathbf{C}/2\pi i\mathbf{Z}$ defined by the flow of the vector field (3). It acts freely on the generic complex invariant level set

$$T_h = \left\{(\Omega, \Gamma) \in \mathbf{C}^6 \, : \, H_1(\Omega,\Gamma)=1, H_2(\Omega,\Gamma)=h_2, H_3(\Omega,\Gamma)=h_3, H_4(\Omega,\Gamma)=h_4\right\}$$

and it is classically known that the quotient manifold $T_h/\mathbf{C}^*$ is an elliptic curve. The starting point of the present article is the observation that, generically, the algebraic manifold T_h *is not isomorphic* to a direct product of the curve $T_h/\mathbf{C}^*$ and $\mathbf{C}^*$ (although as a topological manifold it is). Let us explain first the algebraic structure of the invariant level set T_h. If $\Lambda \subset \mathbf{C}^2$ is a rank three lattice

$$\Lambda = \mathbf{Z}\begin{pmatrix} 2\pi i \\ 0 \end{pmatrix} \oplus \mathbf{Z}\begin{pmatrix} 0 \\ 2\pi i \end{pmatrix} \oplus \mathbf{Z}\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \qquad \mathrm{Re}(\tau_1) < 0 \tag{4}$$

then $\mathbf{C}^2/\Lambda$ is a non-compact *algebraic group* and it can be considered as a (non-trivial) extension of the elliptic curve $\mathbf{C}/\{2\pi i\mathbf{Z} \oplus \tau_1\mathbf{Z}\}$ by $\mathbf{C}^* \sim \mathbf{C}/2\pi i\mathbf{Z}$:

$$0 \to \mathbf{C}/2\pi i\mathbf{Z} \to \mathbf{C}^2/\Lambda \xrightarrow{\phi} \mathbf{C}/\left\{2\pi i\mathbf{Z} \oplus \tau_1\mathbf{Z}\right\} \to 0, \qquad \phi(z_1, z_2) = z_1 . \tag{5}$$

We prove that, for generic h_i, the complex invariant level set T_h of the Lagrange top is biholomorphic to (an affine part of) $\mathbf{C}^2/\Lambda$. The algebraic group $\mathbf{C}^2/\Lambda$ turns out to be the generalized Jacobian of an elliptic curve with two points identified. This curve, say C, is the spectral curve of a Lax pair for the Lagrange top, found first by Adler and van Moerbeke [1] and its Jacobian $\mathrm{Jac}(C) = \mathbf{C}/\{2\pi i\mathbf{Z} \oplus \tau_1\mathbf{Z}\}$ is a curve found first by ... Lagrange. Further we prove that the flows (2), (3) define translation invariant vector fields on $\mathbf{C}^2/\Lambda$ which means that our system is algebraically completely integrable.

Let us compare the above to the classical Lagrange linearization on an elliptic curve [17] (see also [1, 21, 24, 3, 2]). It is well known that, due to the symmetry of the body, the system (2) is invariant under rotations about the axis of symmetry. These rotations are given by the flow of (3) which commutes with the flow of the Lagrange top. Thus we have a well defined $\mathbf{C}^*$ action on the complex invariant level set $T_h \sim \mathbf{C}^2/\Lambda$ and a well defined (factored) flow on $T_h/\mathbf{C}^*$. Lagrange noted around 1788 that this factorization amounts to eliminating the variables $\Omega_1, \Omega_2, \Gamma_1, \Gamma_2$, so he obtained a single autonomous differential equation for the nutation θ, where $\Gamma_3 = \cos\theta$ [17, p. 254] (nutation is the inclination of the symmetry axis of the body to the vertical). Finally it is seen from this equation that $\Gamma_3(t)$ is, up to an addition and a multiplication by a constant, the Weierstrass elliptic function $\wp(t)$. Thus Lagrange linearized the complex flow of the Lagrange top on an elliptic curve. This curve happens to be the Jacobian $J(C)$ of the spectral curve C of Adler and van Moerbeke and is identified with $\mathbf{C}/\{2\pi i\mathbf{Z}\oplus\tau_1\mathbf{Z}\}$ in (5). The kernel of the map ϕ is just the circle action $\mathbf{C}^* \sim \mathbf{C}/2\pi i\mathbf{Z}$ defined by (3), so the linear vector field (3) is projected under ϕ onto the zero vector field on $\mathrm{Jac}(C) = \mathbf{C}/\{2\pi i\mathbf{Z}\oplus\tau_1\mathbf{Z}\}$.

To summarize in modern language, Lagrange's computation shows that *the generic invariant level set T_h of the Lagrange top is an extension of an elliptic curve $C \sim \mathrm{Jac}(C)$ by $\mathbf{C}^*$ and the flow is projected on this curve into a well defined linear flow*. This is, however, a very vague description of $T_h \sim \mathbf{C}^2/\Lambda$. Indeed, although the fibration

$$\mathbf{C}^2/\Lambda \xrightarrow{\phi} \mathrm{Jac}(C) = \mathbf{C}/\{2\pi i\mathbf{Z}\oplus\tau_1\mathbf{Z}\} \tag{6}$$

is topologically trivial, it is not algebraically trivial, and to know its *type* we need the parameter τ_2 defined in (4) (cf. [23]). As the general solution of (2) lives on $\mathbf{C}^2/\Lambda$ then, contrary to what is often asserted, it cannot be expressed in terms of elliptic functions and exponentials. It is even less true that "the flow of the Lagrange top lives on a complex 2-dimensional cylinder with generator the line $z = 0$" as claimed in [21, p. 232].

The algebraic description of the Lagrange top is carried out in Section 2 (Theorem 2.2). The Lax pair is used first in Section 3 where we construct the corresponding Baker-Akhiezer function. This implies explicit formulae for the general solution of the Lagrange top which complete and simplify the classical formulae due to Jacobi [15, p. 503–505] for $\Gamma_1, \Gamma_2, \Gamma_3$ and Klein and Sommerfeld [16, p. 436] for the angular velocities (Theorem 3.6).

In Section 4 we study reality conditions on the (complex) solutions. Besides the usual real structure of the Lagrange top given by complex conjugation there is a second natural real structure induced by the eigenvalue map of the corresponding Lax pair representation. It turns out that these two structures coincide on $\mathrm{Jac}(C)$ but are different on $\mathbf{C}^2/\Lambda$ (and hence on T_h). The corresponding real level sets are described in Theorem 4.2. This makes clear the relation between the real structure of the curve C, its Jacobian $\mathrm{Jac}(C)$ and the real level set $T_h^{\mathbf{R}}$ (a question raised in [2] and [3, p. 37]).

The results obtained in the present paper lead to the following unexpected observation: the real solutions of the Lagrange top corresponding to its two real structures provide one-gap solutions of the nonlinear Schrödinger equation (Proposition 5.1)

$$(NLS^{\pm}) \qquad u_{xx} = iu_t \pm 2|u|^2 u\,.$$

Finally, for the convenience of the reader, we give in the Appendix a brief account of some more or less well known results concerning the linearization of the Lagrange top on an elliptic curve.

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2. ALGEBRAIC STRUCTURE

Let $\check{C}$ be the affine curve $\{\mu^2 = f(\lambda)\}$ where f is a degree 4 polynomial without double roots. We denote by C the completed and normalized curve $\check{C}$. Thus C is a compact Riemann surface, such that $C = \check{C} \cup \infty^+ \cup \infty^-$, where $\infty^{\pm}$ are two distinct "infinite" points on C. Consider the effective divisor $m = \infty^+ + \infty^-$ on C and let $J_m(C)$ be the generalized Jacobian of the elliptic curve C relative to m. Following [23] we shall call m a modulus. We shall denote also $J(C;\infty^{\pm}) = J_m(C)$. Recall that the usual Jacobian

$$J(C) = \mathrm{Div}^0(C) \big/ \sim$$

is the additive group $\mathrm{Div}^0(C)$ of degree zero divisors on C modulo the equivalence relation $\sim$. We have $D_1 \sim D_2$ if and only if there exists a meromorphic function f on C such that $(f) = D_1 - D_2$. Similarly the generalized Jacobian

$$J(C;\infty^{\pm}) = \mathrm{Div}^0(\check{C}) \big/ \overset{m}{\sim}$$

is the additive group $\mathrm{Div}^0(\check{C})$ of degree zero divisors on $\check{C}$ modulo the equivalence relation $\overset{m}{\sim}$. We have $D_1 \overset{m}{\sim} D_2$ if and only if there exists a meromorphic function f on C such that $f(\infty^+) = f(\infty^-) = 1$ and $(f) = D_1 - D_2$. The generalized Jacobian $J(C;\infty^{\pm})$ is thus obtained as a $\mathbf{C}^*$-extension of the usual Jacobian $J(C)$ (isomorphic to C). This means that there is an exact sequence of groups

$$(7) \qquad 0 \xrightarrow{\exp} \mathbf{C}^* \xrightarrow{v} J(C;\infty^{\pm}) \xrightarrow{\phi} J(C) \to 0\,.$$

The map ϕ is induced by the inclusion $\check{C} \subset C$ and $v(r) \in J(C;\infty^{\pm})$, $r \neq 0$, is the divisor of any meromorphic function f on C satisfying $f(\infty^+)/f(\infty^-) = r$ [10, p. 55].

As an analytic manifold $J(C;\infty^{\pm})$ is

$$\mathbf{C}^2/\Lambda \sim H^0\big(C,\Omega^1(\infty^+ + \infty^-)\big)^* / H_1(\check{C},\mathbf{Z}) \tag{8}$$

where the lattice Λ is generated by the three vectors

$$\Lambda_1 = \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda\, d\lambda}{\mu} \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_2} \frac{\lambda\, d\lambda}{\mu} \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} \int_{B_1} \frac{d\lambda}{\mu} \\ \int_{B_1} \frac{\lambda\, d\lambda}{\mu} \end{pmatrix} \tag{9}$$

and the cycles A_1, A_2, B_1 form a basis of the first homology group $H_1(\check{C},\mathbf{Z})$ as in Figure 2. It is seen that the period lattice Λ may be obtained by pinching a non-zero homology cycle of a genus two Riemann surface to a point $\infty^{\pm}$ (Figure 2). This is expressed by saying that $J(C;\infty^{\pm})$ is the Jacobian of the elliptic curve C with two points ∞^+ and ∞^- identified [10].

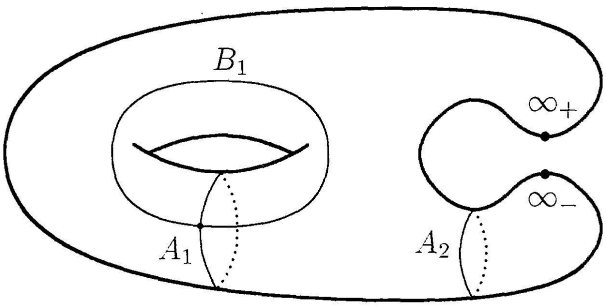

FIGURE 2
The canonical homology basis of the affine curve C

For further use, note also that

$$\phi\colon J(C;\infty^{\pm}) \to J(C), \quad \phi\colon \mathbf{C}^2/\Lambda \to \mathbf{C}/\phi(\Lambda) \tag{10}$$

is just the first projection $\phi(z_1, z_2) = z_1$. As

$$\phi(\Lambda_2) = \int_{A_2} \frac{d\lambda}{\mu} = 0,$$

$\phi(\Lambda)$ is generated by $\phi(\Lambda_1)$ and $\phi(\Lambda_3)$, and

$$\operatorname{Ker} \phi = \mathbf{C}\Big/\Big\{\mathbf{Z}\int_{A_2} \frac{\lambda\, d\lambda}{\mu}\Big\} \sim \mathbf{C}^*.$$

As an analytic manifold the usual Jacobian $J(C)$ is

$$\mathbf{C}/\phi(\Lambda) \sim H^0(C,\Omega^1)^* / H_1(C,\mathbf{Z}).$$

In contrast to the usual Jacobian $J(C)$, the generalized Jacobian $\mathbf{C}^2/\Lambda$ is a *non-compact* algebraic group. For any $p \in J(C)$ define also the divisor $D_p = \phi^{-1}(p) \subset J(C;\infty^{\pm})$.

