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frequencies of binary codings: the frequencies of the factors of given length of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values.

### 6.3 THE $3d$ DISTANCE THEOREM

Let us consider another generalization of the three distance theorem, known as the *3d distance theorem*. This result, conjectured by Graham (see [17] and [34]), was first proved by Chung and Graham in [18] and secondly by Liang who gave a very nice proof in [37]. Geelen and Simpson remark in [29] that their proof uses ideas from Liang's proof.

**THE  $3d$  DISTANCE THEOREM.** *Assume we are given  $0 < \alpha < 1$  irrational,  $\gamma_1, \dots, \gamma_d$  real numbers and  $n_1, \dots, n_d$  positive integers. The points  $\{n\alpha + \gamma_i\}$ , for  $0 \leq n < n_i$  and  $1 \leq i \leq d$ , partition the unit circle into at most  $n_1 + \dots + n_d$  intervals, having at most  $3d$  different lengths.*

We will give a combinatorial proof of this result in Section 8 and express the corresponding result for frequencies of codings of rotations, i.e., that the frequencies of the factors of given length of a coding of a rotation by the unit circle under a partition in  $d$  intervals take ultimately at most  $3d$  values.

### 6.4 OTHER GENERALIZATIONS

Slater has studied in [50] the following generalization of the three gap theorem, which should be compared with Theorem 13: there is a bounded number of gaps between the successive values of the integers  $n$  such that  $\{n(\eta_1, \dots, \eta_d)\} \in C$ , where  $C$  is a closed convex region on the  $d$ -dimensional torus and where  $1, \eta_1, \dots, \eta_d$  are rationally independent. However, Fraenkel and Holzman prove Theorem 13 even in the case where  $\alpha_1, \alpha_2$  and 1 are rationally independent.

Chevallier studies in [16] a  $d$ -generalization of the three distance theorem to  $\mathbf{T}^d$ , where intervals are replaced by Voronoï cells: the number of Voronoï cells (up to isometries) is shown to be connected to the number of sides of a Voronoï cell. The notion of continued fraction expansion is generalized by properties of best approximation.

Finally, note the unsolved problems quoted in [29] concerning further generalizations of the three distance theorem. For instance, an upper bound for the number of distinct lengths in the partition of the unit circle by the points

$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d$ , for  $k_i \leq n_i - 1$  and  $1 \leq i \leq d$  is conjectured to be of the form  $c_d + \prod_{i=1}^{d-1} n_i$ , where  $c_d$  is a constant independent of  $n_1, \dots, n_d$ .

## 7. FREQUENCIES OF FACTORS FOR BINARY CODINGS OF ROTATIONS

We will prove in this section the following result, which corresponds to the case  $\min\{n_1, n_2\} = 2$  in Theorem 14. The idea of using a reflection of the unit circle can also be found in the original proof in [29].

**THEOREM 18.** *Let  $\alpha$  be an irrational number in  $]0, 1[$ ,  $\beta \neq 0$  a real number and  $n$  a non-zero integer. The set of points  $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \dots, \{n\alpha\}, \{\beta + n\alpha\}$  divides the circle into a finite number of intervals, whose lengths take at most five values.*

### 7.1 A COMBINATORIAL PROOF

We will prove Theorem 18 by introducing a coding of the rotation by angle  $\alpha$  with respect to the intervals of the unit circle bounded by the points  $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \dots, \{n\alpha\}, \{\beta + n\alpha\}$ .

Let  $\alpha$  be an irrational number,  $\beta$  a non-zero real number and  $n$  an integer. Let  $I_1, \dots, I_p$  denote the intervals of the unit circle bounded by the points  $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \dots, \{n\alpha\}, \{\beta + n\alpha\}$ . Let  $u = (u_n)_{n \in \mathbb{N}}$  be the sequence defined on the alphabet  $\Sigma = \{a_1, \dots, a_p\}$  as the coding of the orbit of 0 under the rotation  $R$  of angle  $\alpha$  under the partition  $\{I_1, \dots, I_p\}$ :

$$u_n = a_k \iff \{n\alpha\} \in I_k.$$

The frequency of the letter  $a_k$  in the sequence  $u$  is equal to the length of the interval  $I_k$ , by uniform distribution of the sequence  $(\{n\alpha\})_{n \in \mathbb{N}}$ . We must now prove that the frequencies of the letters of  $u$  take at most five values. Let us consider the graph  $\Gamma_1$  of words of  $u$  of length 1. There is one edge from  $a_k$  to  $a_{k'}$  if  $I_{k'}$  is the image of  $I_k$  by the rotation  $R$  or if  $I_{k'}$  contains  $\{-\alpha\}$  or  $\{-\alpha + \beta\}$ . Therefore the graph  $\Gamma_1$  contains  $p$  vertices (one for each letter) and  $p + 2$  edges: indeed, every vertex has only one leaving edge, except the ones associated with the intervals containing  $\{-\alpha\}$  or  $\{-\alpha + \beta\}$ , which have two leaving edges (if both of these points belong to the same interval  $I_k$ , then  $a_k$  has three leaving edges and all the other intervals have only one edge). In other words, we have  $p(1) = p$  and  $p(2) = p + 2$ . As in the proof of Theorem 6, this implies that there are at most 6 branches in  $\Gamma_1$ : indeed, each