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REMARKS.

• Suppose that α is an irrational number. By density of the sequence $(\{n\alpha\})_{n\in\mathbf{N}}$, this theorem still holds when considering the gaps between the successive integers k such that $\{\alpha k\} \in I$, where I denotes any interval of the unit circle of length β.

• Furthermore, the third gap, which is the largest, can have frequency 0, when $\eta_k = \psi$, with the above notation. This means that this gap does not appear at all, as a consequence of the uniform distribution of the sequence $(\{n\alpha\})_{n\in\mathbf{N}}$ in the circle.

• The other two gaps do always appear (infinitely often, in fact, because of their positive frequencies) and are shown to be equal to the smallest positive integers l_1 and l_2 such that $\{l_1\alpha\} < \beta$ and $\{l_2\alpha\} > 1 - \beta$ (see [51]).

• The study of the rational case proves the equivalence between the three distance and the three gap theorems, as observed by Slater [51] in the case of an open interval and by Langevin, for any interval, in [35].

4.1 CONNECTEDNESS INDEX

Let $u = (u_n)_{n\in\mathbf{N}}$ be a coding of a rotation by irrational angle $0 < \alpha < 1$ with respect to the partition

$$\mathcal{P} = \{[\beta_0, \beta_1[,\ [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}\ .$$

We have seen in Section 2.1 that the sets $I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}})$, where $I_k = [\beta_k, \beta_{k+1}[$, for $0 \le j \le p-1$, are connected except for $w_1 \cdots w_n$ of the form a_K^n, where K denotes the index of the interval of $\mathcal{P}$ (if such an interval exists) of length greater than $\sup(\alpha, 1-\alpha)$.

Let us suppose that there exists an interval of $\mathcal{P}$ of length L greater than $1-\alpha$ and index K, say. We deduce from the three gap theorem that the set of integers n such that a_K^n is a factor of the sequence u is bounded. More precisely, let us define $n^{(1)}$ as the largest integer n such that a_K^n is a factor of the sequence u. We will call the integer $n^{(1)}$ the *index of connectedness* of the sequence u. (If every interval of $\mathcal{P}$ has length smaller than or equal to $\sup(\alpha, 1-\alpha)$ then the connectedness index of u is equal to 1.) The three gap theorem enables us to give an exact expression for the connectedness index. Indeed $n^{(1)} + 1$ is the largest gap between the consecutive values of k for which $0 < \{k\alpha\} < 1 - L$. We thus have the following

THEOREM 9. *Let $u = (u_n)_{n \in \mathbf{N}}$ be a coding of the rotation by irrational angle α. Suppose that there exists an interval of $\mathcal{P}$ of length $L > \sup(\alpha, 1-\alpha)$. Let $\left(\frac{p_k}{q_k}\right)_{k \in \mathbf{N}}$ and $(c_k)_{k \in \mathbf{N}}$ be the sequences of convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k(q_k\alpha - p_k)$. Write*

$$1 - L = m\eta_k + \eta_{k+1} + \psi ,$$

with $k \geq 1$, $0 < \psi \leq \eta_k$ and $1 \leq m \leq c_{k+1}$. The connectedness index $n^{(1)}$ of the sequence u satisfies

$$n^{(1)} = q_{k+1} - (m-1)q_k - 1, \text{ if } \psi \neq \eta_k ,$$

$$n^{(1)} = q_{k+1} - mq_k - 1, \text{ if } \psi = \eta_k \text{ and } m < c_{k+1} ,$$

$$n^{(1)} = q_k - 1, \text{ if } \psi = \eta_k \text{ and } m = c_{k+1} .$$

4.2 APPLICATIONS

Precise knowledge of the connectedness index is useful, as shown by the following. Indeed Lemma 1 can be rephrased as follows.

LEMMA 3. *Let u be a coding of an irrational rotation on the unit circle with respect to the partition $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}$. The frequencies of factors of u of length $n \geq n^{(1)}$, where $n^{(1)}$ denotes the connectedness index, are equal to the lengths of the intervals bounded by the points*

$$\{k(1-\alpha) + \beta_i\}, \text{ for } 0 \leq k \leq n-1, \ 0 \leq i \leq p-1 .$$

The complexity of a coding on p letters of an irrational rotation ultimately has the form $p(n) = an + b$, where $a \leq p$, and depends on the algebraic relations between the angle and the lengths of the intervals of the coding. More precisely, we have the following theorem proved in [1].

THEOREM 10. *Let $u = (u_n)_{n \in \mathbf{N}}$ be a coding of the irrational rotation R of irrational angle α with respect to the partition*

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\} .$$

Let $(k_n)_{n \in \mathbf{N}}$ be the sequence defined by

$$k_0 = p = \mathrm{card}(F) ,$$

$$k_n = \mathrm{card}\left\{\beta_i \in F;\ \forall k \in [1, \ldots, n],\ R^{-k}(\beta_i) \notin F\right\} .$$