An explicit embedding of a Zariski open subset of $J(C;\infty^{\pm})$ in $\mathbf{C}^6$ is constructed by the following classical construction due to Jacobi (see Mumford [18]). Let

$$f(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 \tag{11}$$

be a polynomial without double roots and define the polynomials

$$U(\lambda) = \lambda^2 + u_1\lambda + u_2, \qquad V(\lambda) = v_1\lambda + v_2, \qquad W(\lambda) = \lambda^2 + w_1\lambda + w_2. \tag{12}$$

Let T_C be the set of Jacobi polynomials (12) satisfying the relation

$$f(\lambda) - V^2(\lambda) = U(\lambda)W(\lambda). \tag{13}$$

More explicitly, let us expand

$$f - V^2 - UW = \sum_{i=0}^{3} b_i(u_1, u_2, v_1, v_2, w_1, w_2)\,\lambda^i,$$

$$b_3 = a_1 - u_1 - w_1, \qquad b_2 = a_2 - u_2 - w_2 - u_1w_1 - v_1^2,$$

$$b_1 = a_3 - u_1w_2 - u_2w_1 - 2v_1v_2, \qquad b_0 = a_4 - u_2w_2 - v_2^2.$$

If we take u_i, v_j, w_k as coordinates in $\mathbf{C}^6$ then T_C is just the zero locus $V(b_0, b_1, b_2, b_3)$ as a subset of $\mathbf{C}^6$

$$T_C = \big\{(u, v, w) \in \mathbf{C}^6 : u_1 + w_1 = a_1, \quad u_2 + w_2 + u_1w_1 + v_1^2 = a_2,$$
$$u_1w_2 + u_2w_1 + 2v_1v_2 = a_3, \quad u_2w_2 + v_2^2 = a_4\big\}.$$

PROPOSITION 2.1. *If $f(\lambda)$ is a polynomial without double roots, then*

(i) *T_C is a smooth affine variety isomorphic to $J(C;\infty^{\pm}) \setminus D_p$ for some $p \in J(C)$;*

(ii) *any translation invariant vector field on the generalized Jacobian $J(C;\infty^{\pm})$ of the curve C can be written (up to multiplication by a non-zero constant) in the following Lax pair form*

$$2\sqrt{-1}\,\frac{d}{dt}A(\lambda) = \left[A(\lambda), \frac{A(a)}{\lambda - a}\right] \tag{14}$$

where

$$A(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \tag{15}$$

$a \in \mathbf{C}$, and U, V, W are the Jacobi polynomials (12).

Equivalently, if $D = P_1 + P_2 \in \mathrm{Div}^2(\check{C})$, where $P_i = (\lambda_i, \mu_i)$, $i = 1, 2$, then (14) *can be written as*

$$
(16) \qquad \begin{aligned}
\frac{d\lambda_1}{\sqrt{f(\lambda_1)}} + \frac{d\lambda_2}{\sqrt{f(\lambda_2)}} &= -\sqrt{-1}\,dt \\
\frac{\lambda_1 d\lambda_1}{\sqrt{f(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{f(\lambda_2)}} &= -a\sqrt{-1}\,dt\,.
\end{aligned}
$$

REMARK. Note that $a = \infty$ also makes sense. The corresponding vector field is obtained by changing the time as $t \to t/a$ and letting $a \to \infty$. Thus (14) becomes

$$
(17) \qquad 2\sqrt{-1}\,\frac{d}{dt}A(\lambda) = \left[A(\lambda)\,,\,A_\infty\right], \qquad A_\infty = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
$$

and (16) becomes

$$
(18) \qquad \begin{aligned}
\frac{d\lambda_1}{\sqrt{f(\lambda_1)}} + \frac{d\lambda_1}{\sqrt{f(\lambda_1)}} &= 0 \\
\frac{\lambda_1 d\lambda_1}{\sqrt{f(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{f(\lambda_2)}} &= -\sqrt{-1}\,dt\,.
\end{aligned}
$$

The proof of part (i) of the above proposition can be found in Previato [20] (see also Mumford [18]). It is also proved there that a translation invariant vector field $\frac{d}{d\epsilon}$ on the generalized Jacobian $J(C; \infty^\pm)$ which is induced by the tangent vector

$$
(19) \qquad \frac{d}{d\epsilon}\lambda\Big|_{\lambda=a} = \sqrt{f(a)}
$$

on C via the Abel map $C \to J(C; \infty^\pm)$, can be written as

$$
(20) \qquad \frac{d}{d\epsilon}U(\lambda) = \frac{V(a)U(\lambda) - U(a)V(\lambda)}{\lambda - a}
$$

$$
(21) \qquad \frac{d}{d\epsilon}W(\lambda) = -\frac{V(a)W(\lambda) - W(a)V(\lambda)}{\lambda - a}
$$

$$
(22) \qquad \frac{d}{d\epsilon}V(\lambda) = \frac{U(a)W(\lambda) - W(a)U(\lambda)}{2(\lambda - a)}\,.
$$

Our final remark is that the translation invariant vector fields (20), (21) and (22), which we denote further by $\frac{d}{dt}$, can be written in the following Lax pair form (suggested by Beauville [6, Example 1.5]):

$$-2\frac{d}{dt}A(\lambda) = \left[A(\lambda), \frac{A(a)}{\lambda - a}\right],$$

where

$$A(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}.$$

By (19) the direction of the constant tangent vector computed above is

$$\left(\frac{\dot\lambda}{\sqrt{f(a)}}, \frac{a\dot\lambda}{\sqrt{f(a)}}\right) = (1, a),$$

which proves (16). This completes the proof of Proposition 2.1. □

Next we apply Proposition 2.1 to the Lagrange top (2). Let C_h be the curve C as above, where

$$\begin{aligned} a_1 &= 2(1+m)h_4, & a_2 &= 2h_3 + m(m+1)h_4^2, \\ a_3 &= -2h_2, & a_4 &= h_1 = 1, \end{aligned} \tag{23}$$

so

$$\check{C}_h = \left\{\mu^2 = \lambda^4 + 2(1+m)h_4\lambda^3 + \left(2h_3 + m(m+1)h_4^2\right)\lambda^2 - 2h_2\lambda + 1\right\}. \tag{24}$$

Consider the complex invariant level set of the Lagrange top (2)

$$T_h = \left\{(\Omega, \Gamma) \in \mathbf{C}^6 : H_1(\Omega,\Gamma) = 1,\ H_2(\Omega,\Gamma) = h_2,\ H_3(\Omega,\Gamma) = h_3,\ H_4(\Omega,\Gamma) = h_4\right\}$$

and the associated "bifurcation set"

$$\mathbf{B} = \left\{h \in \mathbf{C}^3 : \text{discriminant}\left(f(\lambda)\right) = 0\right\}.$$

It is a straightforward computation to check that the linear change of variables

$$\begin{aligned} u_1 &= (1+m)\Omega_3 - i\Omega_2 & u_2 &= -\Gamma_3 + i\Gamma_2 \\ w_1 &= (1+m)\Omega_3 + i\Omega_2 & w_2 &= -\Gamma_3 - i\Gamma_2 \\ v_1 &= \Omega_1 & v_2 &= -\Gamma_1 \end{aligned} \tag{25}$$

(with $i = \sqrt{-1}$) identifies T_C and T_h. Further, as

$$\left[A(\lambda), \frac{A(a)}{\lambda - a}\right] = \left[A(\lambda), \frac{A(a) - A(\lambda)}{\lambda - a}\right] = \left[A(\lambda), \begin{pmatrix} -v_1 & -a - u_1 - \lambda \\ -a - w_1 - \lambda & v_1 \end{pmatrix}\right],$$

the vector field (2) is obtained by substituting $a = -m\Omega_3$ in (14) and using the change of variables (25) (note that Ω_3 is a constant of motion). Similarly the vector field (3) is obtained by substituting $a = \infty$ (see the remark after Proposition 2.1).

To sum up, we have proved the following

THEOREM 2.2. *If* $h \notin \mathbf{B}$, *then*

(i) *the complex invariant level set* T_h *of the Lagrange top is a smooth complex manifold biholomorphic to* $J(C_h;\infty^{\pm})\backslash D_\infty$ *where* $D_\infty = \phi^{-1}(p)$ *for some* $p \in J(C_h)$ *and* $J(C_h;\infty^{\pm})$ *is the generalized Jacobian of the elliptic curve* C_h *with two points at "infinity" identified;*

(ii) *the Hamiltonian flows of the Lagrange top* (2), (3) *restricted to* T_h *induce linear flows on* $J(C_h;\infty^{\pm})$. *The corresponding vector fields* (2) *and* (3) *have a Lax pair representation obtained from the Lax pair* (14) *by substituting* $a = -m\Omega_3$ *and* $a = \infty$ *respectively, and using the change of variables* (25).

According to the above theorem the Lagrange top is an algebraically completely integrable system in the sense of Mumford [18, p. 353]. Clearly any linear flow on $J(C_h;\infty^{\pm})$ maps under ϕ (7) into a linear flow on the usual Jacobian $J(C_h)$. This is expressed by the fact that the variable Γ_3 which describes the nutation of the body is an elliptic function in time. It was known to Lagrange [17] who deduced the differential equation satisfied by Γ_3. The real version of Theorem 2.2 will be explained in Section 4.

To the end of this section we compare the Lax pair (14) and the Lax pair for the Lagrange top obtained earlier by Adler and van Moerbeke [1]. Namely, if we identify the Lie algebras $(\mathbf{R}^3, \wedge)$ and $\left(\mathfrak{so}(3), [\,.\,,\,.\,]\right)$ by the Lie algebra isomorphism

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix},$$

then (2) can be written in the following equivalent Lax pair form [1]

$$\frac{d}{dt}\left(\lambda^2\chi + \lambda M - \Gamma\right) = \left[\lambda^2\chi + \lambda M - \Gamma\,,\, \lambda\chi + \Omega\right], \tag{26}$$

where

$$\Omega = (\Omega_1, \Omega_2, \Omega_3),\ M = \left(\Omega_1, \Omega_2, (1+m)\Omega_3\right),\ \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3),\ \chi = (0,0,1).$$

The Lax pair representation of (3) is given by

$$\frac{d}{dt}\left(\lambda^2\chi + \lambda M - \Gamma\right) = \left[\lambda^2\chi + \lambda M - \Gamma\,,\, \chi\right]. \tag{27}$$

Both Lax pairs (26), (27) can be also written in the Beauville form

$$-\frac{d}{dt}A(\lambda) = \left[A(\lambda), \frac{A(a)}{\lambda - a}\right] \tag{28}$$

where $A(\lambda) = \lambda^2\chi + \lambda M - \Gamma$. Indeed,

$$\left[A(\lambda), \frac{A(a)}{\lambda - a}\right] = \left[A(\lambda), \frac{A(a) - A(\lambda)}{\lambda - a}\right] = -\left[A(\lambda), \lambda\chi + a\chi + M\right].$$

Now (26) is obtained by replacing as before $a = -m\Omega_3$, and (27) is obtained by letting $a \to \infty$.

Clearly the Lax pair (14) from Proposition 2.1 and (26), (28) are equivalent in the sense that they define one and the same vector field. We can identify them over $\mathbf{C}$ by making use of the isomorphism of the Lie algebras $\mathfrak{so}(3, \mathbf{C})$ and $\mathfrak{sl}(2, \mathbf{C})$ given by

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon x & \epsilon z + \bar{\epsilon} y \\ \epsilon z - \bar{\epsilon} y & -\epsilon x \end{pmatrix}, \qquad \epsilon = \exp\frac{\sqrt{-1}\pi}{4}.$$

Note, however, the following difference. The spectral curve of (26) is reducible

$$\det\left(\lambda^2\chi + \lambda M + \Gamma - \mu I\right) = -\mu\left(\mu^2 + f(\lambda)\right) = 0,$$

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + \left(2h_3 + m(m+1)h_4^2\right)\lambda^2 - 2h_2\lambda + 1,$$

but the spectral curve of (14) is not

$$\det\bigl(A(\lambda) - \mu I\bigr) = \mu^2 - V^2 - UW = \mu^2 - f(\lambda) = 0.$$

The last observation will be of some importance for the next section. Earlier Adler and van Moerbeke [1, p. 351] proposed to linearize the Lagrange top on an elliptic curve by introducing first a small parameter ϵ in the corresponding $\mathfrak{so}(3)$ Lax pair. The new system has the advantage of having an irreducible genus 4 spectral curve C_ϵ which fits the general theory, so we can just "take the limit" $\epsilon \to 0$. This computation, reproduced in [21] and used in [22], is however erroneous.

By abuse of notation we call the curve $\widetilde{C}_h = \{\mu^2 + f(\lambda) = 0\}$ with an antiholomorphic involution $(\lambda, \mu) \to (\bar{\lambda}, \bar{\mu})$, the *spectral curve* of the Lax pair (26). The curve $\widetilde{C}_h$ is real isomorphic to the curve $C_h = \left\{\mu^2 = f(\lambda)\right\}$, equipped with an antiholomorphic involution $(\lambda, \mu) \to (\bar{\lambda}, -\bar{\mu})$, so without loss of generality we shall write $\widetilde{C}_h = C_h$.

