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## REMARKS.

• Suppose that  $\alpha$  is an irrational number. By density of the sequence  $(\{n\alpha\})_{n \in \mathbb{N}}$ , this theorem still holds when considering the gaps between the successive integers  $k$  such that  $\{\alpha k\} \in I$ , where  $I$  denotes any interval of the unit circle of length  $\beta$ .

• Furthermore, the third gap, which is the largest, can have frequency 0, when  $\eta_k = \psi$ , with the above notation. This means that this gap does not appear at all, as a consequence of the uniform distribution of the sequence  $(\{n\alpha\})_{n \in \mathbb{N}}$  in the circle.

• The other two gaps do always appear (infinitely often, in fact, because of their positive frequencies) and are shown to be equal to the smallest positive integers  $l_1$  and  $l_2$  such that  $\{l_1\alpha\} < \beta$  and  $\{l_2\alpha\} > 1 - \beta$  (see [51]).

• The study of the rational case proves the equivalence between the three distance and the three gap theorems, as observed by Slater [51] in the case of an open interval and by Langevin, for any interval, in [35].

## 4.1 CONNECTEDNESS INDEX

Let  $u = (u_n)_{n \in \mathbb{N}}$  be a coding of a rotation by irrational angle  $0 < \alpha < 1$  with respect to the partition

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p\{] \} .$$

We have seen in Section 2.1 that the sets  $I(w_1, \dots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}})$ , where  $I_k = [\beta_k, \beta_{k+1}[,$  for  $0 \leq j \leq p-1$ , are connected except for  $w_1 \cdots w_n$  of the form  $a_K^n$ , where  $K$  denotes the index of the interval of  $\mathcal{P}$  (if such an interval exists) of length greater than  $\sup(\alpha, 1 - \alpha)$ .

Let us suppose that there exists an interval of  $\mathcal{P}$  of length  $L$  greater than  $1 - \alpha$  and index  $K$ , say. We deduce from the three gap theorem that the set of integers  $n$  such that  $a_K^n$  is a factor of the sequence  $u$  is bounded. More precisely, let us define  $n^{(1)}$  as the largest integer  $n$  such that  $a_K^n$  is a factor of the sequence  $u$ . We will call the integer  $n^{(1)}$  the *index of connectedness* of the sequence  $u$ . (If every interval of  $\mathcal{P}$  has length smaller than or equal to  $\sup(\alpha, 1 - \alpha)$  then the connectedness index of  $u$  is equal to 1.) The three gap theorem enables us to give an exact expression for the connectedness index. Indeed  $n^{(1)} + 1$  is the largest gap between the consecutive values of  $k$  for which  $0 < \{k\alpha\} < 1 - L$ . We thus have the following

**THEOREM 9.** *Let  $u = (u_n)_{n \in \mathbb{N}}$  be a coding of the rotation by irrational angle  $\alpha$ . Suppose that there exists an interval of  $\mathcal{P}$  of length  $L > \sup(\alpha, 1 - \alpha)$ . Let  $(\frac{p_k}{q_k})_{k \in \mathbb{N}}$  and  $(c_k)_{k \in \mathbb{N}}$  be the sequences of convergents and partial quotients associated to  $\alpha$  in its continued fraction expansion. Let  $\eta_k = (-1)^k(q_k\alpha - p_k)$ . Write*

$$1 - L = m\eta_k + \eta_{k+1} + \psi,$$

*with  $k \geq 1$ ,  $0 < \psi \leq \eta_k$  and  $1 \leq m \leq c_{k+1}$ . The connectedness index  $n^{(1)}$  of the sequence  $u$  satisfies*

$$\begin{aligned} n^{(1)} &= q_{k+1} - (m - 1)q_k - 1, \text{ if } \psi \neq \eta_k, \\ n^{(1)} &= q_{k+1} - mq_k - 1, \text{ if } \psi = \eta_k \text{ and } m < c_{k+1}, \\ n^{(1)} &= q_k - 1, \text{ if } \psi = \eta_k \text{ and } m = c_{k+1}. \end{aligned}$$

## 4.2 APPLICATIONS

Precise knowledge of the connectedness index is useful, as shown by the following. Indeed Lemma 1 can be rephrased as follows.

**LEMMA 3.** *Let  $u$  be a coding of an irrational rotation on the unit circle with respect to the partition  $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[ \}$ . The frequencies of factors of  $u$  of length  $n \geq n^{(1)}$ , where  $n^{(1)}$  denotes the connectedness index, are equal to the lengths of the intervals bounded by the points*

$$\{k(1 - \alpha) + \beta_i\}, \text{ for } 0 \leq k \leq n - 1, \quad 0 \leq i \leq p - 1.$$

The complexity of a coding on  $p$  letters of an irrational rotation ultimately has the form  $p(n) = an + b$ , where  $a \leq p$ , and depends on the algebraic relations between the angle and the lengths of the intervals of the coding. More precisely, we have the following theorem proved in [1].

**THEOREM 10.** *Let  $u = (u_n)_{n \in \mathbb{N}}$  be a coding of the irrational rotation  $R$  of irrational angle  $\alpha$  with respect to the partition*

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[ \}.$$

*Let  $(k_n)_{n \in \mathbb{N}}$  be the sequence defined by*

$$k_0 = p = \text{card}(F),$$

$$k_n = \text{card} \{ \beta_i \in F; \forall k \in [1, \dots, n], R^{-k}(\beta_i) \notin F \}.$$