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• The three distance theorem is a geometric illustration of the properties of good approximation of the  $n$ -Farey points. Indeed, the two intervals containing 0 are of distinct lengths and their lengths are the two smallest. We thus have

$$\alpha q^{(1)} - p^{(1)} = \inf \{k\alpha, \text{ for } 0 \leq k \leq n\}$$

and

$$p^{(2)} - \alpha q^{(2)} = 1 - \sup \{k\alpha, \text{ for } 0 \leq k \leq n\}.$$

• For a deeper study of the rational case, the reader is referred for instance to [51].

#### 4. THE THREE GAP THEOREM

The following theorem, called the *three gap theorem*, is in some sense the dual of the three distance theorem. This theorem was first proved by Slater (see [49] and see also [50, 51]), in the early fifties, whereas the first proofs of the three distance theorem date back to the late fifties. For other proofs of the three-gap theorem, see also [25], and more recently, [58] and [35].

The formulation of the three gap theorem quoted below is due to Slater. Following the notation of [51], let  $k_i$  be the sequence of integers  $k$  satisfying  $k\alpha < \beta$ . Then any difference  $k_{i+1} - k_i$  is called a *gap*. Moreover, the *frequency* of a gap is defined as its frequency in the sequence of the successive gaps  $(k_{i+1} - k_i)_{i \in \mathbb{N}}$ .

**THREE GAP THEOREM.** *Let  $\alpha$  be an irrational number in  $]0, 1[$  and let  $\beta \in ]0, 1/2[$ . The gaps between the successive integers  $j$  such that  $\{\alpha j\} < \beta$  take at most three values, one being the sum of the other two.*

*More precisely, let  $(\frac{p_k}{q_k})_{k \in \mathbb{N}}$  and  $(c_k)_{k \in \mathbb{N}}$  be the sequences of the convergents and partial quotients associated to  $\alpha$  in its continued fraction expansion. Let  $\eta_k = (-1)^k(q_k\alpha - p_k)$ . There exists a unique expression for  $\beta$  of the form*

$$\beta = m\eta_k + \eta_{k+1} + \psi,$$

*with  $k \geq 0$ ,  $0 < \psi \leq \eta_k$ , and if  $k = 0$  then  $1 \leq m \leq c_1 - 1$ ; otherwise,  $1 \leq m \leq c_{k+1}$ . Then the gaps between two successive  $j$  such that  $\{j\alpha\} \in [0, \beta[$  satisfy the following:*

- *the gap  $q_k$  has frequency  $(m - 1)\eta_k + \eta_{k+1} + \psi$ ,*
- *the gap  $q_{k+1} - mq_k$  has frequency  $\psi$ ,*
- *the gap  $q_{k+1} - (m - 1)q_k$  has frequency  $\eta_k - \psi$ .*

## REMARKS.

• Suppose that  $\alpha$  is an irrational number. By density of the sequence  $(\{n\alpha\})_{n \in \mathbb{N}}$ , this theorem still holds when considering the gaps between the successive integers  $k$  such that  $\{\alpha k\} \in I$ , where  $I$  denotes any interval of the unit circle of length  $\beta$ .

• Furthermore, the third gap, which is the largest, can have frequency 0, when  $\eta_k = \psi$ , with the above notation. This means that this gap does not appear at all, as a consequence of the uniform distribution of the sequence  $(\{n\alpha\})_{n \in \mathbb{N}}$  in the circle.

• The other two gaps do always appear (infinitely often, in fact, because of their positive frequencies) and are shown to be equal to the smallest positive integers  $l_1$  and  $l_2$  such that  $\{l_1\alpha\} < \beta$  and  $\{l_2\alpha\} > 1 - \beta$  (see [51]).

• The study of the rational case proves the equivalence between the three distance and the three gap theorems, as observed by Slater [51] in the case of an open interval and by Langevin, for any interval, in [35].

## 4.1 CONNECTEDNESS INDEX

Let  $u = (u_n)_{n \in \mathbb{N}}$  be a coding of a rotation by irrational angle  $0 < \alpha < 1$  with respect to the partition

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[ \} .$$

We have seen in Section 2.1 that the sets  $I(w_1, \dots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}})$ , where  $I_k = [\beta_k, \beta_{k+1}[$ , for  $0 \leq j \leq p-1$ , are connected except for  $w_1 \cdots w_n$  of the form  $a_K^n$ , where  $K$  denotes the index of the interval of  $\mathcal{P}$  (if such an interval exists) of length greater than  $\sup(\alpha, 1 - \alpha)$ .