3. Explicit solutions

In this section we find explicit solutions for the Lagrange top (2). We compute first the Baker-Akhiezer function of the $\mathfrak{sl}(2,\mathbf{C})$ (or rather $\mathfrak{su}(2)$) Lax pair (14). This implies explicit formulae for the solutions of the Lagrange top in terms of exponentials and theta functions related to the spectral curve C_h (see for example Dubrovin [8], E. D. Belokolos, A. I. Bobenko, V. Z. Enol′skiĭ, A. R. Its, V. B. Matveev [5]).Then we note that the Jacobian $J(C_h)$ of C_h is just the Lagrange elliptic curve used in the classical theory which provides explicit solutions in terms of exponentials and sigma function related to $J(C_h)$.

By performing a unitary operation on the matrix (15) we may put its leading term in diagonal form. Substituting $a=-m\Omega_3$ in (14) and using the change of variables (25) we obtain the following Lax pair representation for the Lagrange top (2)

$$\text{(29)}\qquad \Big[A\,,\,B-2i\frac{d}{dt}\Big]=2i\frac{dA}{dt}+[A,B]=0\,,\qquad \epsilon^2=i\,,\quad i^2=-1$$

where

$$A=A(t,\lambda)=\begin{pmatrix}A_{11}(t,\lambda) & A_{12}(t,\lambda)\\ A_{21}(t,\lambda) & A_{22}(t,\lambda)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\lambda^2+$$

$$+\begin{pmatrix}(1+m)\Omega_3 & \bar\epsilon\,\Omega_1(t)+\epsilon\,\Omega_2(t)\\ \epsilon\,\Omega_1(t)+\bar\epsilon\,\Omega_2(t) & -(m+1)\Omega_3\end{pmatrix}\lambda-\begin{pmatrix}\Gamma_3 & \bar\epsilon\,\Gamma_1(t)+\epsilon\,\Gamma_2(t)\\ \epsilon\,\Gamma_1(t)+\bar\epsilon\,\Gamma_2(t) & -\Gamma_3\end{pmatrix}$$

and

$$B=B(t,\lambda)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\begin{pmatrix}\Omega_3 & \bar\epsilon\,\Omega_1(t)+\epsilon\,\Omega_2(t)\\ \epsilon\,\Omega_1(t)+\bar\epsilon\,\Omega_2(t) & -\Omega_3\end{pmatrix}.$$

The spectral curve of the above Lax representation is defined by

$$\check{C}_h=\left\{\det\big(A(\lambda)-\mu I\big)=\mu^2-f(\lambda)=0\right\}\,,$$

$$f(\lambda)=\lambda^4+2(1+m)\,h_4\lambda^3+\big(2h_3+m(m+1)\,h_4^2\big)\lambda^2-2h_2\lambda+1\,.$$

We shall also denote by C_h the Riemann surface of the compactified affine curve $\check{C}_h$. The reader may note the "similarity" between (29) and the Lax pair of the nonlinear Schrödinger equation (for a rigorous statement see Proposition 5.1).

3.1 The Baker-Akhiezer function

Let us fix a solution $A(t,\lambda)$ of (29) defined in a neighbourhood of $t=0\in\mathbf{C}$. We shall also suppose that the point $P=(\lambda,\mu)$ is such that $(1,-1)$ is not an eigenvector of the matrix $A(0,\lambda)$.

PROPOSITION 3.1. *For any $t \in \mathbf{C}$ in a sufficiently small neighbourhood of the origin, there exists a unique eigenfunction*

$$\Psi = \Psi(t, P) = \begin{pmatrix} \Psi^1(t, P) \\ \Psi^2(t, P) \end{pmatrix}, \qquad P = (\lambda, \mu) \in \check{C} \tag{30}$$

of $A(t, \lambda)$ (called the Baker-Akhiezer function) *satisfying the conditions*

$$2i\frac{d}{dt}\Psi(t, P) = B(t, \lambda)\Psi(t, P) \tag{31}$$

$$A(t, \lambda)\Psi(t, P) = \mu\Psi(t, P) \tag{32}$$

and normalized by

$$\Psi^1(0, P) + \Psi^2(0, P) = 1\,. \tag{33}$$

In terms of the coefficients $A_{ij}(t, \lambda)$ of the matrix $A = (A_{ij})$ we have

$$\Psi^1(0, P) = \frac{A_{12}(0, \lambda)}{A_{12}(0, \lambda) + \mu - A_{11}(0, \lambda)} = \frac{\mu - A_{22}(0, \lambda)}{A_{21}(0, \lambda) + \mu - A_{22}(0, \lambda)} \tag{34}$$

$$\Psi^2(0, P) = \frac{\mu - A_{11}(0, \lambda)}{A_{12}(0, \lambda) + \mu - A_{11}(0, \lambda)} = \frac{A_{21}(0, \lambda)}{A_{21}(0, \lambda) + \mu - A_{22}(0, \lambda)}\,. \tag{35}$$

Proof. Let $\Phi(t, \lambda)$ be a fundamental matrix for the operator $B(t, \lambda) - 2i\frac{d}{dt}$ normalized at $t = 0$. Then the general solution of (31) is written as

$$\Psi(t, P) = \Phi(t, \lambda)\Psi(0, P), \qquad \Phi(0, \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = (\lambda, \mu)\,. \tag{36}$$

As A and $B - 2i\frac{d}{dt}$ commute, we have

$$\left(B(t, \lambda) - 2i\frac{d}{dt}\right) A(t, \lambda)\Phi(t, \lambda) = A(t, \lambda)\left(B(t, \lambda) - 2i\frac{d}{dt}\right) \Phi(t, \lambda) = 0$$

and hence $A(t, \lambda)\Phi(t, \lambda) = \Phi(t, \lambda)M(P)$ for some constant matrix $M(P)$ computed by substituting $t = 0$. Thus $M(P) = A(0, \lambda)$ and

$$A(0, \lambda) = \Phi^{-1}(t, \lambda)A(t, \lambda)\,\Phi(t, \lambda)\,.$$

The constants $\Psi^1(0, P), \Psi^2(0, P)$ are uniquely defined by (32) and (33). Finally,

$$\begin{aligned} A(t, \lambda)\Psi(t, P) &= \Phi(t, \lambda)A(0, \lambda)\,\Phi^{-1}(t, \lambda)\,\Phi(t, \lambda)\,\Psi(0, P) \\ &= \Phi(t, \lambda) \cdot \mu \cdot \Psi(0, P) \\ &= \mu\Psi(t, P)\,. \end{aligned}$$

The formulae (34), (35) follow from (32), (33). □

Denote by ∞^+ (respectively ∞^-) the point on $C_h - \check{C}_h$ such that in its neighbourhood $\mu/\lambda^2 \sim +1$ (resp. (-1)).

PROPOSITION 3.2. *There exists $t_0 > 0$ such that for any fixed $t \in \mathbf{C}$, $|t| < t_0$, the Baker-Akhiezer vector-function $\Psi(t,P)$ is meromorphic in P on the affine curve $\check{C}_h$ and has two poles at $P_1, P_2 \in C_h$ which do not depend on t. In a neighbourhood of the two infinite points $\infty^{\pm}$ on C_h we have*

$$(37) \qquad \Psi^1(t,P) = \begin{cases} \left(1+O(\lambda^{-1})\right)\exp\left(-\frac{i}{2}(\lambda+\Omega_3)t\right), & P\to\infty^+ \\ O(\lambda^{-1})\exp\left(+\frac{i}{2}(\lambda+\Omega_3)t\right), & P\to\infty^- \end{cases}$$

$$(38) \qquad \Psi^2(t,P) = \begin{cases} O(\lambda^{-1})\exp\left(-\frac{i}{2}(\lambda+\Omega_3)t\right), & P\to\infty^+ \\ \left(1+O(\lambda)^{-1}\right)\exp\left(+\frac{i}{2}(\lambda+\Omega_3)t\right), & P\to\infty^-, \end{cases}$$

where $i=\sqrt{-1}$. Moreover, $\Psi^1(t,P)$ $(\Psi^2(t,P))$ has exactly one zero on $\check{C}_h$ and the refined asymptotic estimates of Ψ^1 at ∞^- and of Ψ^2 at ∞^+ read

$$(39) \qquad \Psi^1(t,P)=\left[-\frac{\bar{\epsilon}\,\Omega_1(t)+\epsilon\,\Omega_2(t)}{2\lambda}+O(\lambda^{-2})\right]\exp\left(+\tfrac{i}{2}(\lambda+\Omega_3)t\right), \quad P\to\infty^-$$

$$(40) \qquad \Psi^2(t,P)=\left[+\frac{\epsilon\,\Omega_1(t)+\bar{\epsilon}\,\Omega_2(t)}{2\lambda}+O(\lambda^{-2})\right]\exp\left(-\tfrac{i}{2}(\lambda+\Omega_3)t\right), \quad P\to\infty^+.$$

Proof. According to (32), $(\Psi^1,\Psi^2)\in \operatorname{Ker}(A-\mu I)$ and hence

$$(41) \qquad \frac{\Psi^2(t,P)}{\Psi^1(t,P)} = \frac{\mu-\lambda^2-(1+m)\Omega_3\lambda+\Gamma_3(t)}{\left(\bar{\epsilon}\,\Omega_1(t)+\epsilon\,\Omega_2(t)\right)\lambda-\bar{\epsilon}\,\Gamma_1(t)+\epsilon\,\Gamma_2(t)}.$$

If $P\to\infty^+$ then $\mu-\lambda^2-(1+m)\Omega_3\lambda \sim O(1)$ and using (29), (31), (32) and (41) we compute

$$2i\frac{d}{dt}\ln\Psi^1(t,P) = \lambda+\Omega_3+\left(\bar{\epsilon}\,\Omega_1(t)+\epsilon\,\Omega_2(t)\right)\frac{\Psi^2(t,P)}{\Psi^1(t,P)} = \lambda+\Omega_3+O(\lambda^{-1})$$

and hence

$$\Psi^1(t,P) = \left(1+O(\lambda^{-1}\right)\exp\left(-\tfrac{i}{2}(\lambda+\Omega_3)t\right).$$

In a similar way if $P\to\infty^-$ we obtain

$$\Psi^2(t,P) = \left(1+O(\lambda^{-1})\right)\exp\left(+\tfrac{i}{2}(\lambda+\Omega_3)t\right).$$

To compute the remaining asymptotic estimates we use that if $P\to\infty^-$ then

$$(42) \qquad \frac{\Psi^1(t,P)}{\Psi^2(t,P)} = \frac{A_{12}(t,\lambda)}{\mu-A_{11}(t,\lambda)} = -\frac{\bar{\epsilon}\,\Omega_1(t)+\epsilon\,\Omega_2(t)}{2\lambda}+O(\lambda^{-2})$$

and if $P\to\infty^+$ then

$$\frac{\Psi^2(t,P)}{\Psi^1(t,P)} = \frac{A_{21}(t,\lambda)}{\mu - A_{22}(t,\lambda)} = \frac{\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t)}{2\lambda} + O(\lambda^{-2}). \tag{43}$$

To find the poles of $\Psi(t,P)$ in P we note that according to the proof of Proposition 3.1 (and with the same notations) we have

$$\Psi(t,P) = \Phi(t,\lambda)\Psi(0,P), \qquad \Phi(0,\lambda) = I_2\,. \tag{44}$$

If $|t|$ is sufficiently small, the fundamental matrix $\Phi(t,\lambda)$ has no poles and $\det\Phi(t,\lambda) \neq 0$. It follows that the poles of $\Phi(t,\lambda)$ and $\Phi(0,\lambda)$ coincide, and we can obtain them by solving the following quadratic equation

$$\det A(0,\lambda) = \big(A_{11}(0,\lambda) - A_{12}(0,\lambda)\big)^2 = \mu^2$$

$\big($see (29, (34)$\big)$. One gets two time independent poles $P_1, P_2 \in \check{C}_h$ of $\Psi(t,P)$.

