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• The three distance theorem is a geometric illustration of the properties of good approximation of the n-Farey points. Indeed, the two intervals containing 0 are of distinct lengths and their lengths are the two smallest. We thus have

$$\alpha q^{(1)} - p^{(1)} = \inf \{ k\alpha, \text{ for } 0 \le k \le n \}$$

and

$$p^{(2)} - \alpha q^{(2)} = 1 - \sup\{k\alpha, \text{ for } 0 \le k \le n\}.$$

• For a deeper study of the rational case, the reader is referred for instance to [51].

4. The three gap theorem

The following theorem, called the *three gap theorem*, is in some sense the dual of the three distance theorem. This theorem was first proved by Slater (see [49] and see also [50, 51]), in the early fifties, whereas the first proofs of the three distance theorem date back to the late fifties. For other proofs of the three-gap theorem, see also [25], and more recently, [58] and [35].

The formulation of the three gap theorem quoted below is due to Slater. Following the notation of [51], let k_i be the sequence of integers k satisfying $k\alpha < \beta$. Then any difference $k_{i+1}-k_i$ is called a gap. Moreover, the *frequency* of a gap is defined as its frequency in the sequence of the successive gaps $(k_{i+1}-k_i)_{i\in\mathbb{N}}$.

THREE GAP THEOREM. Let α be an irrational number in]0,1[and let $\beta \in]0,1/2[$. The gaps between the successive integers j such that $\{\alpha j\} < \beta$ take at most three values, one being the sum of the other two.

More precisely, let $\left(\frac{p_k}{q_k}\right)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ be the sequences of the convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k (q_k \alpha - p_k)$. There exists a unique expression for β of the form

$$\beta = m\eta_k + \eta_{k+1} + \psi \,,$$

with $k \ge 0$, $0 < \psi \le \eta_k$, and if k = 0 then $1 \le m \le c_1 - 1$; otherwise, $1 \le m \le c_{k+1}$. Then the gaps between two successive j such that $\{j\alpha\} \in [0, \beta[$ satisfy the following:

- the gap q_k has frequency $(m-1)\eta_k + \eta_{k+1} + \psi$,
- the gap $q_{k+1} mq_k$ has frequency ψ ,
- the gap $q_{k+1} (m-1)q_k$ has frequency $\eta_k \psi$.

REMARKS.

- Suppose that α is an irrational number. By density of the sequence $(\{n\alpha\})_{n\in\mathbb{N}}$, this theorem still holds when considering the gaps between the successive integers k such that $\{\alpha k\} \in I$, where I denotes any interval of the unit circle of length β .
- Furthermore, the third gap, which is the largest, can have frequency 0, when $\eta_k = \psi$, with the above notation. This means that this gap does not appear at all, as a consequence of the uniform distribution of the sequence $(\{n\alpha\})_{n\in\mathbb{N}}$ in the circle.
- The other two gaps do always appear (infinitely often, in fact, because of their positive frequencies) and are shown to be equal to the smallest positive integers l_1 and l_2 such that $\{l_1\alpha\} < \beta$ and $\{l_2\alpha\} > 1 \beta$ (see [51]).
- The study of the rational case proves the equivalence between the three distance and the three gap theorems, as observed by Slater [51] in the case of an open interval and by Langevin, for any interval, in [35].

4.1 Connectedness index

Let $u = (u_n)_{n \in \mathbb{N}}$ be a coding of a rotation by irrational angle $0 < \alpha < 1$ with respect to the partition

$$\mathcal{P} = \{ [\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[]] ..., [\beta_{p-1}, \beta_p[]] ..., [\beta_{p-1}, \beta_p[]] \} ...$$

We have seen in Section 2.1 that the sets $I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}})$, where $I_k = [\beta_k, \beta_{k+1}[$, for $0 \le j \le p-1$, are connected except for $w_1 \cdots w_n$ of the form a_K^n , where K denotes the index of the interval of \mathcal{P} (if such an interval exists) of length greater than $\sup(\alpha, 1 - \alpha)$.

