Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THREE DISTANCE THEOREMS AND COMBINATORICS ON WORDS
Autor: Alessandri, Pascal / Berthé, Valérie
Kapitel: 3. The three distance theorem
DOI: https://doi.org/10.5169/seals-63900

We deduce from Theorem 8 that the lengths of the intervals $I(w_1, \ldots, w_n)$, and thus the lengths of the intervals obtained by placing the points $0, \{1-\alpha\}, \ldots, \{n(1-\alpha)\}$ on the unit circle, take at most three values. Hence Theorem 8 is equivalent to the three distance theorem and provides a combinatorial proof of this result.

REMARK. In fact this point of view, and more precisely the study of the evolution of the graphs of words with respect to the length n of the factors, allows us to give a proof of the most complete version of the three distance theorem as given in [53] (for more details, the reader is referred to [3]).

3. THE THREE DISTANCE THEOREM

The three distance theorem was initially conjectured by Steinhaus, first proved by V. T. Sós (see [53] and also [52]), and then by Świerczkowski [56], Surányi [55], Slater [51], Halton [31]. More recent proofs have also been given by van Ravenstein [44] and Langevin [35]. A survey of the different approaches used by these authors is to be found in [44, 51, 35]. In the literature this theorem is called *the Steinhaus theorem*, *the three length, three gap* or *the three step theorem*. In order to avoid any ambiguity, we will always call it the three distance theorem, reserving the name *three gap* for the theorem introduced in the next section.

THREE DISTANCE THEOREM. *Let $0 < \alpha < 1$ be an irrational number and n a positive integer. The points $\{i\alpha\}$, for $0 \leq i \leq n$, partition the unit circle into $n+1$ intervals, the lengths of which take at most three values, one being the sum of the other two.*

More precisely, let $\left(\frac{p_k}{q_k}\right)_{k\in\mathbf{N}}$ and $(c_k)_{k\in\mathbf{N}}$ be the sequences of convergents and partial quotients associated to α in its continued fraction expansion (if $\alpha = [0, c_1, c_2, \ldots]$, then $\frac{p_n}{q_n} = [0, c_1, \ldots, c_n]$). Let $\eta_k = (-1)^k(q_k\alpha - p_k)$. Let n be a positive integer. There exists a unique expression for n of the form

$$n = mq_k + q_{k-1} + r,$$

with $1 \leq m \leq c_{k+1}$ and $0 \leq r < q_k$. Then the circle is divided by the points $0, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$ into $n+1$ intervals which satisfy:

- $n+1-q_k$ *of them have length* η_k *(which is the largest of the three lengths),*
- $r+1$ *have length* $\eta_{k-1}-m\eta_k$*,*
- $q_k-(r+1)$ *have length* $\eta_{k-1}-(m-1)\eta_k$ *.*

REMARKS.

• One can reformulate this result in terms of *n-Farey points*. Let us recall that an n-Farey point is a rational element $\frac{p}{q}$ of $[0,1]$ such that $p \geq 0$, $1 \leq q \leq n$ and p, q are coprime (see [32] for instance). Note that the two successive n-Farey points, say $\frac{p^{(1)}}{q^{(1)}}$ and $\frac{p^{(2)}}{q^{(2)}}$, satisfying $\frac{p^{(1)}}{q^{(1)}} < \alpha < \frac{p^{(2)}}{q^{(2)}}$ are $\frac{p_k}{q_k}$ and $\frac{mp_k+p_{k-1}}{mq_k+q_{k-1}}$, with the above notation. The three distance theorem states that the lengths of the intervals belong to the set

$$\{p^{(2)}-\alpha q^{(2)},\ \alpha q^{(1)}-p^{(1)},\ \alpha(q^{(1)}-q^{(2)})+p^{(2)}-p^{(1)}\}\ .$$

• As α is irrational, the three lengths are distinct. The third length in the above theorem, which is the largest since it is the sum of the two others, appears if and only if

$$n \neq q^{(1)}+q^{(2)}-1=(m+1)\,q_k+q_{k-1}-1\,.$$

Thus there are infinitely many integers n for which there are only two lengths. The other two lengths do always appear.

• The structure and the transformation rules for the partitioning in two-length intervals are studied in details in [44]. Furthermore, in [45] van Ravenstein, Winley and Tognetti prove the following: for α having as sequence of partial quotients the constant sequence $aaaa\cdots$, label by large and small the lengths of intervals of the partition $\{i\alpha\}$, for $0 \leq i \leq q_n+q_{n-1}-1$, where q_n is the denominator of a reduced convergent of α (there are only two lengths in this case); this binary finite sequence of lengths is a prefix after a permutation of the characteristic sequence of α (i.e., the Sturmian coding of the orbit of α). For a precise study of the limit points of these finite binary sequences (corresponding to the two-length case), see [48].

• In the two-length case, it is easily seen that the largest length is less than or equal to twice the second one. In [14] (see also [15, 16]) Chevallier extends this result to the two-dimensional torus $\mathbf{T}^2$, by studying the notion of best approximation.

• The point $\{(n+1)\alpha\}$ belongs to an interval of largest length in the partition of the unit circle by the points $\{i\alpha\}$, for $0 \leq i \leq n$.

• The three distance theorem is a geometric illustration of the properties of good approximation of the n-Farey points. Indeed, the two intervals containing 0 are of distinct lengths and their lengths are the two smallest. We thus have

$$\alpha q^{(1)} - p^{(1)} = \inf\{k\alpha, \text{ for } 0 \leq k \leq n\}$$

and

$$p^{(2)} - \alpha q^{(2)} = 1 - \sup\{k\alpha, \text{ for } 0 \leq k \leq n\}.$$

• For a deeper study of the rational case, the reader is referred for instance to [51].

4. THE THREE GAP THEOREM

The following theorem, called the *three gap theorem*, is in some sense the dual of the three distance theorem. This theorem was first proved by Slater (see [49] and see also [50, 51]), in the early fifties, whereas the first proofs of the three distance theorem date back to the late fifties. For other proofs of the three-gap theorem, see also [25], and more recently, [58] and [35].

The formulation of the three gap theorem quoted below is due to Slater. Following the notation of [51], let k_i be the sequence of integers k satisfying $k\alpha < \beta$. Then any difference $k_{i+1} - k_i$ is called a *gap*. Moreover, the *frequency* of a gap is defined as its frequency in the sequence of the successive gaps $(k_{i+1} - k_i)_{i \in \mathbf{N}}$.

THREE GAP THEOREM. *Let α be an irrational number in $]0,1[$ and let $\beta \in]0,1/2[$. The gaps between the successive integers j such that $\{\alpha j\} < \beta$ take at most three values, one being the sum of the other two.*

More precisely, let $\left(\frac{p_k}{q_k}\right)_{k \in \mathbf{N}}$ and $(c_k)_{k \in \mathbf{N}}$ be the sequences of the convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k (q_k \alpha - p_k)$. There exists a unique expression for β of the form

$$\beta = m\eta_k + \eta_{k+1} + \psi,$$

with $k \geq 0$, $0 < \psi \leq \eta_k$, and if $k = 0$ then $1 \leq m \leq c_1 - 1$; otherwise, $1 \leq m \leq c_{k+1}$. Then the gaps between two successive j such that $\{j\alpha\} \in [0, \beta[$ satisfy the following:

- *the gap q_k has frequency $(m-1)\eta_k + \eta_{k+1} + \psi$,*
- *the gap $q_{k+1} - mq_k$ has frequency ψ,*
- *the gap $q_{k+1} - (m-1)q_k$ has frequency $\eta_k - \psi$.*