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**Autor:** Alessandri, Pascal / Berthé, Valérie  
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We deduce from Theorem 8 that the lengths of the intervals  $I(w_1, \dots, w_n)$ , and thus the lengths of the intervals obtained by placing the points  $0, \{1 - \alpha\}, \dots, \{n(1 - \alpha)\}$  on the unit circle, take at most three values. Hence Theorem 8 is equivalent to the three distance theorem and provides a combinatorial proof of this result.

REMARK. In fact this point of view, and more precisely the study of the evolution of the graphs of words with respect to the length  $n$  of the factors, allows us to give a proof of the most complete version of the three distance theorem as given in [53] (for more details, the reader is referred to [3]).

### 3. THE THREE DISTANCE THEOREM

The three distance theorem was initially conjectured by Steinhaus, first proved by V. T. Sós (see [53] and also [52]), and then by Świerczkowski [56], Surányi [55], Slater [51], Halton [31]. More recent proofs have also been given by van Ravenstein [44] and Langevin [35]. A survey of the different approaches used by these authors is to be found in [44, 51, 35]. In the literature this theorem is called *the Steinhaus theorem*, *the three length, three gap* or *the three step theorem*. In order to avoid any ambiguity, we will always call it the three distance theorem, reserving the name *three gap* for the theorem introduced in the next section.

THREE DISTANCE THEOREM. *Let  $0 < \alpha < 1$  be an irrational number and  $n$  a positive integer. The points  $\{i\alpha\}$ , for  $0 \leq i \leq n$ , partition the unit circle into  $n + 1$  intervals, the lengths of which take at most three values, one being the sum of the other two.*

*More precisely, let  $(\frac{p_k}{q_k})_{k \in \mathbb{N}}$  and  $(c_k)_{k \in \mathbb{N}}$  be the sequences of convergents and partial quotients associated to  $\alpha$  in its continued fraction expansion (if  $\alpha = [0, c_1, c_2, \dots]$ , then  $\frac{p_n}{q_n} = [0, c_1, \dots, c_n]$ ). Let  $\eta_k = (-1)^k(q_k\alpha - p_k)$ . Let  $n$  be a positive integer. There exists a unique expression for  $n$  of the form*

$$n = mq_k + q_{k-1} + r,$$

*with  $1 \leq m \leq c_{k+1}$  and  $0 \leq r < q_k$ . Then the circle is divided by the points  $0, \{\alpha\}, \{2\alpha\}, \dots, \{n\alpha\}$  into  $n + 1$  intervals which satisfy:*

- $n + 1 - q_k$  of them have length  $\eta_k$  (which is the largest of the three lengths),
- $r + 1$  have length  $\eta_{k-1} - m\eta_k$ ,
- $q_k - (r + 1)$  have length  $\eta_{k-1} - (m - 1)\eta_k$ .

## REMARKS.

• One can reformulate this result in terms of  $n$ -Farey points. Let us recall that an  $n$ -Farey point is a rational element  $\frac{p}{q}$  of  $[0, 1]$  such that  $p \geq 0$ ,  $1 \leq q \leq n$  and  $p, q$  are coprime (see [32] for instance). Note that the two successive  $n$ -Farey points, say  $\frac{p^{(1)}}{q^{(1)}}$  and  $\frac{p^{(2)}}{q^{(2)}}$ , satisfying  $\frac{p^{(1)}}{q^{(1)}} < \alpha < \frac{p^{(2)}}{q^{(2)}}$  are  $\frac{p_k}{q_k}$  and  $\frac{mp_k + p_{k-1}}{mq_k + q_{k-1}}$ , with the above notation. The three distance theorem states that the lengths of the intervals belong to the set

$$\{p^{(2)} - \alpha q^{(2)}, \alpha q^{(1)} - p^{(1)}, \alpha(q^{(1)} - q^{(2)}) + p^{(2)} - p^{(1)}\}.$$

• As  $\alpha$  is irrational, the three lengths are distinct. The third length in the above theorem, which is the largest since it is the sum of the two others, appears if and only if

$$n \neq q^{(1)} + q^{(2)} - 1 = (m + 1)q_k + q_{k-1} - 1.$$

Thus there are infinitely many integers  $n$  for which there are only two lengths. The other two lengths do always appear.

