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frequencies of binary codings: the frequencies of the factors of given length of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values.

6.3 The 3d distance theorem

Let us consider another generalization of the three distance theorem, known as the 3d distance theorem. This result, conjectured by Graham (see [17] and [34]), was first proved by Chung and Graham in [18] and secondly by Liang who gave a very nice proof in [37]. Geelen and Simpson remark in [29] that their proof uses ideas from Liang's proof.

THE 3d DISTANCE THEOREM. Assume we are given $0 < \alpha < 1$ irrational, $\gamma_1, \ldots, \gamma_d$ real numbers and n_1, \ldots, n_d positive integers. The points $\{n\alpha + \gamma_i\}$, for $0 \le n < n_i$ and $1 \le i \le d$, partition the unit circle into at most $n_1 + \cdots + n_d$ intervals, having at most 3d different lengths.

We will give a combinatorial proof of this result in Section 8 and express the corresponding result for frequencies of codings of rotations, i.e., that the frequencies of the factors of given length of a coding of a rotation by the unit circle under a partition in d intervals take ultimately at most 3d values.

6.4 OTHER GENERALIZATIONS

Slater has studied in [50] the following generalization of the three gap theorem, which should be compared with Theorem 13: there is a bounded number of gaps between the successive values of the integers n such that $\{n(\eta_1,\ldots,\eta_d)\}\in C$, where C is a closed convex region on the d-dimensional torus and where $1,\eta_1\ldots,\eta_d$ are rationally independent. However, Fraenkel and Holzman prove Theorem 13 even in the case where α_1 , α_2 and 1 are rationally independent.

Chevallier studies in [16] a d-generalization of the three distance theorem to \mathbf{T}^d , where intervals are replaced by Voronoï cells: the number of Voronoï cells (up to isometries) is shown to be connected to the number of sides of a Voronoï cell. The notion of continued fraction expansion is generalized by properties of best approximation.

Finally, note the unsolved problems quoted in [29] concerning further generalizations of the three distance theorem. For instance, an upper bound for the number of distinct lengths in the partition of the unit circle by the points

 $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d$, for $k_i \leq n_i - 1$ and $1 \leq i \leq d$ is conjectured to be of the form $c_d + \prod_{i=1}^{d-1} n_i$, where c_d is a constant independent of n_1, \ldots, n_d .

7. Frequencies of factors for binary codings of rotations

We will prove in this section the following result, which corresponds to the case $\min\{n_1, n_2\} = 2$ in Theorem 14. The idea of using a reflection of the unit circle can also be found in the original proof in [29].

THEOREM 18. Let α be an irrational number in]0,1[, $\beta \neq 0$ a real number and n a non-zero integer. The set of points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \ldots, \{n\alpha\}, \{\beta + n\alpha\}$ divides the circle into a finite number of intervals, whose lengths take at most five values.

7.1 A COMBINATORIAL PROOF

We will prove Theorem 18 by introducing a coding of the rotation by angle α with respect to the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \dots, \{n\alpha\}, \{\beta+n\alpha\}.$

Let α be an irrational number, β a non-zero real number and n an integer. Let I_1, \ldots, I_p denote the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$. Let $u = (u_n)_{n \in \mathbb{N}}$ be the sequence defined on the alphabet $\Sigma = \{a_1, \ldots, a_p\}$ as the coding of the orbit of 0 under the rotation R of angle α under the partition $\{I_1, \ldots, I_p\}$:

$$u_n = a_k \iff \{n\alpha\} \in I_k$$
.

The frequency of the letter a_k in the sequence u is equal to the length of the interval I_k , by uniform distribution of the sequence $(\{n\alpha\})_{n\in\mathbb{N}}$. We must now prove that the frequencies of the letters of u take at most five values. Let us consider the graph Γ_1 of words of u of length 1. There is one edge from a_k to $a_{k'}$ if $I_{k'}$ is the image of I_k by the rotation R or if $I_{k'}$ contains $\{-\alpha\}$ or $\{-\alpha+\beta\}$. Therefore the graph Γ_1 contains p vertices (one for each letter) and p+2 edges: indeed, every vertex has only one leaving edge, except the ones associated with the intervals containing $\{-\alpha\}$ or $\{n-\alpha+\beta\}$, which have two leaving edges (if both of these points belong to the same interval I_k , then a_k has three leaving edges and all the other intervals have only one edge). In other words, we have p(1) = p and p(2) = p+2. As in the proof of Theorem 6, this implies that there are at most 6 branches in Γ_1 : indeed, each