Finally, the meromorphic one-form $d\ln\Psi^1$ has a simple pole at ∞^- with residue $+1$ and is holomorphic in a neighbourhood of ∞^+. On the other hand $\Psi^1(t,P)$ has exactly two poles on $\check{C}_h$ and hence it has one zero on $\check{C}_h$. The same arguments hold for $\Psi^2(t,P)$. □

Let A_1, A_2, B_1 be a basis of $H_1(\check{C}_h, \mathbf{Z})$ as shown in Figure 2 ($A_1 \circ B_1 = 1$), and let ω_1, ω_2 be a basis of $H^0\big(C, \Omega^1(\infty^+ + \infty^-)\big)$, normalized by the conditions

$$\left(\int_{A_i} \omega_j\right)_{i,j=1,2} = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \end{pmatrix}.$$

We shall also suppose that ω_1 is a holomorphic form on the elliptic curve C_h. Define now the period matrix

$$\Pi = \begin{pmatrix} 2\pi i & 0 & \tau_1 \\ 0 & 2\pi i & \tau_2 \end{pmatrix},$$

where

$$\tau_1 = \int_{B_1} \omega_1, \qquad \tau_2 = \int_{B_1} \omega_2, \qquad \mathrm{Re}(\tau_1) < 0\,.$$

Recall that the generalized Jacobian $J(C_h; \infty^\pm)$ of C_h relative to the modulus $m = \infty^+ + \infty^-$ is identified with $\mathbf{C}^2/\Lambda$ where Λ is the lattice in $\mathbf{C}^2$ generated by the columns of Π. Let

$$\theta_{11}(z) = \theta_{11}(z \mid \tau_1) = \sum_{n=-\infty}^{\infty} \exp\{\tfrac{1}{2}\tau_1(n+\tfrac{1}{2})^2 + (z+\pi\sqrt{-1})(n+\tfrac{1}{2})\}, \qquad z \in \mathbf{C}$$

be the Jacobi theta function with characteristics $\left[\frac{1}{2}, \frac{1}{2}\right]$,

$$\theta_{11}(0) = 0, \quad \theta_{11}(z + 2\pi i) = -\theta_{11}(z), \quad \theta_{11}(z+\tau_1) = -\exp\big(-z - \tfrac{1}{2}\tau_1\big)\,\theta_{11}(z)\,.$$

Denote by Ω the unique Abelian differential of second kind on C_h with poles at $\infty^{\pm}$, principal parts $\pm\frac{i}{2}\,d\lambda$ where $P=(\lambda,\mu)$, $i=\sqrt{-1}$, and normalized by $\int_{A_1}\Omega=0$. Let $P_0\in\check{C}_h$ be a fixed initial point, $c^{\pm}$, U be the constants defined by

$$(45)\qquad \int_{P_0}^{P}\Omega=\begin{cases}-\frac{i}{2}\lambda+c^{-}+0(\lambda^{-1}), & P\to\infty^{+}\\ +\frac{i}{2}\lambda+c^{+}+0(\lambda^{-1}), & P\to\infty^{-}\end{cases}\quad,\qquad \int_{B_1}\Omega=U\,.$$

Define the Abel-Jacobi map

$$\mathcal{A}\colon\ \mathrm{Div}^0(C_h)\to J(C_h)\ :\ \sum P_i-\sum Q_i\ \mapsto\ \int_{\Sigma\ Q_i}^{\Sigma\ P_i}\omega_1\,.$$

Here, and henceforth, we make the convention that the paths of integration between divisors are taken within C_h cut along its homology basis A_1, B_1, which we assume does not contain points of these divisors.

PROPOSITION 3.3. *The Baker-Akhiezer function is explicitly given by*

$$(46)\quad \Psi^1(t,P)=\mathrm{const}_1\cdot\exp\Big[t\Big(\int_{P_0}^{P}\Omega-c^{-}-\tfrac{i}{2}\Omega_3\Big)\Big]\frac{\theta_{11}\big(\mathcal{A}(P+\infty^{-}-P_1-P_2)+tU\big)}{\theta_{11}\big(\mathcal{A}(\infty^{+}+\infty^{-}-P_1-P_2)+tU\big)}$$

$$(47)\quad \Psi^2(t,P)=\mathrm{const}_2\cdot\exp\Big[t\Big(\int_{P_0}^{P}\Omega-c^{+}+\tfrac{i}{2}\Omega_3\Big)\Big]\frac{\theta_{11}\big(\mathcal{A}(P+\infty^{+}-P_1-P_2)+tU\big)}{\theta_{11}\big(\mathcal{A}(\infty^{+}+\infty^{-}-P_1-P_2)+tU\big)}$$

where

$$\mathrm{const}_1=\frac{\theta_{11}\big(\mathcal{A}(P-\infty^{-})\big)}{\theta_{11}\big(\mathcal{A}(\infty^{+}-\infty^{-})\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{+}-P_1)\big)}{\theta_{11}\big(\mathcal{A}(P-P_1)\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{+}-P_2)\big)}{\theta_{11}\big(\mathcal{A}(P-P_2)\big)}$$

$$\mathrm{const}_2=\frac{\theta_{11}\big(\mathcal{A}(P-\infty^{+})\big)}{\theta_{11}\big(\mathcal{A}(\infty^{-}-\infty^{+})\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{-}-P_1)\big)}{\theta_{11}\big(\mathcal{A}(P-P_1)\big)}\cdot\frac{\theta_{11}\big(\mathcal{A}(\infty^{-}-P_2)\big)}{\theta_{11}\big(\mathcal{A}(P-P_2)\big)}$$

and P_1, P_2 are the poles of Ψ.

The proof of the above proposition is based on a general fact: the properties of Ψ enumerated in Proposition 3.2 define it uniquely. Indeed, if Ψ and $\widetilde{\Psi}$ are vector functions both satisfying the assumptions of Proposition 3.2, then the functions Ψ^1 and $\widetilde{\Psi}^1$ (resp. Ψ^2 and $\widetilde{\Psi}^2$) meromorphic on C_h have the same poles. Using this and the asymptotic estimates at infinity we conclude that $\Psi^1/\widetilde{\Psi}^1$ and $\Psi^2/\widetilde{\Psi}^2$ are meromorphic functions on C_h which have one pole (at $\widetilde{\Psi}^i=0$). Moreover

$$\Psi_1(t,\infty^{-})/\widetilde{\Psi}_1(t,\infty^{-})=1,\qquad \Psi_2(t,\infty^{-})/\widetilde{\Psi}_2(t,\infty^{-})=1$$

and hence $\Psi = \widetilde{\Psi}$. Finally, the reader may check that the functions (46) and (47) have the analyticity properties from Proposition 3.2 and hence they coincide with the Baker-Akhiezer function defined in Proposition 3.1. □

3.2 Solutions of the Lagrange top

Let $z = (z_1, z_2) \in J(C_h; \infty^{\pm})$. It is easy to check that the functions

$$\frac{\theta_{11}(z_1 \pm \tau_2)}{\theta_{11}(z_1)}\, e^{\mp z_2}$$

live on $J(C_h; \infty^{\pm})$. We shall see that they give solutions of the Lagrange top. By (16) we compute that $\frac{d}{dt} z = \text{constant}$, where

$$\frac{dz}{dt} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 2\pi i \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} & \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda d\lambda}{\mu} & \int_{A_2} \frac{\lambda d\lambda}{\mu} \end{pmatrix}^{-1} \begin{pmatrix} -i \\ -ai \end{pmatrix},$$

$$\int_{A_2} \frac{d\lambda}{\mu} = 0, \qquad \int_{A_2} \frac{\lambda d\lambda}{\mu} = -2\pi i$$

so

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left(\int_{A_1} \tfrac{d\lambda}{\mu} \right)^{-1} \begin{pmatrix} 2\pi \\ -i \int_{A_1} \frac{\lambda d\lambda}{\mu} + ai \int_{A_1} \frac{d\lambda}{\mu} \end{pmatrix}, \qquad a = -m\Omega_3 .$$

THEOREM 3.4. *The following equations hold*

$$\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t) = \text{const}_3 \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)}\, e^{-z_2}, \tag{48}$$

$$\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t) = \text{const}_4 \frac{\theta_{11}(z_1 + \tau_2)}{\theta_{11}(z_1)}\, e^{+z_2}, \tag{49}$$

where

$$\begin{aligned} z_2 &= tV_2, \quad z_1 = tV_1 + \mathcal{A}(\infty^+ + \infty^- - P_1 - P_2), \\ \tau_2 &= \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2 \end{aligned} \tag{50}$$

and

$$\text{const}_3 = \frac{2i\, V_1\, \theta'_{11}(0)}{\theta_{11}\big(\mathcal{A}(\infty^- - \infty^+)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^+ - P_1)\big)}{\theta_{11}\big(\mathcal{A}(\infty^- - P_1)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^+ - P_2)\big)}{\theta_{11}\big(\mathcal{A}(\infty^- - P_2)\big)},$$

$$\text{const}_4 = \frac{2i\, V_1\, \theta'_{11}(0)}{\theta_{11}\big(\mathcal{A}(\infty^+ - \infty^-)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^- - P_1)\big)}{\theta_{11}\big(\mathcal{A}(\infty^+ - P_1)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^- - P_2)\big)}{\theta_{11}\big(\mathcal{A}(\infty^+ - P_2)\big)}.$$

Let us denote

$$\omega_1 = \pm\big(\omega_1^0 + O(\lambda^{-1})\big)\, d\big(\lambda^{-1}\big), \qquad P = (\lambda, \mu) \to \infty^{\pm},$$
$$\omega_2 = \pm\big(\omega_2^1\lambda + \omega_2^0 + O(\lambda^{-1})\big)\, d\big(\lambda^{-1}\big), \qquad P = (\lambda, \mu) \to \infty^{\pm}.$$

To prove Theorem 3.4 we shall need the following

LEMMA 3.5. *The above defined differentials are such that*

$$\omega_1^0 = -i\int_{B_1}\Omega = -iV_1\,, \qquad \omega_2^0 = i(c^+ - c^-)\,,$$
$$V_2 = -c^+ + c^- + i\Omega_3\,, \qquad \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1}\omega_2\,.$$

Proof. The identity $\omega_1^0 = -i\int_{B_1}\Omega$ is a reciprocity law between the differential of the first kind ω_1 and the differential of the second kind Ω [13]. It is obtained by integrating $\pi(P)\omega_1$, where $\pi(P) = \int_{P_0}^{P}\Omega$, along the border of C_h cut along its homology basis A_1, B_1. On the other hand

$$\omega_1 = 2\pi i\left(\int_{A_1}\frac{d\lambda}{\mu}\right)^{-1}\frac{d\lambda}{\mu}$$

and hence

$$\omega_1^0 = -2\pi i\left(\int_{A_1}\frac{d\lambda}{\mu}\right)^{-1} = -iV_1\,.$$

Similarly the identity $\omega_2^0 = i(c^+ - c^-)$ is a reciprocity law between the differential of the third kind ω_2 and the differential of the second kind Ω, and $\mathcal{A}(\infty^+ - \infty^-) = \int_{B_1}\omega_2$ is a reciprocity law between the differential of the third kind ω_2 and the differential of the first kind ω_1. Finally, as $\omega_2 = \frac{\int_{A_1}\frac{\lambda d\lambda}{\mu}}{\int_{A_1}\frac{d\lambda}{\mu}}\frac{d\lambda}{\mu} - \frac{\lambda\, d\lambda}{\mu}$ we have $\omega_2^0 = -\frac{\int_{A_1}\frac{\lambda d\lambda}{\mu}}{\int_{A_1}\frac{d\lambda}{\mu}} - (1+m)\Omega_3 = -iV_1 - \Omega_3$ and hence $V_2 = -c^+ + c^- + i\Omega_3$. □

Proof of Theorem 3.4. According to (42), (43)

$$\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t) = -2\lim_{P\to\infty^-}\frac{\lambda\Psi^1(t,P)}{\Psi^2(t,P)}$$

and

$$\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t) = +2\lim_{P\to\infty^+}\frac{\lambda\Psi^2(t,P)}{\Psi^1(t,P)}\,.$$

To compute the limit we use (46), (47) and

$$\lim_{P\to\infty^-} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^-)\big) = \theta'_{11}(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta'_{11}(0)$$

$$\lim_{P\to\infty^+} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^+)\big) = \theta'_{11}(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta'_{11}(0)$$

(see Lemma 3.5). □

3.3 Effectivization

Let $\wp, \zeta, \sigma$ be the Weierstrass functions related to the elliptic curve Γ defined by

(51) $$\eta^2 = 4\xi^3 - g_2\xi - g_3$$

(we use the standard notations of [4]).

Consider also the *real* elliptic curve C with affine equation

(52) $$\mu^2 + \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

and natural anti-holomorphic involution $(\lambda,\mu) \to (\overline{\lambda},\overline{\mu})$, and put

(53) $$g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^4 - 4\frac{a_1}{4}\frac{a_3}{4}, \qquad g_3 = \det\begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}.$$

It is well known that the curves C and Γ are isomorphic over $\mathbf{C}$ and that under this isomorphism

(54) $$\frac{d\lambda}{\mu} = \frac{d\xi}{\eta}.$$

Following Weil [25] we call Γ the Jacobian $J(C)$ of the elliptic curve C and we write $J(C) = \Gamma$. Note that $J(C)$ and Γ are real isomorphic and that $J(C)$ and C are not real isomorphic.

