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frequencies of binary codings: the frequencies of the factors of given length of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values.

6.3 THE $3d$ DISTANCE THEOREM

Let us consider another generalization of the three distance theorem, known as the $3d$ *distance theorem*. This result, conjectured by Graham (see [17] and [34]), was first proved by Chung and Graham in [18] and secondly by Liang who gave a very nice proof in [37]. Geelen and Simpson remark in [29] that their proof uses ideas from Liang's proof.

THE $3d$ DISTANCE THEOREM. *Assume we are given* $0 < \alpha < 1$ *irrational,* $\gamma_1, \ldots, \gamma_d$ *real numbers and* $n_1, \ldots, n_d$ *positive integers. The points* $\{n\alpha + \gamma_i\}$, *for* $0 \leq n < n_i$ *and* $1 \leq i \leq d$, *partition the unit circle into at most* $n_1 + \cdots + n_d$ *intervals, having at most* $3d$ *different lengths.*

We will give a combinatorial proof of this result in Section 8 and express the corresponding result for frequencies of codings of rotations, i.e., that the frequencies of the factors of given length of a coding of a rotation by the unit circle under a partition in d intervals take ultimately at most $3d$ values.

6.4 OTHER GENERALIZATIONS

Slater has studied in [50] the following generalization of the three gap theorem, which should be compared with Theorem 13: there is a bounded number of gaps between the successive values of the integers n such that $\{n(\eta_1, \ldots, \eta_d)\} \in C$, where C is a closed convex region on the d-dimensional torus and where $1, \eta_1 \ldots, \eta_d$ are rationally independent. However, Fraenkel and Holzman prove Theorem 13 even in the case where α_1, α_2 and 1 are rationally independent.

Chevallier studies in [16] a d-generalization of the three distance theorem to $\mathbf{T}^d$, where intervals are replaced by Voronoï cells: the number of Voronoï cells (up to isometries) is shown to be connected to the number of sides of a Voronoï cell. The notion of continued fraction expansion is generalized by properties of best approximation.

Finally, note the unsolved problems quoted in [29] concerning further generalizations of the three distance theorem. For instance, an upper bound for the number of distinct lengths in the partition of the unit circle by the points

$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d$, for $k_i \leq n_i - 1$ and $1 \leq i \leq d$ is conjectured to be of the form $c_d + \prod_{i=1}^{d-1} n_i$, where c_d is a constant independent of $n_1, \ldots, n_d$.

7. FREQUENCIES OF FACTORS FOR BINARY CODINGS OF ROTATIONS

We will prove in this section the following result, which corresponds to the case $\min\{n_1, n_2\} = 2$ in Theorem 14. The idea of using a reflection of the unit circle can also be found in the original proof in [29].

THEOREM 18. *Let α be an irrational number in $]0,1[$, $\beta \neq 0$ a real number and n a non-zero integer. The set of points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \ldots, \{n\alpha\}, \{\beta + n\alpha\}$ divides the circle into a finite number of intervals, whose lengths take at most five values.*

7.1 A COMBINATORIAL PROOF

We will prove Theorem 18 by introducing a coding of the rotation by angle α with respect to the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \ldots, \{n\alpha\}, \{\beta + n\alpha\}$.

Let α be an irrational number, β a non-zero real number and n an integer. Let $I_1, \ldots, I_p$ denote the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \ldots, \{n\alpha\}, \{\beta + n\alpha\}$. Let $u = (u_n)_{n \in \mathbf{N}}$ be the sequence defined on the alphabet $\Sigma = \{a_1, \ldots, a_p\}$ as the coding of the orbit of 0 under the rotation R of angle α under the partition $\{I_1, \ldots, I_p\}$:

$$u_n = a_k \iff \{n\alpha\} \in I_k .$$

The frequency of the letter a_k in the sequence u is equal to the length of the interval I_k, by uniform distribution of the sequence $(\{n\alpha\})_{n \in \mathbf{N}}$. We must now prove that the frequencies of the letters of u take at most five values. Let us consider the graph Γ_1 of words of u of length 1. There is one edge from a_k to $a_{k'}$ if $I_{k'}$ is the image of I_k by the rotation R or if $I_{k'}$ contains $\{-\alpha\}$ or $\{-\alpha + \beta\}$. Therefore the graph Γ_1 contains p vertices (one for each letter) and $p+2$ edges: indeed, every vertex has only one leaving edge, except the ones associated with the intervals containing $\{-\alpha\}$ or $\{n - \alpha + \beta\}$, which have two leaving edges (if both of these points belong to the same interval I_k, then a_k has three leaving edges and all the other intervals have only one edge). In other words, we have $p(1) = p$ and $p(2) = p+2$. As in the proof of Theorem 6, this implies that there are at most 6 branches in Γ_1 : indeed, each