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We deduce from Theorem 8 that the lengths of the intervals  $I(w_1, \ldots, w_n)$ , and thus the lengths of the intervals obtained by placing the points  $0, \{1-\alpha\}, \ldots, \{n(1-\alpha)\}$  on the unit circle, take at most three values. Hence Theorem 8 is equivalent to the three distance theorem and provides a combinatorial proof of this result.

REMARK. In fact this point of view, and more precisely the study of the evolution of the graphs of words with respect to the length n of the factors, allows us to give a proof of the most complete version of the three distance theorem as given in [53] (for more details, the reader is referred to [3]).

## 3. The three distance theorem

The three distance theorem was initially conjectured by Steinhaus, first proved by V. T. Sós (see [53] and also [52]), and then by Świerczkowski [56], Surányi [55], Slater [51], Halton [31]. More recent proofs have also been given by van Ravenstein [44] and Langevin [35]. A survey of the different approaches used by these authors is to be found in [44, 51, 35]. In the literature this theorem is called *the Steinhaus theorem*, *the three length*, *three gap* or *the three step theorem*. In order to avoid any ambiguity, we will always call it the three distance theorem, reserving the name *three gap* for the theorem introduced in the next section.

THREE DISTANCE THEOREM. Let  $0 < \alpha < 1$  be an irrational number and n a positive integer. The points  $\{i\alpha\}$ , for  $0 \le i \le n$ , partition the unit circle into n+1 intervals, the lengths of which take at most three values, one being the sum of the other two.

More precisely, let  $\left(\frac{p_k}{q_k}\right)_{k\in\mathbb{N}}$  and  $(c_k)_{k\in\mathbb{N}}$  be the sequences of convergents and partial quotients associated to  $\alpha$  in its continued fraction expansion (if  $\alpha=[0,c_1,c_2,\ldots]$ , then  $\frac{p_n}{q_n}=[0,c_1,\ldots,c_n]$ ). Let  $\eta_k=(-1)^k(q_k\alpha-p_k)$ . Let n be a positive integer. There exists a unique expression for n of the form

$$n = mq_k + q_{k-1} + r,$$

with  $1 \le m \le c_{k+1}$  and  $0 \le r < q_k$ . Then the circle is divided by the points  $0, \{\alpha\}, \{2\alpha\}, \dots, \{n\alpha\}$  into n+1 intervals which satisfy:

- $n+1-q_k$  of them have length  $\eta_k$  (which is the largest of the three lengths),
- r+1 have length  $\eta_{k-1}-m\eta_k$ ,
- $q_k (r+1)$  have length  $\eta_{k-1} (m-1)\eta_k$ .

### REMARKS.

• One can reformulate this result in terms of *n-Farey points*. Let us recall that an *n*-Farey point is a rational element  $\frac{p}{q}$  of [0,1] such that  $p \geq 0$ ,  $1 \leq q \leq n$  and p, q are coprime (see [32] for instance). Note that the two successive *n*-Farey points, say  $\frac{p^{(1)}}{q^{(1)}}$  and  $\frac{p^{(2)}}{q^{(2)}}$ , satisfying  $\frac{p^{(1)}}{q^{(1)}} < \alpha < \frac{p^{(2)}}{q^{(2)}}$  are  $\frac{p_k}{q_k}$  and  $\frac{mp_k+p_{k-1}}{mq_k+q_{k-1}}$ , with the above notation. The three distance theorem states that the lengths of the intervals belong to the set

$$\left\{p^{(2)} - \alpha q^{(2)}, \ \alpha q^{(1)} - p^{(1)}, \ \alpha (q^{(1)} - q^{(2)}) + p^{(2)} - p^{(1)}\right\}$$

 $\bullet$  As  $\alpha$  is irrational, the three lengths are distinct. The third length in the above theorem, which is the largest since it is the sum of the two others, appears if and only if

$$n \neq q^{(1)} + q^{(2)} - 1 = (m+1)q_k + q_{k-1} - 1$$
.

