

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THREE DISTANCE THEOREMS AND COMBINATORICS ON WORDS
Autor: Alessandri, Pascal / Berthé, Valérie
Kapitel: 4.3 Beatty sequences
DOI: <https://doi.org/10.5169/seals-63900>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Let a be the limit of this sequence, $n^{(2)}$ the smallest index such that $k_n = a$, and let

$$b = \sum_{i=0}^{n^{(2)}-1} (k_i - a).$$

Let $n^{(1)}$ denote the connectedness index of u .

If $n \geq \max(n^{(1)}, n^{(2)})$, then the complexity of the sequence u satisfies

$$p(n) = an + b.$$

REMARKS.

- Note that if $1, \alpha, \beta_1, \dots, \beta_p$ are rationally independent, then $n^{(2)} = 0$, $b = 0$ and $a = p$.

- Theorem 10 answers the question of the existence of sequences of ultimately affine complexity (for more details, the reader is referred to [1], see also the result of Cassaigne in [11]).

4.3 BEATTY SEQUENCES

The connections between the three gap theorem and the Beatty sequences have been investigated by Fraenkel and Holzman in [26]. Let us recall that a Beatty sequence is a sequence $u(\alpha, \rho) = (u_n)_{n \in \mathbf{N}}$ of the form $u_n = \lfloor \alpha n + \rho \rfloor$, where α and ρ are real numbers such that $\alpha \geq 1$. The number α is called the *modulus* and ρ is called the *residue* or *intercept*. For an impressive bibliography on the subject, we refer the reader to [27] and [54]. Fraenkel and Holzman have noticed in [26] that the three gap theorem answers the question of the gaps in the intersection of a Beatty sequence and an arithmetical sequence $(an + c)_{n \in \mathbf{N}}$, for a a positive integer and c an integer. This result has been obtained independently by Wolff and Pitman in [58]. By intersection of the two Beatty sequences $s = (s_n)_{n \in \mathbf{N}}$ and $t = (t_n)_{n \in \mathbf{N}}$, we mean the strictly increasing sequence u defined as:

$$\{u_n, n \in \mathbf{N}\} = \{u, \exists k, l \in \mathbf{N} \text{ such that } u = s_k = t_l\}.$$

Hence a gap in the intersection denotes the difference between two distinct elements of the intersection.

Note that Beatty sequences and Sturmian sequences are related: let u be a Beatty sequence of modulus α and residue ρ ; the characteristic sequence $(v_n)_{n \in \mathbf{N}}$ of u defined as

$$v_n = 1 \text{ if and only if there exists } m \text{ such that } n = \lfloor \alpha m + \rho \rfloor$$

is the Sturmian sequence obtained as the coding of the orbit of $-\rho/\alpha$ under the rotation by angle $1/\alpha$, with respect to the partition

$$\{]0, 1 - 1/\alpha],]1 - 1/\alpha, 1]\}.$$

Indeed, if $n = \lfloor \alpha m + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = m+1 = 1 + \lceil n/\alpha - \rho/\alpha \rceil$, and if $\lfloor \alpha m + \rho \rfloor < n < \lfloor \alpha(m+1) + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = \lceil n/\alpha - \rho/\alpha \rceil$.

5. THE RECURRENCE FUNCTION

Let us deduce now from the three distance and three gap theorems a simple proof of the following result originally due to Morse and Hedlund concerning the recurrence function of a Sturmian sequence (see [40]).

Recall that a sequence u is called *minimal* or *uniformly recurrent* if every factor of u appears infinitely often and with bounded gaps or, equivalently, if for any integer n , there exists an integer m such that every factor of u of length m contains every factor of u of length n . Note that it is equivalent (see [30]) to the *minimality* of the dynamical system $(\overline{\mathcal{O}(u)}, T)$, i.e., the orbit of every element of $\overline{\mathcal{O}(u)}$ is dense, or equivalently every sequence in the orbit closure of u has the same set of factors as u .

The recurrence function φ of a minimal sequence u is defined by

$$\varphi(n) = \min \{m \in \mathbf{N} \text{ such that } \forall B \in L_m, \forall A \in L_n, A \text{ is a factor of } B\},$$

where L_n denotes the set of factors of u of length n , i.e., $\varphi(n)$ is the size of the smallest window that contains all factors of length n whatever its position in the sequence.

THEOREM 11. *Let u be a Sturmian sequence with angle α . Let $(q_k)_{k \in \mathbf{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of α . The recurrence function φ of this sequence satisfies for any non zero integer n :*

$$\varphi(n) = n - 1 + q_k + q_{k-1}, \text{ where } q_{k-1} \leq n < q_k.$$

Proof of Theorem 11. Let $u \in \{0, 1\}^{\mathbf{N}}$ be a Sturmian sequence. There exist a real number x and an irrational number α in $]0, 1[$ such that $u_n = 0 \Leftrightarrow \{x + n\alpha\} \in I_0$, with $I_0 = [0, \alpha[$ or $I_0 =]0, \alpha]$ (see Section 2.1). Let $I_1 = [\alpha, 1[$ (respectively, $]\alpha, 1]$) if $I_0 = [0, \alpha[$ (respectively, $I_0 =]0, \alpha]$). Let us denote by R the rotation of the circle by angle α . Assume we are given