Further we make the substitution (23) and C becomes the spectral curve $\widetilde{C}_h$ of Adler and van Moerbeke $\{\mu^2 + f(\lambda) = 0\}$, where

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + \left(2h_3 + m(m+1)h_4^2\right)\lambda^2 - 2h_2\lambda + 1$$

and Γ becomes the Lagrange curve Γ_h. Recall that, as we explained at the end of Section 2, the curve C_h with an equation $\{\mu^2 = f(\lambda)\}$ and antiholomorphic involution $(\lambda,\mu) \to (\overline{\lambda}, -\overline{\mu})$, is isomorphic over $\mathbf{R}$ to $\widetilde{C}_h$, so we write $C_h = \widetilde{C}_h$. The Jacobian curve $J(C_h) = \Gamma_h$ was computed by

Lagrange [17], while C_h appeared first in [1, 21] as a spectral curve of a Lax pair associated to the Lagrange top.

Recall that $\sigma(z)$ is an entire function in z related to $\zeta(z)$, $\wp(z)$ and the already defined function $\theta_{11}(z \mid \tau_1)$ on C_h as follows:

$$\zeta'(z) = -\wp(z)\,, \qquad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z)\,, \qquad ' = \frac{d}{dz}$$

$$\sigma(z) = \frac{\theta_{11}(zU)}{U\theta'_{11}(0)} \exp\left\{\frac{z^2U^2\theta'''_{11}(0)}{6\theta'_{11}(0)}\right\} = z - \frac{g_2 z^5}{240} + \cdots\,, \tag{55}$$

where U is a constant depending on g_2 and g_3. We shall also use the "addition formula"

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)} = \wp(v) - \wp(u)\,.$$

To state our result let us introduce the notations

$$\begin{aligned}
&2x_1 = \epsilon\,\Omega_1 + \bar\epsilon\,\Omega_2\,, && 2x_2 = \bar\epsilon\,\Omega_1 + \epsilon\,\Omega_2\,, && \epsilon^2 = \sqrt{-1}\\
&2y_1 = \epsilon^3\Gamma_1 + \epsilon\,\Gamma_2\,, && 2y_2 = \epsilon\,\Gamma_1 + \epsilon^3\Gamma_2\,, && i^2 = -1\\
&\rho_1 = -i\,m\,\Omega_3\,, && \rho_2 = -i\,\Omega_3\,.
\end{aligned} \tag{56}$$

The system (2) is equivalent to

$$\begin{aligned}
&\dot x_1 = +\rho_1 x_1 - y_1\,, && \dot y_1 = -\rho_2 y_1 + x_1\Gamma_3\\
&\dot x_2 = -\rho_1 x_2 + y_2\,, && \dot y_2 = +\rho_2 y_2 - x_2\Gamma_3\\
&\rho_1\,,\ \rho_2 = \text{constants}\,, && \dot\Gamma_3 = 2x_1y_2 - 2x_2y_1
\end{aligned} \tag{57}$$

with first integrals $I_0 = 4x_1x_2 - 2\Gamma_3$, $I_1 = 4x_1y_2 + 4x_2y_1 - 2(\rho_1+\rho_2)\Gamma_3$ and $I_2 = \Gamma_3^2 - 4y_1y_2$.

THEOREM 3.6. *The general solution of the Lagrange top* (2) *can be written in the form*

$$x_1(t) = -\frac{\sigma(t-k-l)}{\sigma(t)\,\sigma(k+l)}\,e^{at+b} \qquad x_2(t) = -\frac{\sigma(t+k+l)}{\sigma(t)\,\sigma(k+l)}\,e^{-at-b}$$

$$y_1(t) = \frac{\sigma(t-k)\,\sigma(t-l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}\,e^{at+b} \qquad y_2(t) = \frac{\sigma(t+k)\,\sigma(t+l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}\,e^{-at-b}$$

$$\Gamma_3(t) = \frac{\sigma(t+k)\,\sigma(t-k)}{\sigma^2(k)\,\sigma^2(t)} + \frac{\sigma(t+l)\,\sigma(t-l)}{\sigma^2(l)\,\sigma^2(t)} = -2\wp(t) + \wp(l) + \wp(k)$$

$$\rho_1 = a - \zeta(l) - \zeta(k) \qquad \rho_2 = -a - \zeta(k) - \zeta(l) + 2\zeta(k+l)\,,$$

where $g_2,\ g_3,\ a,\ b,\ k,\ l$ *are arbitrary constants subject to the relation* $g_2^3 - 27g_3^2 \neq 0$.

REMARK. The non-general solutions of the Lagrange top are obtained from the above formulae by taking the limit $g_2^3 - 27g_3^2 \to 0$. The formulae for the position of the body in space, and in particular for $\Gamma_3(t)$, $y_1(t)$, $y_2(t)$, are due to Jacobi [15]. The expressions for $x_1(t)$, $x_2(t)$ were first deduced by Klein and Sommerfeld [16, p. 436]. Note however that in [16] the constant a, and hence the invariant level set on which the solution lives, is not arbitrary.

Proof. To make the solutions of the Lagrange top effective we use the following 4-dimensional Lie group of transformations preserving the system (57):

$$\begin{array}{lll} x_1 \to U x_1 e^{at+b}, & x_2 \to U x_2 e^{-at-b}, & t \to \dfrac{t}{U} + T \\ y_1 \to U^2 y_1\, e^{at+b}, & y_2 \to U^2 y_2\, e^{-at-b}, & \Gamma_3 \to U^2\, \Gamma_3 \\ \rho_1 \to U\rho_1 + a, & \rho_2 \to U\rho_2 - a & \end{array} \tag{58}$$

where $U \neq 0$, T, a, b are constants.

The group (58) transforms x_1 from (48) (see also (56), (55)), where $z_1 = tU - TU$, $z_1 - \tau_2 = (t-k-l)U$ as follows

$$x_1(t) = \text{const}\, \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} = -\frac{\sigma(t-k-l)}{\sigma(t)\,\sigma(k+l)}\, e^{at+b}.$$

(we used the fact that

$$\frac{\theta_{11}(z_1 - \tau_2)\,\sigma(t)}{\theta_{11}(z_1)\,\sigma(t-k-l)}$$

is a constant). The variable x_2 is computed in the same way.

If we define the constant k by the condition $y_1(t-k) = 0$, then the first equation of (57) gives

$$\frac{y_1(t)}{x_1(t)} = \rho_1 - \frac{x_1'(t)}{x_1(t)} = \frac{\sigma(t-k)\,h(t)}{\sigma(t)\,\sigma(t-k-l)}$$

where $h(t)$ is a meromorphic function on $\mathbf{C}$, such that $y_1(t)/x_1(t)$ is single valued with poles at $t = 0$ and $t = k + l$, and residues (-1) and $(+1)$ respectively. These three conditions define $h(t)$ uniquely:

$$h(t) = \frac{\sigma(t-l)\,\sigma(k+l)}{\sigma(k)\,\sigma(l)},$$

which implies the formula for $y_1(t)$. The expression for $y_2(t)$ is obtained in the same way.

To deduce an expression for $\Gamma_3(t)$ we use the fact that

$$\Gamma_3(t) = 2x_1x_2 - \tfrac{1}{2}I_0 = -2\wp(t) + 2\wp(k+l) - \tfrac{1}{2}I_0\,.$$

The value of I_0 is easily computed by using the third equation of (57) and the formulae deduced for x_1, y_1. By substituting $t = k$ we obtain

$$\Gamma_3(k) = \frac{\sigma(k-l)\,\sigma(k+l)}{\sigma^2(k)\,\sigma^2(l)} = \wp(l) - \wp(k)$$

and in a similar way $\Gamma_3(l) = \wp(k) - \wp(l)$. We conclude that

$$\Gamma_3(t) = -2\wp(t) + \wp(l) + \wp(k)\,.$$

Finally, to compute ρ_1, ρ_2 we shall use once again (57). As $y_1(k) = 0$ we have

$$\begin{aligned}\rho_1 &= \frac{\dot{x}_1(k)}{x_1(k)} = \frac{d}{dt}\ln x_1(t)\Big|_{t=k} \\ &= \frac{d}{dt}\ln\sigma(t-k-l)\Big|_{t=k} - \frac{d}{dt}\ln\sigma(t)\Big|_{t=k} + a \\ &= a - \zeta(l) - \zeta(k)\,.\end{aligned}$$

In a quite similar way we obtain

$$\rho_2 = -\frac{d}{dt}\ln y_1(t)\Big|_{t=k+l} = -a - \zeta(k) - \zeta(l) + 2\zeta(k+l)\,.$$

Theorem 3.6 is proved. □

REMARK. If we impose the condition

$$\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = \Gamma_3^2 - 4y_1y_2 = 1\,,$$

then

$$\left(\frac{\sigma(t+k)\,\sigma(t-k)}{\sigma^2(k)\,\sigma^2(t)} + \frac{\sigma(t+l)\,\sigma(t-l)}{\sigma^2(l)\,\sigma^2(t)}\right)^2 - \frac{\sigma(t-k)\,\sigma(t-l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}\,\frac{\sigma(t+k)\,\sigma(t+l)}{\sigma^2(t)\,\sigma(k)\,\sigma(l)}$$

$$= \left(\frac{\sigma(t+k)\,\sigma(t-k)}{\sigma^2(k)\,\sigma^2(t)} - \frac{\sigma(t+l)\,\sigma(t-l)}{\sigma^2(l)\,\sigma^2(t)}\right)^2 = \big(\wp(k) - \wp(l)\big)^2 = 1$$

and hence $\wp(k) - \wp(l) = \pm 1$.

4. REAL STRUCTURES

Recall that a *real algebraic variety* is a pair (X, S) where X is a complex algebraic variety and $S\colon X \to X$ is an anti-holomorphic involution on it. The set of fixed points of S is the *real part* of (X, S). S acts on the group of divisors $\mathrm{Div}(X)$: if $D \in \mathrm{Div}(X)$ is defined locally by analytic functions f_α, then $S(D)$ is defined by the analytic functions $\overline{f_\alpha \circ S}$. Thus it is natural to define an involution S^* on the sheaf of analytic functions $\mathcal{O}_X$

$$S^*\colon \Gamma\big(S(U), \mathcal{O}_X\big) \to \Gamma\big(U, \mathcal{O}_X\big) \; : f \mapsto \overline{f \circ S}\,.$$

This also induces an involution on the groups of one-forms and one-cycles. If $\omega \in H^0(X, \Omega^1)$, $c \in H_1(X, \mathbf{Z})$, then $\int_c S^*\omega = \overline{\int_{S(c)} \omega}$. A form ω is S-real if and only if $S^*\omega = \omega$ and one may always choose a basis of S-real forms. In the case when $X = C_h$ is the spectral curve of the Lagrange top, the action of S on $\mathrm{Div}(X)$ induces an involution on $J(C_h; \infty^\pm)$. This, however, does not suffice to determine the real structure of the invariant manifold $T_h \sim J(C_h; \infty^\pm) \setminus \phi^{-1}(p)$ (Theorem 2.2), as it will also depend on the point $p \in J(C_h)$. Recall that the symmetric product $S^2\breve{C}_h$ is bi-rational to T_h. Thus the generalized Jacobian and the invariant manifold T_h are identified by the Abel map

$$\mathcal{A}\colon S^2\breve{C}_h \to J(C_h; \infty^\pm) \; : \; P_1 + P_2 \mapsto \int_{W_1+W_2}^{P_1+P_2} \omega \;, \qquad \omega = (\omega_1, \omega_2)\,. \tag{59}$$

This induces an involution on $J(C_h; \infty^\pm)$, $z \to S(z)$, where

$$z = \int_{W_1+W_2}^{P_1+P_2} \omega \;, \qquad S(z) = \int_{W_1+W_2}^{S(P_1+P_2)} \omega\,.$$

Of course this depends on the fixed points $W_1, W_2 \in J(C_h; \infty^\pm)$. Let ω_1, ω_2 be S-real . Then

$$S(z) = \int_{W_1+W_2}^{S(W_1+W_2)} \omega + \int_{S(W_1+W_2)}^{S(P_1+P_2)} \omega = \int_{W_1+W_2}^{S(W_1+W_2)} \omega + \overline{\int_{W_1+W_2}^{P_1+P_2} \omega} = S(0) + \bar{z}\,.$$

If S has a fixed point on $J(C_h; \infty^\pm)$ (this does not depend on W_1, W_2) then one may always choose it for origin, and hence $S(z) = \bar{z}$ becomes a group homomorphism.

Denote by S the anti-holomorphic involution on the spectral curve C_h defined by $S(\lambda, \mu) = (\overline{\lambda}, -\overline{\mu})$. This involution comes from the real Lax pair of Adler and van Moerbeke defined in Section 2. We shall also suppose that the real polynomial $f(\lambda)$ has distinct roots. S induces an involution on the usual Jacobian $J(C_h)$ which we also denote by S, and an involution on the generalized Jacobian $J(C_h;\infty^\pm)$ which we denote by S^+. If we use (59), then in terms of the Jacobi polynomials U, V, W, it is given by

$$S^+ : (U, V, W) \mapsto (\overline{U}, -\overline{V}, \overline{W}) .$$

There is another natural anti-holomorphic involution on T_h given by the usual complex conjugation

$$\left(\Omega_i, \Gamma_i\right) \mapsto \left(\overline{\Omega}_i, \overline{\Gamma}_i\right) ,$$

which we denote by S^-. In terms of the Jacobi polynomials (12) it is

$$S^- : (U, V, W) \mapsto (\overline{W}, \overline{V}, \overline{U}) .$$

PROPOSITION 4.1. *The holomorphic involution* $S^+ \circ S^- = S^- \circ S^+$ *on* $J(C_h;\infty^\pm)$ *is a translation on the half-period* $\frac{1}{2}\Lambda_2$, *where* $\phi\left(\frac{1}{2}\Lambda_2\right) = 0 \in J(C_h)$ (*see* (7), (9)).

