Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THREE DISTANCE THEOREMS AND COMBINATORICS ON WORDS
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Kapitel: 7.2 Application to binary codings
DOI: https://doi.org/10.5169/seals-63900

7.2 APPLICATION TO BINARY CODINGS

A more natural coding of the rotation R would have been with respect to the partition $[0,\beta[$, $[\beta,1[$. The points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \ldots, \{n\alpha\}$, $\{\beta+n\alpha\}$ are the endpoints of the sets $I(w_1,\ldots,w_n)$, following the notation of Section 2. But these sets might not be connected. Thus the frequencies of factors of length n are the sums of the lengths of the connected components of the sets $I(w_1,\ldots,w_n)$. Despite this disadvantage, this coding allows us to deduce the following result from Lemma 3.

THEOREM 19. *Let u be a coding of an irrational rotation with respect to the partition into two intervals $\{[0,\beta[, [\beta,1[\}$, where $0<\beta<1$. Let $n^{(1)}$ denote the connectedness index of u. The frequencies of factors of given length $n \geq n^{(1)}$ of u take at most 5 values. Furthermore, the set of factors of u is stable by mirror image, i.e., if the word $a_1 \cdots a_n$ is a factor of the sequence u, then $a_n \cdots a_1$ is also a factor and furthermore, both words have the same frequency.*

Proof. It remains to prove the part of this theorem concerning the stability by mirror image. Assume we are given a fixed positive integer n. Let s_n be the reflection of the circle defined by $s_n \colon x \rightarrow \{\beta-(n-1)\alpha-x\}$. We have $s_n(R^{-k}(I_j)) = R^{(-n+1+k)}(I_j)$, for $j=0,1$, following the previous notation. The image of $I(w_1,\ldots,w_n)$ by s_n is $I(w_n,\ldots,w_1)$; they thus have the same length, which gives the result.

REMARK. A study of the topology of the graph of words for a binary coding of an irrational rotation of complexity satisfying ultimately $p(n+1)-p(n)=2$ can be found in [24] or in [46].

8. THE $3d$ DISTANCE THEOREM

Following the idea of the above proof, let us give a combinatorial proof of the $3d$ *distance theorem.*

THE $3d$ DISTANCE THEOREM. *Assume we are given $0<\alpha<1$ irrational, $\gamma_1,\ldots,\gamma_d$ real numbers and $n_1,\ldots,n_d$ positive integers. The points $\{n\alpha+\gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$, partition the unit circle into at most $n_1+\cdots+n_d$ intervals, having at most $3d$ different lengths.*