Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THREE DISTANCE THEOREMS AND COMBINATORICS ON WORDS

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Bibliographie

DOI: https://doi.org/10.5169/seals-63900

Proof. Let us consider a coding of the rotation by angle α under the left-closed and right-open partition of the unit circle bounded by all the points of the form $\{n\alpha + \gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$; let $\beta_0, \ldots, \beta_{p-1}$ denote these consecutive points. The letter associated with the interval $I_k = [\beta_k, \beta_{k+1}[$ has a unique right extension, except when I_k contains points of the form $\{\beta_i - \alpha\}$. Suppose there are $q \geq 2$ points of this form; the associated letter has $q+1$ right extensions. Since there are at most d points of this type, we obtain $p(2) - p(1) \leq d$. We deduce from Theorem 6 that there are at most $3d$ different frequencies for the letters of the coding, i.e., there are at most $3d$ different lengths for the intervals I_k.

REMARK. The start and finish intervals as introduced by Liang in his proof in [37] correspond exactly to the beginning of the branches in the graph of words. Indeed, Liang shows that any interval is associated either with a start point $\{\gamma_i\}$ (i.e., with one extension of a factor having more than one right extension) or with a finish point $\{(n_i - 1)\alpha + \gamma_i\}$ (i.e., with a factor having more than one left extension). Counting the finish and start points defined in [37] (there are $3d$ such points) is equivalent to counting the number of branches in the graph of words.

As in the remark of the previous section, we can consider a coding of the rotation by irrational angle $1 - \alpha$ under the partition $\{[\gamma_1, \gamma_2[, \ldots, [\gamma_d, \gamma_1[\}$. For such a coding, the $3d$ distance theorem can be rephrased as follows.

THEOREM 20. *The frequencies of the factors of given length $n \geq n^{(1)}$ of a coding of a rotation by irrational angle under a partition in d intervals take at most $3d$ values, where $n^{(1)}$ denotes the connectedness index.*

ACKNOWLEDGEMENTS. We would like to thank Arnoux, Berstel, Daudé, Dumont, Liardet and Shallit for their bibliographic help. Furthermore, we are greatly indebted to Allouche for his many useful comments and to Ferenczi and Jenkinson who read carefully a previous version of this paper.

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(Reçu le 10 janvier 1997; version révisée reçue le 12 février 1998)

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