Thus there are infinitely many integers n for which there are only two lengths. The other two lengths do always appear.

- The structure and the transformation rules for the partitioning in two-length intervals are studied in details in [44]. Furthermore, in [45] van Ravenstein, Winley and Tognetti prove the following: for  $\alpha$  having as sequence of partial quotients the constant sequence  $aaaa\cdots$ , label by large and small the lengths of intervals of the partition  $\{i\alpha\}$ , for  $0 \le i \le q_n + q_{n-1} 1$ , where  $q_n$  is the denominator of a reduced convergent of  $\alpha$  (there are only two lengths in this case); this binary finite sequence of lengths is a prefix after a permutation of the characteristic sequence of  $\alpha$  (i.e., the Sturmian coding of the orbit of  $\alpha$ ). For a precise study of the limit points of these finite binary sequences (corresponding to the two-length case), see [48].
- In the two-length case, it is easily seen that the largest length is less than or equal to twice the second one. In [14] (see also [15, 16]) Chevallier extends this result to the two-dimensional torus  $T^2$ , by studying the notion of best approximation.
- The point  $\{(n+1)\alpha\}$  belongs to an interval of largest length in the partition of the unit circle by the points  $\{i\alpha\}$ , for  $0 \le i \le n$ .

• The three distance theorem is a geometric illustration of the properties of good approximation of the n-Farey points. Indeed, the two intervals containing 0 are of distinct lengths and their lengths are the two smallest. We thus have

$$\alpha q^{(1)} - p^{(1)} = \inf \{ k\alpha, \text{ for } 0 \le k \le n \}$$

and

$$p^{(2)} - \alpha q^{(2)} = 1 - \sup\{k\alpha, \text{ for } 0 \le k \le n\}.$$

• For a deeper study of the rational case, the reader is referred for instance to [51].

# 4. The three gap theorem

The following theorem, called the *three gap theorem*, is in some sense the dual of the three distance theorem. This theorem was first proved by Slater (see [49] and see also [50, 51]), in the early fifties, whereas the first proofs of the three distance theorem date back to the late fifties. For other proofs of the three-gap theorem, see also [25], and more recently, [58] and [35].

The formulation of the three gap theorem quoted below is due to Slater. Following the notation of [51], let  $k_i$  be the sequence of integers k satisfying  $k\alpha < \beta$ . Then any difference  $k_{i+1}-k_i$  is called a gap. Moreover, the frequency of a gap is defined as its frequency in the sequence of the successive gaps  $(k_{i+1}-k_i)_{i\in\mathbb{N}}$ .

THREE GAP THEOREM. Let  $\alpha$  be an irrational number in ]0,1[ and let  $\beta \in ]0,1/2[$ . The gaps between the successive integers j such that  $\{\alpha j\} < \beta$  take at most three values, one being the sum of the other two.

More precisely, let  $\left(\frac{p_k}{q_k}\right)_{k\in\mathbb{N}}$  and  $(c_k)_{k\in\mathbb{N}}$  be the sequences of the convergents and partial quotients associated to  $\alpha$  in its continued fraction expansion. Let  $\eta_k = (-1)^k (q_k \alpha - p_k)$ . There exists a unique expression for  $\beta$  of the form

$$\beta = m\eta_k + \eta_{k+1} + \psi \,,$$

with  $k \ge 0$ ,  $0 < \psi \le \eta_k$ , and if k = 0 then  $1 \le m \le c_1 - 1$ ; otherwise,  $1 \le m \le c_{k+1}$ . Then the gaps between two successive j such that  $\{j\alpha\} \in [0, \beta[$  satisfy the following:

- the gap  $q_k$  has frequency  $(m-1)\eta_k + \eta_{k+1} + \psi$ ,
- the gap  $q_{k+1} mq_k$  has frequency  $\psi$ ,
- the gap  $q_{k+1} (m-1)q_k$  has frequency  $\eta_k \psi$ .