The proof of the above Proposition will be given later in this section. If ϕ is the projection homomorphism defined in (7), then it implies

$$\phi \circ S^+ = \phi \circ S^- = S \circ \phi .$$

In other words the anti-holomorphic involutions S^+ and S^- "look alike" in the same way on the usual Jacobian $J(C_h)$ and differ in a half-period in the "vertical" direction with respect to ϕ on the generalized Jacobian $J(C_h;\infty^\pm)$.

An important feature of S^+ is that the S^+-real part of the invariant level set T_h is preserved by the flow of (2). Indeed, changing the variables as

$$\begin{aligned} &\Omega_1 \to i\Omega_1 , && \Omega_2 \to i\Omega_2 , && \Omega_3 \to \Omega_3 , \\ &\Gamma_1 \to i\Gamma_1 , && \Gamma_2 \to i\Gamma_2 , && \Gamma_3 \to \Gamma_3 , \end{aligned}$$

we obtain a new system

$$\begin{aligned} &\dot{\Omega}_1 = -m\,\Omega_2\Omega_3 - \Gamma_2 , && \dot{\Gamma}_1 = \Gamma_2\Omega_3 - \Gamma_3\Omega_2 , \\ &\dot{\Omega}_2 = m\,\Omega_3\Omega_1 + \Gamma_1 , && \dot{\Gamma}_2 = \Gamma_3\Omega_1 - \Gamma_1\Omega_3 , \\ &\dot{\Omega}_3 = 0 , && \dot{\Gamma}_3 = \Gamma_2\Omega_1 - \Gamma_1\Omega_2 , \end{aligned} \tag{60}$$

with first integrals

$$H_1 = -\Gamma_1^2 - \Gamma_2^2 + \Gamma_3^2 , \qquad H_2 = -\Omega_1\Gamma_1 - \Omega_2\Gamma_2 + (1+m)\Omega_3\Gamma_3 ,$$
$$H_3 = \tfrac{1}{2}\left(- \Omega_1^2 - \Omega_2^2 + (1+m)\Omega_3^2\right) - \Gamma_3 , \qquad H_4 = \Omega_3 .$$

The anti-holomorphic involution S^+ in these coordinates is given again by the complex conjugation.

THEOREM 4.2. *In each of the three connected subdomains of the complement to the discriminant locus of $f(\lambda)$ the topological type of the real part of the algebraic varieties $\left(J(C_h;\infty^\pm), S^\pm\right)$ and $(T_h, S^\pm)$ is one and the same and is given in the following table, where $T^2 = S^1 \times S^1$.*

roots of $f(\lambda)$	*no real roots*	*two real roots*	*four real roots*
real part of $\left(J(C_h;\infty^\pm), S^+\right)$	T^2	T^2	$T^2 \times (\mathbf{Z}/2)$
real part of $\left(J(C_h;\infty^\pm), S^-\right)$	T^2	$\varnothing$	$\varnothing$
real part of (T_h, S^+)	$S^1 \times \mathbf{R}$	$S^1 \times \mathbf{R}$	$T^2 \cup (S^1 \times \mathbf{R})$
real part of (T_h, S^-)	T^2	$\varnothing$	$\varnothing$

REMARK. It is easy to check that when the real invariant level set $T_h^{\mathbf{R}}$ of the Lagrange top is non-empty, then the polynomial $f(\lambda)$ has no real roots. If we do not use the generalized Jacobian $J(C_h;\infty^\pm)$, then it might be difficult to understand the relation between $T_h^{\mathbf{R}}$ (which has one connected component), $C_h^{\mathbf{R}}$ (which is empty) and $J(C_h)^{\mathbf{R}}$ (which has two connected components) (cf. [2], [3, p. 37]).

Proof of Proposition 4.1. We have $S^+ \circ S^-\colon (U,V,W) \mapsto (W,-V,U)$. The involution $(U,V,W) \mapsto (U,-V,W)$ is obviously induced by the elliptic involution $i\colon (\lambda,\mu) \mapsto (\lambda,-\mu)$ on C_h so it is a reflexion. This means that if a fixed point of i is taken for origin in $J(C_h;\infty^\pm)$ then $i = -$identity. It remains to prove that $j\colon (U,V,W) \mapsto (W,V,U)$ is a reflexion too. The involution j has the following simple geometrical interpretation. Let P_1, P_2 be two generic points in the (λ,μ) plane and lying on the affine curve $\check{C}_h = \{\mu^2 = f(\lambda)\}$. If $\{\mu = V(\lambda)\}$ is the straight line through P_1 and P_2 then it intersects C_h in four points P_1, P_2, P_3, P_4 and then $j(P_1+P_2) = P_3+P_4$. Indeed, if the zero divisor of the Jacobi polynomial $U(\lambda)$ on C_h is $P_1+P_2+i(P_1)+i(P_2)$, then by (13) the zero divisor of $W(\lambda)$ is $P_3+P_4+i(P_3)+i(P_4)$ and the involution $P_1+P_2 \mapsto P_3+P_4$ amounts to exchanging the roots of $U(\lambda)$ and $V(\lambda)$.

Let W_i, $i = 1,\dots,4$ be the Weierstrass points on C_h. Then

$$\left(\frac{\mu - V(\lambda)}{\mu}\right) = \sum_{i=1}^{4} P_i - \sum_{i=1}^{4} W_i\,, \qquad \frac{\mu - V(\lambda)}{\mu} \overset{\infty}{\sim} 1$$

and hence on $J(C_h;\infty^\pm) \sim \mathrm{Div}^0(\check{C}_h)/\overset{m}{\sim}$ we have $P_1+P_2 = -P_3-P_4+$ constant. This implies that j is a reflexion. Thus we have proved that $S^+ \circ S^-$

is a translation $\left(S^+ \circ S^-\right)(z) = z + a$. Finally, a is easily computed. We have $i(W_k) = W_k$, $j(W_1 + W_2) = W_3 + W_4$ and hence $a \overset{m}{\sim} W_1 + W_2 - W_3 - W_4$. Further if λ_1, λ_2 are zeros of $f(\lambda)$, then $(g) = W_1 + W_2 - W_3 - W_4$, where $g(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)/\mu$. Moreover $g(\infty^\pm) = \pm 1$, $g^2(\infty^\pm) = 1$ and hence

$$W_1+W_2-W_3-W_4 \sim 0\,, \quad W_1+W_2-W_3-W_4 \overset{m}{\not\sim} 0, \quad 2(W_1+W_2-W_3-W_4) \overset{m}{\sim} 0\,.$$

This shows that a is a half-period and $\phi(a) = 0 \in J(C_h)$. □

Proof of Theorem 4.2. The proof will consist of two steps. First we determine the action of $S^\pm$ on $H_1(\check{C}_h, \mathbf{Z})$ and hence on the period lattice Λ. From that we deduce the first two lines of the table. Second, we determine the action of $S^\pm : D_\infty \mapsto D_\infty$ on the infinity divisor $D_\infty = \phi^{-1}(p) = \mathbf{C}^2/\Lambda_2 \sim C^*$ and then we use that

$$\text{real part of } (T_h, S^\pm) = \text{real part of } (J(C_h; \infty^\pm), S^\pm) - \text{real part of } D_\infty\,.$$

It is easier to determine the action of S^+ on Λ. Indeed, S^+ is induced by an anti-holomorphic involution on C_h, $S^+ : (\lambda, \mu) \mapsto (\overline{\lambda}, -\overline{\mu})$. Note that S^+ always has fixed points on $J(C_h; \infty^\pm)$: if W_1, W_2 are two Weierstrass points on C_h such that either $W_1 = \overline{W}_2$, or W_1 and W_2 are S^+-real, then $S^+(W_1 + W_2) = W_1 + W_2$. On the other hand S^- has fixed points only if $f(\lambda)$ has no real roots. Indeed, in this last case let W_i, $i = 1, \ldots, 4$, be the Weierstrass points of C_h where $W_1 = \overline{W}_2$, $W_3 = \overline{W}_4$. Then $j(W_1 + W_3) = W_2 + W_4$ (see the proof of Proposition 4.1) and hence $S^-(W_1 + W_3) = W_1 + W_3$. On the other hand if $U = \overline{W}$ and $V = \overline{V}$, then

$$V^2(\lambda) + U(\lambda)W(\lambda) = |V(\lambda)|^2 + |U(\lambda)|^2 = f(\lambda) > 0 \qquad \forall\, \lambda \in \mathbf{R}\,,$$

and hence $f(\lambda)$ has no real roots.

Suppose first that $f(\lambda)$ has no real roots and let us choose a basis A_1, B_1, A_2 of $H_1(\check{C}_h, \mathbf{Z})$ as shown in Figure 2 and in Figure 3 overleaf.

Then $S^+(A_1) = A_1$, $S^+(A_2) = A_2$ and it is easily seen that $S^+(B_1) + B_1$ is homologous to A_2 on $H_1(\check{C}_h, \mathbf{Z})$. Thus in the basis A_1, A_2, B_1 the matrix of the involution $S^+ : H_1(\check{C}_h, \mathbf{Z}) \to H_1(\check{C}_h, \mathbf{Z})$ takes the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

From this and the fact that $\left(J(C_h; \infty^\pm), S^+\right)$ is not empty we conclude that the real part of $\left(J(C_h; \infty^\pm), S^+\right)$ is a torus with generators the periods $\int_{B_1} \omega$ and

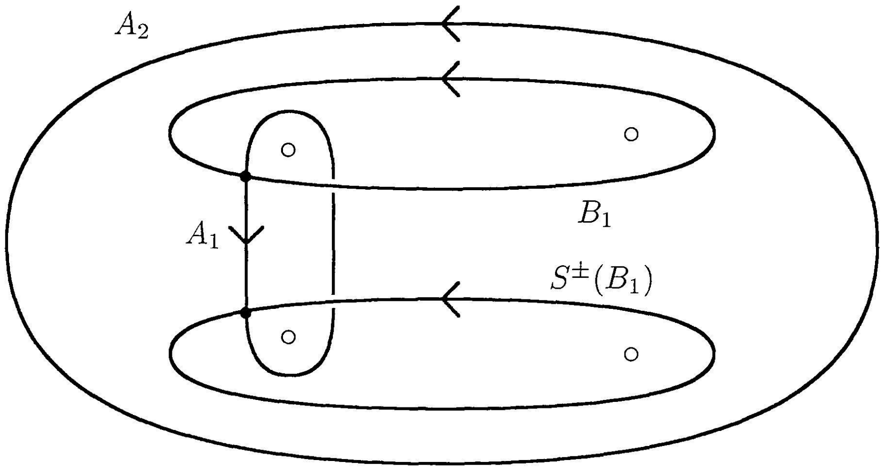

FIGURE 3
Projection of the cycles $A_1, B_1, A_2, S^{\pm}(B_1)$ on the λ–plane

$\int_{A_2} \omega$. On the other hand the real part of $\big(J(C_h;\infty^{\pm}), S^{-}\big)$ is also non-empty and $S^{+} \circ S^{-}$ is a translation. We conclude that the real part of $\big(J(C_h;\infty^{\pm}), S^{-}\big)$ is just a translation of the real part of $\big(J(C_h;\infty^{\pm}), S^{+}\big)$ and in particular it is generated by the same periods.

In a similar way we find the real part of $\big(J(C_h;\infty^{\pm}), S^{+}\big)$ in the remaining cases. Note that in an appropriate $\mathbf{Z}$ basis of $H_1(\check{C}_h, \mathbf{Z})$ the matrix of the involution $S^{\pm}\colon H_1(\check{C}_h, \mathbf{Z}) \to H_1(\check{C}_h, \mathbf{Z})$ takes the same form if $f(\lambda)$ has two real roots, and it is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

if $f(\lambda)$ has four real roots. This implies the first two lines of the table.