Let us suppose that there exists an interval of  $\mathcal{P}$  of length  $L$  greater than  $1 - \alpha$  and index  $K$ , say. We deduce from the three gap theorem that the set of integers  $n$  such that  $a_K^n$  is a factor of the sequence  $u$  is bounded. More precisely, let us define  $n^{(1)}$  as the largest integer  $n$  such that  $a_K^n$  is a factor of the sequence  $u$ . We will call the integer  $n^{(1)}$  the *index of connectedness* of the sequence  $u$ . (If every interval of  $\mathcal{P}$  has length smaller than or equal to  $\sup(\alpha, 1 - \alpha)$  then the connectedness index of  $u$  is equal to 1.) The three gap theorem enables us to give an exact expression for the connectedness index. Indeed  $n^{(1)} + 1$  is the largest gap between the consecutive values of  $k$  for which  $0 < \{k\alpha\} < 1 - L$ . We thus have the following

THEOREM 9. Let  $u = (u_n)_{n \in \mathbb{N}}$  be a coding of the rotation by irrational angle  $\alpha$ . Suppose that there exists an interval of  $\mathcal{P}$  of length  $L > \sup(\alpha, 1 - \alpha)$ . Let  $(\frac{p_k}{q_k})_{k \in \mathbb{N}}$  and  $(c_k)_{k \in \mathbb{N}}$  be the sequences of convergents and partial quotients associated to  $\alpha$  in its continued fraction expansion. Let  $\eta_k = (-1)^k(q_k\alpha - p_k)$ . Write

$$1 - L = m\eta_k + \eta_{k+1} + \psi,$$

with  $k \geq 1$ ,  $0 < \psi \leq \eta_k$  and  $1 \leq m \leq c_{k+1}$ . The connectedness index  $n^{(1)}$  of the sequence  $u$  satisfies

$$\begin{aligned} n^{(1)} &= q_{k+1} - (m-1)q_k - 1, \text{ if } \psi \neq \eta_k, \\ n^{(1)} &= q_{k+1} - mq_k - 1, \text{ if } \psi = \eta_k \text{ and } m < c_{k+1}, \\ n^{(1)} &= q_k - 1, \text{ if } \psi = \eta_k \text{ and } m = c_{k+1}. \end{aligned}$$

## 4.2 APPLICATIONS

Precise knowledge of the connectedness index is useful, as shown by the following. Indeed Lemma 1 can be rephrased as follows.

LEMMA 3. Let  $u$  be a coding of an irrational rotation on the unit circle with respect to the partition  $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[ \}$ . The frequencies of factors of  $u$  of length  $n \geq n^{(1)}$ , where  $n^{(1)}$  denotes the connectedness index, are equal to the lengths of the intervals bounded by the points

$$\{k(1 - \alpha) + \beta_i\}, \text{ for } 0 \leq k \leq n - 1, \quad 0 \leq i \leq p - 1.$$

The complexity of a coding on  $p$  letters of an irrational rotation ultimately has the form  $p(n) = an + b$ , where  $a \leq p$ , and depends on the algebraic relations between the angle and the lengths of the intervals of the coding. More precisely, we have the following theorem proved in [1].

THEOREM 10. Let  $u = (u_n)_{n \in \mathbb{N}}$  be a coding of the irrational rotation  $R$  of irrational angle  $\alpha$  with respect to the partition

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[ \}.$$

Let  $(k_n)_{n \in \mathbb{N}}$  be the sequence defined by

$$k_0 = p = \text{card}(F),$$

$$k_n = \text{card} \{ \beta_i \in F; \forall k \in [1, \dots, n], R^{-k}(\beta_i) \notin F \}.$$

Let  $a$  be the limit of this sequence,  $n^{(2)}$  the smallest index such that  $k_n = a$ , and let

$$b = \sum_{i=0}^{n^{(2)}-1} (k_i - a).$$

Let  $n^{(1)}$  denote the connectedness index of  $u$ .

If  $n \geq \max(n^{(1)}, n^{(2)})$ , then the complexity of the sequence  $u$  satisfies

$$p(n) = an + b.$$

#### REMARKS.

- Note that if  $1, \alpha, \beta_1, \dots, \beta_p$  are rationally independent, then  $n^{(2)} = 0$ ,  $b = 0$  and  $a = p$ .

- Theorem 10 answers the question of the existence of sequences of ultimately affine complexity (for more details, the reader is referred to [1], see also the result of Cassaigne in [11]).