Let us suppose that there exists an interval of $\mathcal P$ of length L greater than $1-\alpha$ and index K, say. We deduce from the three gap theorem that the set of integers n such that a_K^n is a factor of the sequence u is bounded. More precisely, let us define $n^{(1)}$ as the largest integer n such that a_K^n is a factor of the sequence u. We will call the integer $n^{(1)}$ the index of connectedness of the sequence u. (If every interval of $\mathcal P$ has length smaller than or equal to $\sup(\alpha, 1-\alpha)$ then the connectedness index of u is equal to 1.) The three gap theorem enables us to give an exact expression for the connectedness index. Indeed $n^{(1)}+1$ is the largest gap between the consecutive values of k for which $0<\{k\alpha\}<1-L$. We thus have the following

THEOREM 9. Let $u = (u_n)_{n \in \mathbb{N}}$ be a coding of the rotation by irrational angle α . Suppose that there exists an interval of \mathcal{P} of length $L > \sup(\alpha, 1-\alpha)$. Let $\left(\frac{p_k}{q_k}\right)_{k \in \mathbb{N}}$ and $(c_k)_{k \in \mathbb{N}}$ be the sequences of convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k (q_k \alpha - p_k)$. Write

$$1 - L = m\eta_k + \eta_{k+1} + \psi,$$

with $k \ge 1$, $0 < \psi \le \eta_k$ and $1 \le m \le c_{k+1}$. The connectedness index $n^{(1)}$ of the sequence u satisfies

$$n^{(1)} = q_{k+1} - (m-1)q_k - 1$$
, if $\psi \neq \eta_k$,
 $n^{(1)} = q_{k+1} - mq_k - 1$, if $\psi = \eta_k$ and $m < c_{k+1}$,
 $n^{(1)} = q_k - 1$, if $\psi = \eta_k$ and $m = c_{k+1}$.

4.2 APPLICATIONS

Precise knowledge of the connectedness index is useful, as shown by the following. Indeed Lemma 1 can be rephrased as follows.

LEMMA 3. Let u be a coding of an irrational rotation on the unit circle with respect to the partition $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, ..., [\beta_{p-1}, \beta_p[]\}. The frequencies of factors of u of length <math>n \geq n^{(1)}$, where $n^{(1)}$ denotes the connectedness index, are equal to the lengths of the intervals bounded by the points

$$\{k(1-\alpha)+\beta_i\}$$
, for $0 \le k \le n-1$, $0 \le i \le p-1$.

The complexity of a coding on p letters of an irrational rotation ultimately has the form p(n) = an + b, where $a \le p$, and depends on the algebraic relations between the angle and the lengths of the intervals of the coding. More precisely, we have the following theorem proved in [1].

THEOREM 10. Let $u = (u_n)_{n \in \mathbb{N}}$ be a coding of the irrational rotation R of irrational angle α with respect to the partition

$$\mathcal{P} = \{ [\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[]]] ... \}]$$

Let $(k_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$k_0 = p = \operatorname{card}(F),$$

$$k_n = \operatorname{card}\left\{\beta_i \in F; \ \forall k \in [1, \dots, n], \ R^{-k}(\beta_i) \notin F\right\}.$$

Let a be the limit of this sequence, $n^{(2)}$ the smallest index such that $k_n = a$, and let

$$b = \sum_{i=0}^{n^{(2)}-1} (k_i - a).$$

Let $n^{(1)}$ denote the connectedness index of u.

If $n \ge \max(n^{(1)}, n^{(2)})$, then the complexity of the sequence u satisfies

$$p(n) = an + b.$$

REMARKS.

- Note that if $1, \alpha, \beta_1, \dots, \beta_p$ are rationally independent, then $n^{(2)} = 0$, b = 0 and a = p.
- Theorem 10 answers the question of the existence of sequences of ultimately affine complexity (for more details, the reader is referred to [1], see also the result of Cassaigne in [11]).

