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frequencies of binary codings: the frequencies of the factors of given length of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values.

### 6.3 THE $3d$ DISTANCE THEOREM

Let us consider another generalization of the three distance theorem, known as the *3d distance theorem*. This result, conjectured by Graham (see [17] and [34]), was first proved by Chung and Graham in [18] and secondly by Liang who gave a very nice proof in [37]. Geelen and Simpson remark in [29] that their proof uses ideas from Liang's proof.

**THE  $3d$  DISTANCE THEOREM.** *Assume we are given  $0 < \alpha < 1$  irrational,  $\gamma_1, \dots, \gamma_d$  real numbers and  $n_1, \dots, n_d$  positive integers. The points  $\{n\alpha + \gamma_i\}$ , for  $0 \leq n < n_i$  and  $1 \leq i \leq d$ , partition the unit circle into at most  $n_1 + \dots + n_d$  intervals, having at most  $3d$  different lengths.*

We will give a combinatorial proof of this result in Section 8 and express the corresponding result for frequencies of codings of rotations, i.e., that the frequencies of the factors of given length of a coding of a rotation by the unit circle under a partition in  $d$  intervals take ultimately at most  $3d$  values.

### 6.4 OTHER GENERALIZATIONS

Slater has studied in [50] the following generalization of the three gap theorem, which should be compared with Theorem 13: there is a bounded number of gaps between the successive values of the integers  $n$  such that  $\{n(\eta_1, \dots, \eta_d)\} \in C$ , where  $C$  is a closed convex region on the  $d$ -dimensional torus and where  $1, \eta_1, \dots, \eta_d$  are rationally independent. However, Fraenkel and Holzman prove Theorem 13 even in the case where  $\alpha_1, \alpha_2$  and 1 are rationally independent.

Chevallier studies in [16] a  $d$ -generalization of the three distance theorem to  $\mathbf{T}^d$ , where intervals are replaced by Voronoï cells: the number of Voronoï cells (up to isometries) is shown to be connected to the number of sides of a Voronoï cell. The notion of continued fraction expansion is generalized by properties of best approximation.

Finally, note the unsolved problems quoted in [29] concerning further generalizations of the three distance theorem. For instance, an upper bound for the number of distinct lengths in the partition of the unit circle by the points