Let us determine now the real part of $(D_\infty, S^{\pm})$. As $D_\infty = \mathbf{C}^*/\Lambda_2$ then we have to compute $S^{\pm}(\Lambda_2)$. Note that, as the real invariant manifold T_h is compact, then (D_∞, S^{-}) is always empty. On the other hand (D_∞, S^{+}) is never empty. Indeed, if $S^{+}(\lambda, \mu) = (\overline{\lambda}, -\overline{\mu})$ then for $Q \in C_h$ the point $Q + S^{+}(Q)$ is S^{+}-real on $J(C_h;\infty^{\pm})$. As $S^{+}(\infty^{+}) = \infty^{-}$ we see that an S^{+}-real point of $\phi^{-1}(p)$ is obtained by taking the limit $Q \mapsto \infty^{+}$ in $S^{+}(Q) + Q$ along an appropriate real analytic curve on $\check{C}_h$. Finally, from the computation of the action of S^{+} on Λ we get $S^{+}(\Lambda_2) = \Lambda_2$ which shows that the S^{+}-real part of $(\phi^{-1}(p), S^{+})$ is always a circle $\mathbf{R}/\Lambda_2$. This gives the last two lines in the table. $\square$

5. The Lagrange top and the non-linear Schrödinger equation

Our final remark concerns a previously unknown relation between the real solutions of the Lagrange top and the one-gap solutions of the nonlinear Schrödinger equation

$$(NLS^{\pm}) \qquad u_{xx} = iu_t \pm 2|u|^2 u .$$

In the physical applications both forms of (NLS) are of interest. Comparing Theorem 2.2 to the results of Previato [20] we note that the invariant manifolds of one-gap solutions of the NLS equation are isomorphic to the invariant manifolds of the Lagrange top. This relation can be made explicit if we compare the expressions for the solutions found in Theorem 3.4 to the well known formulae for $u(x, t)$ (cf. [5, 20]). We shall see that the $S^{\pm}$-real solutions of the Lagrange top give also one-gap solutions of $NLS^{\pm}$ equation. Recall that, according to the preceding section, an S^{-}-real solution is a usual real solution of the Lagrange top (2), and that an S^{+}-real solution is a real solution of the system (60).

Let X_E, X_{Ω_3} be the Hamiltonian vector fields (2) and (3) respectively and put

$$\frac{\partial}{\partial x} = \tfrac{1}{2}X_E, \qquad \frac{\partial}{\partial t} = \tfrac{1}{4}(m-1)\Omega_3 X_E + \tfrac{1}{8}\left(2h_3 - (3m+1)\Omega_3^2\right)X_{\Omega_3} .$$

As $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ define translation invariant vector fields on the generalized Jacobian $J(C_h; \infty^{\pm})$ then fixing an arbitrary point for origin we may introduce (x, t) coordinates on $J(C_h; \infty^{\pm})$ (and hence on the complex invariant manifold T_h). If the real part $T_h^{\mathbf{R}}$ of T_h is not empty, then we shall choose for origin a real point. As the real vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ are tangent to the Liouville torus $T_h^{\mathbf{R}}$, then (x, t) provide real affine coordinates on it. Denote, lastly, by $u^{-}(x, t)$ the restriction of the function $\bar{\epsilon}\,\Omega_1 + \epsilon\,\Omega_2$ on the Liouville torus $T_h^{\mathbf{R}}$ of the Lagrange top (2).

Similarly, let $u^{+}(x, t)$ be the restriction of the function $\bar{\epsilon}\,\Omega_1 + \epsilon\,\Omega_2$ on a connected component of the S^{+}-real part of $J(C_h; \infty^{\pm})$. If the origin belongs to this component too, then as above we conclude that $x, t \in \mathbf{R}$.

Proposition 5.1. *The functions $u^{+}(x, t)$ and $u^{-}(x, t)$ satisfy NLS^{+} and NLS^{-} respectively.*

The proof of the above Proposition is a straightforward computation (compare with [20], Theorem 2.2). From the definition of $u^{\pm}$ we get

$\bar{u}^- = \bar{\epsilon}\,\Omega_2 + \epsilon\,\Omega_1$ and $\bar{u}^+ = -\bar{\epsilon}\,\Omega_2 - \epsilon\,\Omega_1$. It follows that $|u^\pm|^2 = \mp(\Omega_1^2 + \Omega_2^2)$ and it is easy to check that

$$u_{xx}^\pm = iu_t^\pm \pm 2|u^\pm|^2 u^\pm$$

is equivalent to the system

$$\begin{aligned}(\Omega_1)_{xx} + (\Omega_2)_t &= \pm 2\Omega_1(\Omega_1^2 + \Omega_2^2)\\ (\Omega_2)_{xx} - (\Omega_1)_t &= \pm 2\Omega_2(\Omega_1^2 + \Omega_2^2)\end{aligned}$$

where Ω_1, Ω_2 are defined on the $S^\pm$-real part of T_h respectively. Using (2) we get for the derivatives along X_E

$$\ddot{\Omega}_1 + (m-1)\Omega_3\dot{\Omega}_2 = -m\Omega_1\Omega_3^2 - \Omega_1\Gamma_3$$

and as

$$\Gamma_3 = \tfrac{1}{2}\left(\Omega_1^2 + \Omega_2^2 + (1+m)\Omega_3^2\right) - E\,,$$

then

$$\ddot{\Omega}_1 + (m-1)\Omega_3\dot{\Omega}_2 = -\tfrac{1}{2}\Omega_1(\Omega_1^2 + \Omega_2^2) + \Omega_1\left(E - \tfrac{3m+1}{2}\Omega_3^2\right).$$

Finally, as $X_{\Omega_3}\Omega_2 = -\Omega_1$ we conclude that

$$\begin{aligned}(\Omega_1)_{xx} + (\Omega_2)_t &= -2\Omega_1(\Omega_1^2 + \Omega_2^2)\\ (\Omega_2)_{xx} - (\Omega_1)_t &= -2\Omega_2(\Omega_1^2 + \Omega_2^2)\,.\end{aligned}$$

This proves also that u^+ is a solution of NLS^+ (we just have to substitute $\Omega_1 \mapsto i\Omega_1$, $\Omega_2 \mapsto i\Omega_2$). □

APPENDIX: LINEARIZATION OF THE LAGRANGE TOP ON AN ELLIPTIC CURVE

The purpose of the present Appendix is to give a brief account of some "well known" facts concerning the linearization of the Lagrange top on an elliptic curve. All algebraic varieties below come equipped with real structures. We shall make the following convention. If the complex algebraic varieties V_1 and V_2 are isomorphic over $\mathbf{R}$, then we shall simply write $V_1 = V_2$.

Further we shall suppose that the invariant complex level set

$$T_h = \left\{ (\Omega, \Gamma) \in \mathbf{C}^6 \ : \ H_1 = 1\,,\ H_2 = h_2\,,\ H_3 = h_3\,,\ H_4 = h_4 \right\}$$

of the Lagrange top (2) is smooth, and moreover $h = (h_2, h_3, h_4) \in \mathbf{R}^3$. Thus T_h has a natural real structure, and if $T_h^{\mathbf{R}}$ is its real part we make the assumption $T_h^{\mathbf{R}} \neq \varnothing$. Recall that to T_h we associate the following smooth algebraic curves:

(i) the Lagrange curve $\Gamma_h = \{\eta^2 = 4\xi^3 - g_2\xi - g_3\}$ where $g_2 = g_2(h)$, $g_3 = g_3(h)$ are given by (53) and (23). The polynomial $4\xi^3 - g_2\xi - g_3$ has three real roots, so the curve Γ_h has two ovals. Denote by $\overline{\Gamma}_h$ the completed curve Γ_h.

(ii) the spectral curve $\widetilde{C}_h = \{\mu^2 + f(\lambda) = 0\}$ of the Lax pair of Adler and van Moerbeke (26), with the natural anti-holomorphic involution $(\lambda, \mu) \mapsto (\overline{\lambda}, \overline{\mu})$, where $f(\lambda)$ is given by (24). It is isomorphic over $\mathbf{R}$ to the curve $C_h = \{\mu^2 = f(\lambda)\}$ with an anti-holomorphic involution $(\lambda, \mu) \mapsto (\overline{\lambda}, -\overline{\mu})$, so $\widetilde{C}_h = C_h$. The polynomial $f(\lambda)$ has two pairs of complex conjugate roots.

(iii) the Jacobian $J(C_h) = \mathrm{Pic}^2(C_h)$ of C_h which is identified, via the Euler-Weil map ([25]), to the Lagrange curve $\overline{\Gamma}_h$, so $J(C_h) = \overline{\Gamma}_h$.

According to the context the curves $\widetilde{C}_h$, C_h will be considered either as affine, or as completed and normalized curves.

Recall also that the generalized Jacobian $J(C_h; \infty^{\pm}) = \mathbf{C}^2/\Lambda$ of the elliptic curve C_h with two points identified is defined as an extension of $J(C_h)$ by C^*

$$0 \xrightarrow{\exp} \mathbf{C}^* \xrightarrow{\iota} J(C_h; \infty^{\pm}) \xrightarrow{\phi} J(C_h) \to 0\,.$$

By Theorem 2.2 the invariant complex level set T_h identifies with $J(C_h; \infty^{\pm}) - D_\infty$, where $D_\infty = \phi^{-1}(p)$, $p = \infty \in \Gamma_h$, so we obtain the following exact sequence

$$0 \xrightarrow{\exp} \mathbf{C}^* \xrightarrow{\iota} T_h \xrightarrow{\phi} \Gamma_h \to 0\,. \tag{61}$$

Denote by $\bar{T}_h$ the variety T_h completed by the curve D_∞, so $\bar{T}_h = J(C_h;\infty^\pm)$. It follows from Theorem 3.6 that a point $t \in J(C_h)$ is defined by $\Gamma_3(t)$ and its derivative in t, and hence

$$(62)\qquad \begin{aligned} &\phi\colon T_h \to \Gamma_h : (\Omega,\Gamma) \mapsto (\eta,\xi)\\ &\xi = -\tfrac{1}{2}\Gamma_3, \qquad \eta = -\tfrac{1}{2}\tfrac{d}{dt}\Gamma_3(t) = -\tfrac{1}{2}(\Gamma_1\Omega_2 - \Gamma_2\Omega_1)\,. \end{aligned}$$

The map ι in (61) defines a $\mathbf{C}^*$-action on T_h which is just the action of the linear complex flow of (3). The latter is obviously given by

$$(63)\qquad \begin{aligned} &\Omega_1 \pm i\Omega_2 \mapsto e^{\pm b}(\Omega_1 \pm i\Omega_2)\,, \qquad (M_3,\Gamma_3) \mapsto (M_3,\Gamma_3)\\ &\Gamma_1 \pm i\Gamma_2 \mapsto e^{\pm b}(\Gamma_1 \pm i\Gamma_2)\,, \qquad e^b \in \mathbf{C}^*\,. \end{aligned}$$

This C^*-action is free and compatible with the projection map ϕ so we have a well defined quotient map

$$\phi\colon T_h/\mathbf{C}^* \to \Gamma_h\,,$$

which is an isomorphism. It is obviously prolonged to the isomorphism

$$\phi\colon \bar{T}_h/\mathbf{C}^* \to \bar{\Gamma}_h\,.$$

As Γ_3 is a first integral of (3), then the corresponding flow is projected on $\bar{\Gamma}_h$ to the identity. According to Theorem 3.6 we have $\Gamma_3(t) = -2\wp(t) + \text{constant}$, and hence the flow of the Lagrange top is projected to a linear flow on the Lagrange curve Γ_h. The real part of T_h is a torus $T_h^{\mathbf{R}} \sim S_1 \times S_1$ on which the real flow of (3) defines a free circle action $\Re = S^1$ compatible with ϕ. $T_h^{\mathbf{R}}$ is compact and connected so is $\phi(T_h^{\mathbf{R}})$. It follows that $\phi\big(T_h^{\mathbf{R}}\big) = \phi\big(T_h^{\mathbf{R}}/\Re\big)$ is contained in the compact oval of the Lagrange curve Γ_h. In fact, ϕ provides an isomorphism between $T_h^{\mathbf{R}}/\Re$ and this oval. Indeed, the only thing we need to check is that the pre-image of a point on this compact oval, under the map $\phi\colon T_h^{\mathbf{R}} \to \Gamma_h$, is a single orbit of the system (3), that is to say a circle. But a point t on Γ_h is determined by $\Gamma_3(t)$ and $\frac{d}{dt}\Gamma_3(t) = \Gamma_1\Omega_2 - \Gamma_2\Omega_1$. This combined with the first integrals amounts to fixing Ω_3, Γ_3, the lengths

$$\Omega_1^2 + \Omega_2^2\,, \qquad \Gamma_1^2 + \Gamma_2^2\,,$$

the scalar product

$$\Omega_1\Gamma_1 + \Omega_2\Gamma_2$$

and the vector product

$$\Gamma_1\Omega_2 - \Gamma_2\Omega_1$$

of the real vectors (Ω_1,Ω_2), (Γ_1,Γ_2), which defines a circle. To sum up, we have

THEOREM A.1 (Lagrange linearization).
(i) $\phi\colon T_h/\mathbf{C}^* \to \Gamma_h$ *is an isomorphism.*
(ii) $\phi\colon \overline{T}_h/\mathbf{C}^* \to \overline{\Gamma}_h$ *is an isomorphism.*
(iii) *The image of the flow of* (3) *on* $\overline{\Gamma}_h$ *is the identity, and that of* (2) *is linear.*
(iv) *The map* ϕ *provides an isomorphism between* $T_h^{\mathbf{R}}/\Re$ *and the compact oval of the affine real curve* Γ_h.