4.3 BEATTY SEQUENCES

The connections between the three gap theorem and the Beatty sequences have been investigated by Fraenkel and Holzman in [26]. Let us recall that a Beatty sequence is a sequence $u(\alpha, \rho) = (u_n)_{n \in \mathbb{N}}$ of the form $u_n = \lfloor \alpha n + \rho \rfloor$, where α and ρ are real numbers such that $\alpha \geq 1$. The number α is called the *modulus* and ρ is called the *residue* or *intercept*. For an impressive bibliography on the subject, we refer the reader to [27] and [54]. Fraenkel and Holzman have noticed in [26] that the three gap theorem answers the question of the gaps in the intersection of a Beatty sequence and an arithmetical sequence $(an + c)_{n \in \mathbb{N}}$, for a a positive integer and c an integer. This result has been obtained independently by Wolff and Pitman in [58]. By intersection of the two Beatty sequences $s = (s_n)_{n \in \mathbb{N}}$ and $t = (t_n)_{n \in \mathbb{N}}$, we mean the strictly increasing sequence u defined as:

$$\{u_n, n \in \mathbb{N}\} = \{u, \exists k, l \in \mathbb{N} \text{ such that } u = s_k = t_l\}$$
.

Hence a gap in the intersection denotes the difference between two distinct elements of the intersection.

Note that Beatty sequences and Sturmian sequences are related: let u be a Beatty sequence of modulus α and residue ρ ; the characteristic sequence $(v_n)_{n \in \mathbb{N}}$ of u defined as

 $v_n = 1$ if and only if there exists m such that $n = |\alpha m + \rho|$

is the Sturmian sequence obtained as the coding of the orbit of $-\rho/\alpha$ under the rotation by angle $1/\alpha$, with respect to the partition

$$\{]0, 1 - 1/\alpha],]1 - 1/\alpha, 1]\}$$
.

Indeed, if $n = \lfloor \alpha m + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = m+1 = 1 + \lceil n/\alpha - \rho/\alpha \rceil$, and if $\lfloor \alpha m + \rho \rfloor < n < \lfloor \alpha(m+1) + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = \lceil n/\alpha - \rho/\alpha \rceil$.

5. The recurrence function

Let us deduce now from the three distance and three gap theorems a simple proof of the following result originally due to Morse and Hedlund concerning the recurrence function of a Sturmian sequence (see [40]).

Recall that a sequence u is called *minimal* or *uniformly recurrent* if every factor of u appears infinitely often and with bounded gaps or, equivalently, if for any integer n, there exists an integer m such that every factor of u of length m contains every factor of u of length n. Note that it is equivalent (see [30]) to the *minimality* of the dynamical system $(\overline{\mathcal{O}(u)}, T)$, i.e., the orbit of every element of $\overline{\mathcal{O}(u)}$ is dense, or equivalently every sequence in the orbit closure of u has the same set of factors as u.

The recurrence function φ of a minimal sequence u is defined by

$$\varphi(n) = \min \{ m \in \mathbb{N} \text{ such that } \forall B \in L_m, \ \forall A \in L_n, \ A \text{ is a factor of } B \}$$

where L_n denotes the set of factors of u of length n, i.e., $\varphi(n)$ is the size of the smallest window that contains all factors of length n whatever its position in the sequence.

THEOREM 11. Let u be a Sturmian sequence with angle α . Let $(q_k)_{k \in \mathbb{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of α . The recurrence function φ of this sequence satisfies for any non zero integer n:

$$\varphi(n) = n - 1 + q_k + q_{k-1}, \text{ where } q_{k-1} \le n < q_k.$$

Proof of Theorem 11. Let $u \in \{0,1\}^N$ be a Sturmian sequence. There exist a real number x and an irrational number α in]0,1[such that $u_n = 0 \Leftrightarrow \{x + n\alpha\} \in I_0$, with $I_0 = [0,\alpha[$ or $I_0 =]0,\alpha[$ (see Section 2.1). Let $I_1 = [\alpha,1[$ (respectively, $]\alpha,1]$) if $I_0 = [0,\alpha[$ (respectively, $I_0 =]0,\alpha[$). Let us denote by R the rotation of the circle by angle α . Assume we are given