### 4.3 BEATTY SEQUENCES

The connections between the three gap theorem and the Beatty sequences have been investigated by Fraenkel and Holzman in [26]. Let us recall that a Beatty sequence is a sequence  $u(\alpha, \rho) = (u_n)_{n \in \mathbf{N}}$  of the form  $u_n = \lfloor \alpha n + \rho \rfloor$ , where  $\alpha$  and  $\rho$  are real numbers such that  $\alpha \geq 1$ . The number  $\alpha$  is called the *modulus* and  $\rho$  is called the *residue* or *intercept*. For an impressive bibliography on the subject, we refer the reader to [27] and [54]. Fraenkel and Holzman have noticed in [26] that the three gap theorem answers the question of the gaps in the intersection of a Beatty sequence and an arithmetical sequence  $(an + c)_{n \in \mathbf{N}}$ , for  $a$  a positive integer and  $c$  an integer. This result has been obtained independently by Wolff and Pitman in [58]. By intersection of the two Beatty sequences  $s = (s_n)_{n \in \mathbf{N}}$  and  $t = (t_n)_{n \in \mathbf{N}}$ , we mean the strictly increasing sequence  $u$  defined as:

$$\{u_n, n \in \mathbf{N}\} = \{u, \exists k, l \in \mathbf{N} \text{ such that } u = s_k = t_l\}.$$

Hence a gap in the intersection denotes the difference between two distinct elements of the intersection.

Note that Beatty sequences and Sturmian sequences are related: let  $u$  be a Beatty sequence of modulus  $\alpha$  and residue  $\rho$ ; the characteristic sequence  $(v_n)_{n \in \mathbf{N}}$  of  $u$  defined as

$$v_n = 1 \text{ if and only if there exists } m \text{ such that } n = \lfloor \alpha m + \rho \rfloor$$

is the Sturmian sequence obtained as the coding of the orbit of  $-\rho/\alpha$  under the rotation by angle  $1/\alpha$ , with respect to the partition

$$\{]0, 1 - 1/\alpha], ]1 - 1/\alpha, 1]\}.$$

Indeed, if  $n = \lfloor \alpha m + \rho \rfloor$ , then  $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = m+1 = 1 + \lceil n/\alpha - \rho/\alpha \rceil$ , and if  $\lfloor \alpha m + \rho \rfloor < n < \lfloor \alpha(m+1) + \rho \rfloor$ , then  $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = \lceil n/\alpha - \rho/\alpha \rceil$ .

## 5. THE RECURRENCE FUNCTION

Let us deduce now from the three distance and three gap theorems a simple proof of the following result originally due to Morse and Hedlund concerning the recurrence function of a Sturmian sequence (see [40]).

Recall that a sequence  $u$  is called *minimal* or *uniformly recurrent* if every factor of  $u$  appears infinitely often and with bounded gaps or, equivalently, if for any integer  $n$ , there exists an integer  $m$  such that every factor of  $u$  of length  $m$  contains every factor of  $u$  of length  $n$ . Note that it is equivalent (see [30]) to the *minimality* of the dynamical system  $(\overline{\mathcal{O}(u)}, T)$ , i.e., the orbit of every element of  $\overline{\mathcal{O}(u)}$  is dense, or equivalently every sequence in the orbit closure of  $u$  has the same set of factors as  $u$ .

The recurrence function  $\varphi$  of a minimal sequence  $u$  is defined by

$$\varphi(n) = \min \{m \in \mathbf{N} \text{ such that } \forall B \in L_m, \forall A \in L_n, A \text{ is a factor of } B\},$$

where  $L_n$  denotes the set of factors of  $u$  of length  $n$ , i.e.,  $\varphi(n)$  is the size of the smallest window that contains all factors of length  $n$  whatever its position in the sequence.

**THEOREM 11.** *Let  $u$  be a Sturmian sequence with angle  $\alpha$ . Let  $(q_k)_{k \in \mathbf{N}}$  denote the sequence of denominators of the convergents of the continued fraction expansion of  $\alpha$ . The recurrence function  $\varphi$  of this sequence satisfies for any non zero integer  $n$ :*

$$\varphi(n) = n - 1 + q_k + q_{k-1}, \text{ where } q_{k-1} \leq n < q_k.$$

*Proof of Theorem 11.* Let  $u \in \{0, 1\}^{\mathbf{N}}$  be a Sturmian sequence. There exist a real number  $x$  and an irrational number  $\alpha$  in  $]0, 1[$  such that  $u_n = 0 \Leftrightarrow \{x + n\alpha\} \in I_0$ , with  $I_0 = [0, \alpha[$  or  $I_0 = ]0, \alpha]$  (see Section 2.1). Let  $I_1 = [\alpha, 1[$  (respectively,  $]\alpha, 1]$ ) if  $I_0 = [0, \alpha[$  (respectively,  $I_0 = ]0, \alpha]$ ). Let us denote by  $R$  the rotation of the circle by angle  $\alpha$ . Assume we are given