The above theorem may be attributed to Lagrange [17, p. 254] who computed the differential equation satisfied by the nutation $\Gamma_3(t)$. It worth noting that this computation was published in 1813 (the year when Lagrange died) by Poisson [19] as completely new, and without mentioning Lagrange.

There is another more sophisticated way to linearize the Lagrange top on the elliptic curve Γ_h, by making use of the Lax pair representation (26) (see [1, 21, 24, 2, 3])

$$\frac{d}{dt}\left(\lambda^2\chi+\lambda M-\Gamma\right)=\left[\lambda^2\chi+\lambda M-\Gamma\,,\,\lambda\chi+\Omega\right].$$

Namely, let $\overset{\circ}{C}_h$ be the affine curve $\widetilde{C}_h$ with its Weierstrass points removed (they correspond to the roots of $f(\lambda)$), and put $A(\lambda)=\lambda^2\chi+\lambda M-\Gamma$. As $-\mu\left(\mu^2+f(\lambda)\right)=\det(A(\lambda)-\mu I)$, then for $(\lambda,\mu)\in\overset{\circ}{C}_h$ we have $\dim\operatorname{Ker}\left(A(\lambda)-\mu I\right)=1$. It follows that the variety

$$\left\{(\lambda,\mu)\in\overset{\circ}{C},\,[v_0,v_1,v_2]\in\mathbf{CP}^2\,:\,(v_0,v_1,v_2)\in\operatorname{Ker}(A(\lambda)-\mu I)\right\}\subset\widetilde{C}_h\times\mathbf{CP}^2$$

is smooth and it is easy to check that its closure in $\left\{\widetilde{C}_h\cup\infty^+\cup\infty^-\right\}\times\mathbf{CP}^2$ is also smooth, so we have a holomorphic line bundle on the compactified and normalized curve $\left\{\widetilde{C}_h\cup\infty^+\cup\infty^-\right\}$ (this also follows from [12, Proposition 2.2]). One computes further that the degree of this bundle is 4 and there is always a meromorphic section with a pole divisor $D=R_++R_-+\infty^++\infty^-$. Of course, the divisor D depends on the coefficients of the polynomial matrix $A(\lambda)$, and hence on (Ω,Γ). Consider now the map

$$\widetilde{\phi}\colon T_h\to\operatorname{Pic}^2(\widetilde{C}_h)=J(\widetilde{C}_h)=\overline{\Gamma}_h$$
$$(\Omega,\Gamma)\mapsto[R_++R_-]$$

where the divisor $R_\pm=\left(\lambda(R_\pm),\mu(R_\pm)\right)\in\widetilde{C}_h$ is equal to

$$\lambda(R_\pm)=\frac{\Gamma_1\mp i\Gamma_2}{\Omega_1\mp i\Omega_2}\,,\qquad \mu(R_\pm)=\pm i\left(-\Gamma_3+(1+m)h_4\lambda(R_\pm)+\lambda^2(R_\pm)\right).$$

Note that, according to Theorem 3.6, the map $\tilde{\phi}$ is prolonged to a holomorphic map

$$\tilde{\phi}\colon \overline{T}_h \to \mathrm{Pic}^2(\tilde{C}_h) = J(\tilde{C}_h) = \overline{\Gamma}_h\,.$$

We shall show that the map $\tilde{\phi}$ provides a linearization of the Lagrange top on $\overline{\Gamma}_h$. It is obvious that $\tilde{\phi}$ is compatible with the $\mathbf{C}^*$ action (63) on $T_h, \tilde{T}_h$, so we have the holomorphic maps

$$\tilde{\phi}\colon \overline{T}_h/\mathbf{C}^* \to \overline{\Gamma}_h\,, \qquad \phi\colon \overline{T}_h/\mathbf{C}^* \to \overline{\Gamma}_h\,, \qquad \tilde{\phi}\circ\phi^{-1}\colon \overline{\Gamma}_h \to \overline{\Gamma}_h\,.$$

Remembering that $\overline{\Gamma}_h$ is a complex torus, we conclude that if $z \in \mathbf{C}/\Lambda \sim \overline{\Gamma}_h$, then $\tilde{\phi}\circ\phi^{-1}(z) = kz$, for some $k \in \mathbf{Z}$, and hence $\tilde{\phi}$ provides a linearization on $\overline{\Gamma}_h$ too. The map $\tilde{\phi}$ is a non-ramified covering of degree k^2 and it is easy to check that $k^2 = 4$. Indeed, if $R_+ + R_-$ is linearly equivalent on $\tilde{C}_h$ to $\infty^+ + \infty^-$, then $R_+ = \sigma(R_-)$, where $\sigma(\lambda,\mu) = (\lambda,-\mu)$ is the elliptic involution. It follows that

$$\frac{\Gamma_1 + i\Gamma_2}{\Omega_1 + i\Omega_2} = \frac{\Gamma_1 - i\Gamma_2}{\Omega_1 - i\Omega_2} \iff \Omega_1\Gamma_2 - \Omega_2\Gamma_1 = \frac{d}{dt}\Gamma_3(t) = 0\,,$$

which shows that the pre-image of the divisor class $\infty^+ + \infty^-$ on $\overline{\Gamma}_h$ with respect to $\tilde{\phi}\circ\phi^{-1}$ consists of the four Weierstrass points on $\overline{\Gamma}_h$. Finally, we note that $\tilde{\phi}(T_h^{\mathbf{R}}/\Re)$, as before, is contained in an oval of $\overline{\Gamma}_h$. In this case, however, $\tilde{\phi}$ provides a double non-ramified covering of $T_h^{\mathbf{R}}/\Re$ to its image – the oval of the curve $\overline{\Gamma}_h = \mathrm{Pic}^2(\tilde{C}_h)$ containing the point ∞. Indeed, note that the divisor class of $\infty^+ + \infty^-$ represents a real point on $\mathrm{Pic}^2(\tilde{C}_h)$. It has exactly two real pre-images: the two Weirstrass points contained in the compact oval of Γ_h, and the remaining two Weirstrass points are not real. Thus we have proved the following

THEOREM A.2 (Linearization by making use of a Lax pair). *Let Γ_h be the affine curve defined above, and*

$$\overset{\circ}{T}_h = T_h \setminus \left\{(\omega,\Gamma) \in \mathbf{C}^6 \,:\, \Omega_1\Gamma_2 - \Omega_2\Gamma_1 = 0\right\}.$$

Then

(i) $\tilde{\phi}\colon \overset{\circ}{T}_h/\mathbf{C}^* \to \Gamma_h$ *is a non-ramified covering of degree* 4.

(ii) $\tilde{\phi}\colon \overline{T}_h/\mathbf{C}^* \to \overline{\Gamma}_h$ *is a non-ramified covering of degree* 4.

(iii) *The image of the flow of* (3) *on* $\overline{\Gamma}_h$ *is the identity, and that of* (2) *is linear.*

(iv) *The map* $\tilde{\phi}$ *provides a double non-ramified covering of* $T_h^{\mathbf{R}}/\Re$ *to its image – the oval of the compactified and normalized curve* $\overline{\Gamma}_h = \Gamma_h \cup \infty$ *containing the point* ∞.

Statement (iv) is due to M. Audin. In this form it appeared first in [2] (Proposition 3.3.2) but the proof is not correct. Earlier Verdier [24] wrongly claimed that the map $\widetilde{\phi}$ provides an isomorphism between $T_h^{\mathbf{R}}/\Re$ and its image. Statement (iii) is a well known fact, though it does not seem to have ever been rigorously proved. Thus Adler and van Moerbeke [1] and then Ratiu and van Moerbeke [21] proposed a "proof" based on a general scheme for linearizing the flow defined by a Lax pair with a spectral parameter (e.g. Adler and van Moerbeke [1], Theorem 1, p. 337). The Lax pair (26) does not fit, however, the general procedure, as its spectral curve is always reducible. Of course this is only a minor technical difficulty as we may also use the Lax pair (14). It was proposed in [1, p. 351] and [21] to consider, instead of the Lax pair (26), another Lax pair

$$\frac{dA^\epsilon}{dt} = [A^\epsilon, B^\epsilon]\,, \tag{64}$$

where in the notation of [1] we have

$$A^\epsilon = A^\epsilon(h) = \begin{pmatrix} \epsilon h^2 & \beta & i\beta^* \\ -\beta^* & -\omega & 0 \\ i\beta & 0 & \omega \end{pmatrix}, \quad B^\epsilon = B^\epsilon(h) = \frac{1}{I_1}\,[h^{-1}A^\epsilon(h)]_+\,,$$

$[\,.\,]_+$ means "polynomial part" and

$$\begin{aligned} \beta &= y + hx\,, & y &= \tfrac{1}{\sqrt{2}}(\gamma_1 - i\gamma_2)\,, & x &= \tfrac{I_1}{\sqrt{2}}(\Omega_1 - i\Omega_2) \\ \beta^* &= \bar{y} + h\bar{x}\,, & \bar{y} &= \tfrac{1}{\sqrt{2}}(\gamma_1 + i\gamma_2)\,, & \bar{x} &= \tfrac{I_1}{\sqrt{2}}(\Omega_1 + i\Omega_2) \end{aligned}$$

$$i\omega = z_0 I_1 h^2 + I_3\Omega_3 h + \gamma_3\,.$$

To obtain our notations from those of [1], we just replace

$$\gamma_i = -\Gamma_i\,, \quad z_0 = I_1 = 1\,, \quad I_3 = 1 + m\,, \quad h = \lambda\,.$$

For the spectral curve X_ϵ of A^ϵ we obtain

$$\begin{aligned} \det\left(A^\epsilon(h) - zI\right) &= (\epsilon h^2 - z)(z^2 - \omega^2) - 2\beta\beta^* z \\ &= -z^3 + \epsilon h^2 z^2 + (-2\beta\beta^* + \omega^2)z - \epsilon h^2\omega^2 = 0\,. \end{aligned} \tag{65}$$

This is generically a smooth irreducible genus 4 curve, so the Lax pair (64) fits to Theorem 1, p. 337 in [1]. Thus the flow of (64) linearizes on $\mathrm{Jac}(X_\epsilon)$ and when $\epsilon \to 0$ it goes over into a linear flow on the compact piece of $\mathrm{Jac}(X_0)$ which is just the Lagrange elliptic curve. On the other hand the differential equation (64) for $\epsilon = 0$ is, modulo a linear change of the variables, the original system (2) which establishes once again Theorem A.2, (ii). It is easy to see, however, that the above approach does not work as for $\epsilon \neq 0$ the Lax pair (64) *does not define a differential equation.* Indeed, note that (64) is equivalent to the Lax pair

$$\frac{dA^0}{dt} = [A^0, B^0] - \frac{\epsilon h}{I_1} \begin{pmatrix} 0 & y & i\bar{y} \\ -\bar{y} & i\gamma_3 & 0 \\ iy & 0 & -i\gamma_3 \end{pmatrix}. \tag{66}$$

Its $(1,2)$ entry is computed to be

$$\frac{d\beta}{dt} = \frac{i}{I_1}\left(yI_3\Omega_3 - x\gamma_3 + hz_0I_1y\right) - \frac{\epsilon h y}{I_1}$$

and the $(3,1)$ entry is

$$i\frac{d\beta}{dt} = \frac{1}{I_1}\left(- yI_3\Omega_3 + x\gamma_3 - hz_0I_1y\right) + \frac{\epsilon^3 h y}{I_1},$$

so $y \equiv 0$ and in a similar way $\bar{y} \equiv 0$.

More generally, it is seen from the coefficients of the spectral curve X_ϵ, $\epsilon \neq 0$, that the functions

$$\Omega_1^2 + \Omega_2^2, \qquad \gamma_1^2 + \gamma_2^2, \qquad \Omega_1\gamma_1 + \Omega_2\gamma_2, \qquad \gamma_3, \qquad \Omega_3$$

are invariants for *any* isospectral deformation of the matrix A^ϵ. By continuity these five functions are invariants for $\epsilon = 0$ too, so the vector field in $\mathbf{C}^6$ obtained as $\epsilon \to 0$ is collinear to the linear vector field of (3). Of course there is no analytic change of variables in $\mathbf{C}^6$ which sends the orbits of (3) to orbits of (2).

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