• The structure and the transformation rules for the partitioning in two-length intervals are studied in details in [44]. Furthermore, in [45] van Ravenstein, Winley and Tognetti prove the following: for  $\alpha$  having as sequence of partial quotients the constant sequence  $aaaa \dots$ , label by large and small the lengths of intervals of the partition  $\{i\alpha\}$ , for  $0 \leq i \leq q_n + q_{n-1} - 1$ , where  $q_n$  is the denominator of a reduced convergent of  $\alpha$  (there are only two lengths in this case); this binary finite sequence of lengths is a prefix after a permutation of the characteristic sequence of  $\alpha$  (i.e., the Sturmian coding of the orbit of  $\alpha$ ). For a precise study of the limit points of these finite binary sequences (corresponding to the two-length case), see [48].

• In the two-length case, it is easily seen that the largest length is less than or equal to twice the second one. In [14] (see also [15, 16]) Chevallier extends this result to the two-dimensional torus  $\mathbf{T}^2$ , by studying the notion of best approximation.

• The point  $\{(n + 1)\alpha\}$  belongs to an interval of largest length in the partition of the unit circle by the points  $\{i\alpha\}$ , for  $0 \leq i \leq n$ .

• The three distance theorem is a geometric illustration of the properties of good approximation of the  $n$ -Farey points. Indeed, the two intervals containing 0 are of distinct lengths and their lengths are the two smallest. We thus have

$$\alpha q^{(1)} - p^{(1)} = \inf \{k\alpha, \text{ for } 0 \leq k \leq n\}$$

and

$$p^{(2)} - \alpha q^{(2)} = 1 - \sup \{k\alpha, \text{ for } 0 \leq k \leq n\}.$$

• For a deeper study of the rational case, the reader is referred for instance to [51].

#### 4. THE THREE GAP THEOREM

The following theorem, called the *three gap theorem*, is in some sense the dual of the three distance theorem. This theorem was first proved by Slater (see [49] and see also [50, 51]), in the early fifties, whereas the first proofs of the three distance theorem date back to the late fifties. For other proofs of the three-gap theorem, see also [25], and more recently, [58] and [35].

The formulation of the three gap theorem quoted below is due to Slater. Following the notation of [51], let  $k_i$  be the sequence of integers  $k$  satisfying  $k\alpha < \beta$ . Then any difference  $k_{i+1} - k_i$  is called a *gap*. Moreover, the *frequency* of a gap is defined as its frequency in the sequence of the successive gaps  $(k_{i+1} - k_i)_{i \in \mathbb{N}}$ .

**THREE GAP THEOREM.** *Let  $\alpha$  be an irrational number in  $]0, 1[$  and let  $\beta \in ]0, 1/2[$ . The gaps between the successive integers  $j$  such that  $\{\alpha j\} < \beta$  take at most three values, one being the sum of the other two.*

*More precisely, let  $(\frac{p_k}{q_k})_{k \in \mathbb{N}}$  and  $(c_k)_{k \in \mathbb{N}}$  be the sequences of the convergents and partial quotients associated to  $\alpha$  in its continued fraction expansion. Let  $\eta_k = (-1)^k(q_k\alpha - p_k)$ . There exists a unique expression for  $\beta$  of the form*

$$\beta = m\eta_k + \eta_{k+1} + \psi,$$

*with  $k \geq 0$ ,  $0 < \psi \leq \eta_k$ , and if  $k = 0$  then  $1 \leq m \leq c_1 - 1$ ; otherwise,  $1 \leq m \leq c_{k+1}$ . Then the gaps between two successive  $j$  such that  $\{j\alpha\} \in [0, \beta[$  satisfy the following:*

- *the gap  $q_k$  has frequency  $(m - 1)\eta_k + \eta_{k+1} + \psi$ ,*
- *the gap  $q_{k+1} - mq_k$  has frequency  $\psi$ ,*
- *the gap  $q_{k+1} - (m - 1)q_k$  has frequency  $\eta_k - \psi$ .*